

Surjective Functions

Definition: A function $f : A \rightarrow B$ is **surjective** or **onto** if $f(A) = B$, in other words, if

$$\forall b \in B \exists a \in A \ni f(a) = b .$$

A function $f : A \rightarrow B$ is **bijective** if it is both injective and surjective.

Assume for a little while that both A and B are finite:

$$|A| = m \text{ and } |B| = n .$$

If $m < n$ then there are no surjective functions $f : A \rightarrow B$.

If $m = n$ then a function $f : A \rightarrow B$ is surjective iff it is injective, and there are $m! = n!$ of them. (why?)

Theorem: The number of ways you can distribute m distinguishable widgets over n indistinguishable bins in such a way that every bin contains at least one widget is

$$S_{m,n} = \begin{cases} 0 & \text{if } m < n \\ \frac{1}{n!} \sigma_{m,n} & \text{if } m \geq n \end{cases}$$

The numbers $S_{m,n}$ are called the **Stirling Numbers of the Second Kind**.

Example: The new associate brings with her 11 files on which she has been working for years. She is given 5 secretaries from the pool. On the first day, when she has had no chance yet to meet them, she wants to dole out the 11 files to them in such a way that every one of them gets at least one.

In how many ways can she do that?

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$$\text{In } S_{11,5} = \frac{1}{5!} \sigma_{11,5} \text{ ways.}$$

By X-mas she has learned their names and wishes to distribute \$100 in \$5 bills to them in such a way that every one gets at least \$5. In how many ways can she do that?

$$\text{In } \binom{15 + (5-1)}{(5-1)} \text{ ways.}$$

Theorem:

$$S_{m+1,n} = S_{m,n-1} + nS_{m,n} .$$

Proof: $S_{m,n-1}$ counts the distributions in which the first of the $m+1$ widgets is in a container by itself, $nS_{m,n}$ counts the dis-

tributions gotten by putting it into a container that has already some widgets in it.

```
(defun SNSK (m n)
  "Stirling Number of the Second Kind"
  (cond
    ((< m n) 0) ((= n 0) 0) ((= m n) 1)
    (t (+ (SNSK (1- m) (1- n))
          (* n (SNSK (1- m) n))))))
```

Special Functions

Definition: Let $A, B, C \subseteq \mathcal{U}$. A function

$$\diamond : A \times B \rightarrow C$$

is called a **Binary Operation**. Of particular interest is the case $A = B = C$. A **Binary Operation on A** is a function

$$\diamond : A \times A \rightarrow A, (a, a') \mapsto a \diamond a'.$$

Examples: $(\mathbb{Z}, +)$, (\mathbb{Z}, \times) , $(\mathbb{Z}, -)$, $(\mathbb{Q}, +)$, (\mathbb{Q}, \times) , $(\mathbb{Q}, -)$, $(\mathbb{R}, +)$, (\mathbb{R}, \times) , $(\mathbb{R}, -)$, **etc.**

Consider for a while a fixed binary operation $\diamond : A \times A \rightarrow A$

Definition: $e \in A$ is an **identity** or a **neutral element** for \diamond if

$$a \diamond e = e \diamond a = a \quad \forall a \in A .$$

Lemma: \diamond has at most one identity.

Proof: Suppose e, \bar{e} are both identities. Then

$$e = e \diamond \bar{e} = \bar{e} . \quad \text{QED}$$

Example: Let $A = \{a, b, c, d, e\}$.

1) How many binary operations are there on A ?

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1) How many binary operations are there on A ?

As many as you can make.

\diamond	a	b	c	d	e
a	*	*	*	*	*
b	*	*	*	*	*
c	*	*	*	*	*
d	*	*	*	*	*
e	*	*	*	*	*

$* \in \{a, b, c, d, e\}$

$$5^{5 \times 5} .$$

2) How many binary operations are there on A for which a is a neutral element?

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As many as you can make.

\diamond a b c d e

a a b c d e

b b $*$ $*$ $*$ $*$

c c $*$ $*$ $*$ $*$

d d $*$ $*$ $*$ $*$

e e $*$ $*$ $*$ $*$

$*$ \in $\{a, b, c, d, e\}$

$5^{4 \times 4}$.

3) How many binary operations are there on A that have an identity?

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$$5 \times 5^{4 \times 4} .$$

Definition: \diamond is **commutative** if

$$a \diamond b = b \diamond a \quad \forall a, b \in A .$$

4) How many commutative binary operations are there on A ?

4) How many commutative binary operations are there on A ?

As many as you can make.

\diamond a b c d e

a $*$ $*$ $*$ $*$ $*$

b $*$ $*$ $*$ $*$

c $*$ $*$ $*$

d $*$ $*$

e $*$

$*$ \in $\{a, b, c, d, e\}$

$$5^{5 \times 6 / 2} .$$

5) How many commutative binary operations are there on A for which a is a neutral element?

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As many as you can make.

\diamond	a	b	c	d	e
a	a	b	c	d	e
b	b	*	*	*	*
c	c		*	*	*
d	d			*	*
e	e				*

$* \in \{a, b, c, d, e\}$

$$5^{4 \times 5 / 2} .$$

6) How many commutative binary operations are there on A that have an identity?

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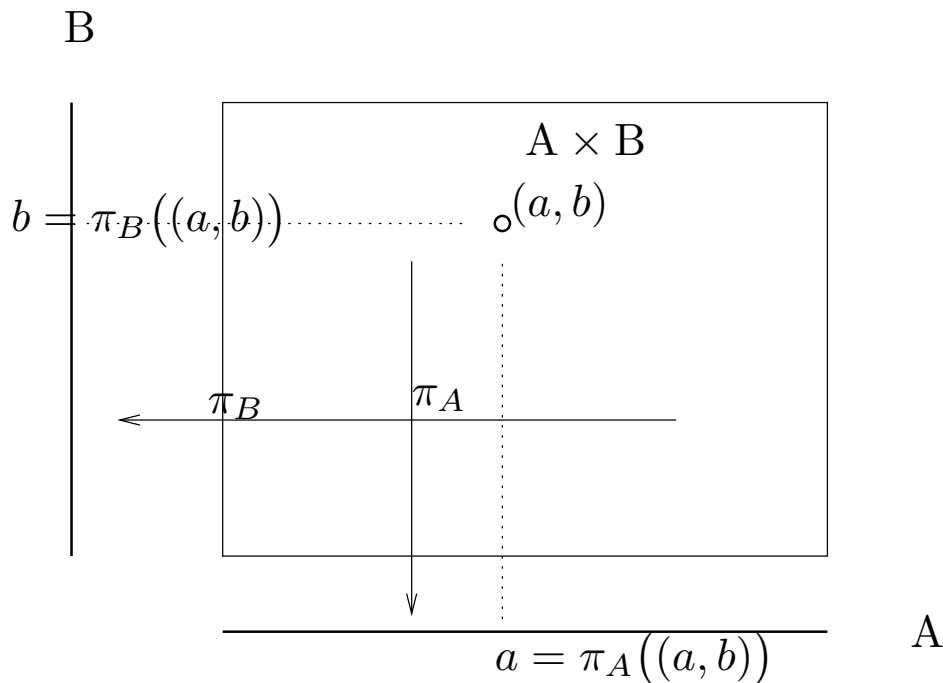
$$5 \times 5^{4 \times 5/2} .$$

Projections

Definition: Let A, B be sets. The functions

$$\begin{aligned} \pi_A : A \times B &\rightarrow A, (a, b) \mapsto a && \text{and} \\ \pi_B : A \times B &\rightarrow B, (a, b) \mapsto b \end{aligned}$$

are called the **projections** of $A \times B$ onto A, B , respectively.



Identity Functions

Let A be a set. **The Identity Function on A** is the function

$$I_A : A \rightarrow A, a \mapsto a \quad \forall a \in A.$$

I_A is obviously bijective.