Relations Revisited

We investigate here relations $\mathcal{R} : A \to A$ from a set A to itself, "Relations on A."
Of particular interest are those that have one or more of the following properties:

Definition: **Let $\mathcal{R} \subseteq A \times A$ be a relation on A.**

a) $\mathcal{R}$ is reflexive if $a\mathcal{R}a \; \forall a \in A$.

b) $\mathcal{R}$ is transitive if

$$a\mathcal{R}b \wedge b\mathcal{R}c \implies a\mathcal{R}c, \qquad \forall a, b, c \in A \,.$$

c) $\mathcal{R}$ is symmetric if

$$a\mathcal{R}b \implies b\mathcal{R}a \qquad \forall a, b \in A \,.$$

d) $\mathcal{R}$ is antisymmetric if

$$a\mathcal{R}b \wedge b\mathcal{R}a \implies a = b, \qquad \forall a, b \in A \,.$$

Discussion and Examples:

a): The equality relation on A

$$=_A \stackrel{\text{def}}{=} \{(a,a) : a \in A\}$$

is sometimes called the diagonal and denoted by Δ. The reflexivity a) means but

$$=_A \subseteq \mathcal{R}\ .$$

b): Given relations $\mathcal{R} : A \to B$ and $\mathcal{S} : B \to C$ their composition $\mathcal{R} \circ \mathcal{S} : A \to C$, pronounced "$\mathcal{R}$ before $\mathcal{S}$," is defined by

$$\begin{aligned} \mathcal{R} \circ \mathcal{S} \stackrel{\text{def}}{=} & \{(a,c) \in A \times C : \\ & \exists b \in B \ni (a,b) \in \mathcal{R} \wedge (b,c) \in \mathcal{S} \\ a\mathcal{R} \circ \mathcal{S}c \iff & \exists b \in B \ni a\mathcal{R}b \wedge b\mathcal{S}c\ . \end{aligned}$$

In these terms, transitivity is equivalent to

$$\mathcal{R} \circ \mathcal{R} \subseteq \mathcal{R}\ .$$

Proof: **Assume** $\mathcal{R}$ **is transitive. Then for all** $a, b, c, \ldots \in A$

$$\begin{aligned}(a,c) \in \mathcal{R} \circ \mathcal{R} &\implies \exists b \ni (a,b) \in \mathcal{R} \wedge (b,c) \in \mathcal{R} \\ &\implies (a,c) \in \mathcal{R}\,; \qquad \textbf{hence}\end{aligned}$$

$$\mathcal{R} \circ \mathcal{R} \subseteq \mathcal{R}\,.$$

Conversely, assume $\mathcal{R} \circ \mathcal{R} \subseteq \mathcal{R}$**. Then, with** $\mathcal{U} = A$

$$\begin{aligned}(a,b) \in \mathcal{R} \wedge (b,c) \in \mathcal{R} &\implies (a,c) \in \mathcal{R} \circ \mathcal{R} \\ &\implies (a,c) \in \mathcal{R}\end{aligned}$$

Hence $\mathcal{R}$ **is transitive.**

c): Recall the transpose $\mathcal{R}^T$ **of a relation** $\mathcal{R} : A \to B$**:**

$$\mathcal{R}^T = \{(b,a) \in B \times A :\ (a,b) \in \mathcal{R}\}\,.$$

Using this term, $\mathcal{R} : A \to B$ **is symmetric iff** $\mathcal{R} = \mathcal{R}^T$**.**

d): $\mathcal{R}$ is antisymmetric iff $\mathcal{R} \cap \mathcal{R}^T \subseteq =_A$

Proof: **Let $\mathcal{U} = A$. First assume $\mathcal{R}$ is antisymmetric. If $(a, b) \in \mathcal{R} \cap \mathcal{R}^T$ then $(a, b) \in \mathcal{R}$ and $(b, a) \in \mathcal{R}$, which implies $a = b$ or $(a, b) \subseteq =_A$.**

Next assume $\mathcal{R} \cap \mathcal{R}^T \subseteq =_A$. Then if $(a, b) \in \mathcal{R}$ and $(b, a) \in \mathcal{R}$ then $(a, b) \in \mathcal{R} \cap \mathcal{R}^T \subseteq =_A$.

The equality relation $=_A$ on A is reflexive, transitive, symmetric, and antisymmetric; it is the only relation having these four properties; it is a function (which?).

A relation $\mathcal{R}$ on A is both symmetric and antisymmetric iff it is a subset of $=_A$.

A relation $\mathcal{R}$ on A that is both symmetric and antisymmetric is transitive.

If $|A| = m$, how many relations on A are

there that are reflexive? symmetric? antisymmetric? both symmetric and antisymmetric? reflexive and symmetric? reflexive and antisymmetric? transitive?

Definition: **Let $A \subset \mathcal{U}$.**
I) A relation $\preceq: A \to A$ is a <Partial> Order if it is reflexive, transitive, and antisymmetric. The pair $(A, \preceq)$ is often given the unattractive moniker Poset.
Ib) A partial order $\preceq$ on A is a Total Order or a Linear Order if $\forall a, b \in A$

$$(a \preceq b) \text{ or } (b \preceq a) \text{ holds.}$$

II) A relation $\approx: A \to A$ is an Equivalence Relation if it is reflexive, transitive, and

symmetric.

Examples: **Let $\leq$ denote the natural order on numbers. Then $(\mathbb{N}, \leq)$, $(\mathbb{Z}, \leq)$, $(\mathbb{Q}, \leq)$, $(\mathbb{R}, \leq)$ are posets. Actually, $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ are linearly ordered by $\leq$.**
Let $B \subset \mathcal{U}$ and set $A = \mathcal{P}[B]$. Then $(A, \subseteq)$ is a poset, not totally ordered unless ??
Let A denote the tasks that go into building a house. Define $a \preceq b$ to mean that task a must be done before task b; for example, the electric cables must be laid before the wallboard goes on. This is a partial and not a total order (why?). This is an example of a poset entering industrial design – more later.

Define a relation on this class by

$a \approx b \iff$ a **and** b **have the same hair color.**

This is an equivalence relation on this class. So is the relation

$$a \simeq b \iff a \text{ and } b \text{ weigh the same.}$$

So is the relation

$$a = b \iff a \text{ and } b \text{ are the same.}$$

On $\mathbb{Z}$ **define the relation** $\cong_{36}$ **by**

$a \cong_{36} b \iff$ a, b **have same remainder mod 36.**

The relation $\cong_n$ **on** $\mathbb{Z}$ **is very important in Number Theory and** $a \cong_n b$ **is usually written**

$$a \cong b \textbf{ mod } n \textbf{ or } a \cong b \; (n) \; .$$

Exercise: **If $\preceq$ is a partial order on a set A then so is its transpose $\preceq^T$. It is called the Reverse Order.**

Example: **Let $A \stackrel{\text{def}}{=} \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ and define $\preceq$ by**

$$a \preceq b \iff a|b\ .$$

This is a partial order on A.