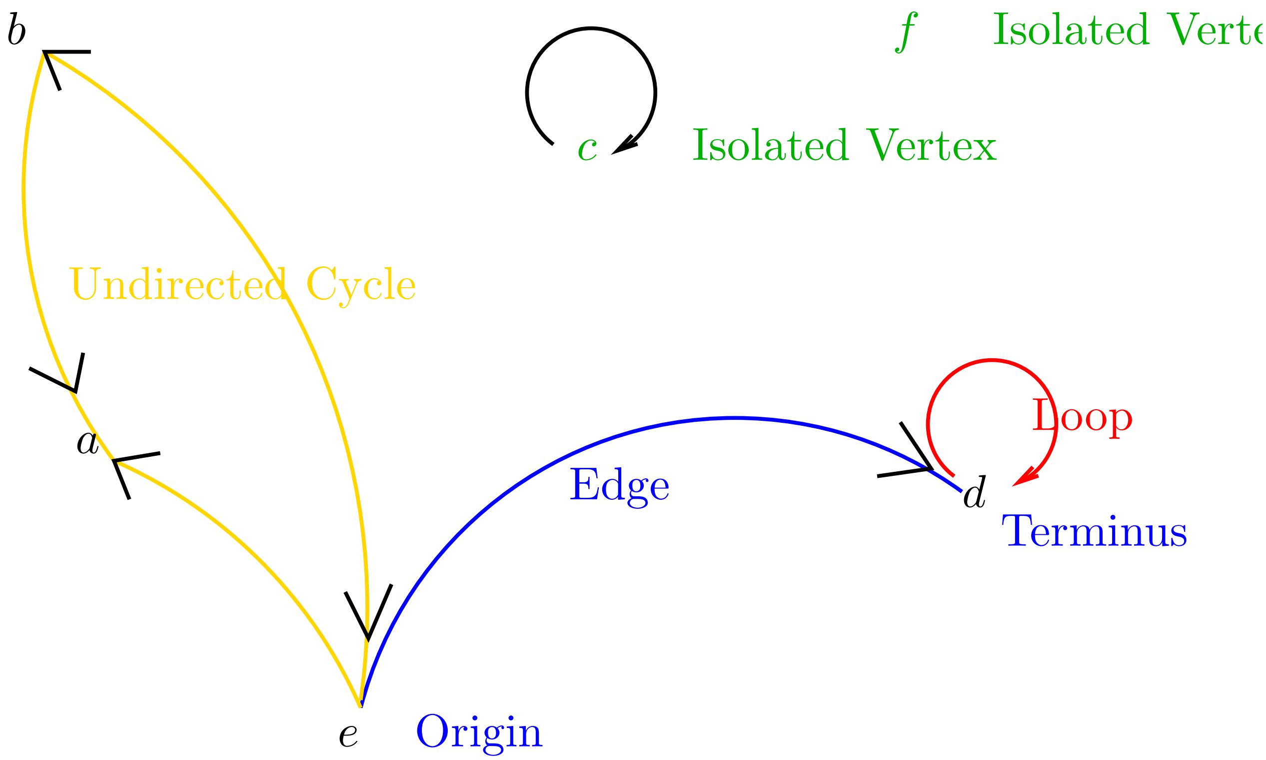
# Depicting Relations

**Let $A = \{a_1, a_2, \ldots, a_n\}$ be a set of modest size $n$ and $\mathcal{R}$ a relation on $A$. Here is a way of depicting $\mathcal{R}$ by a Directed Graph: Call the members of $A$ Vertices and plunk them down on a sheet of paper or a blackboard; whenever a pair $(a_i, a_j)$ is in $\mathcal{R}$ draw a line from the vertex $a_i$ to the vertex $a_j$ with an arrow at the end next to $a_j$; call this line an Edge.**

Directed Graph of

$\mathcal{R} = \{(b,a),(c,c),(e,b),(e,a),(e,d),(b,e),(d,d)\}$

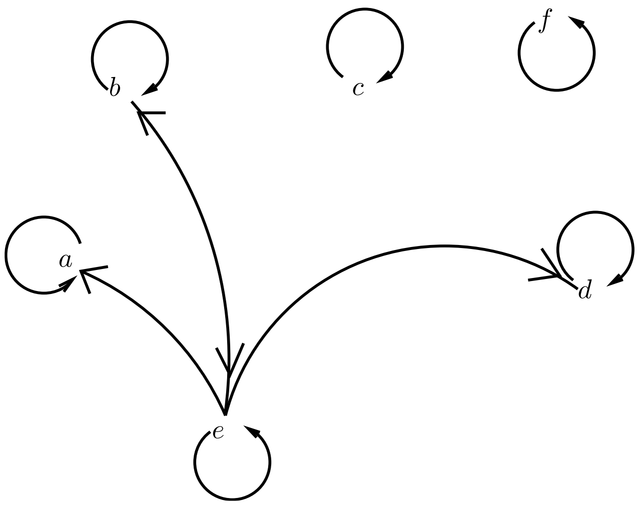
on $A = \{a,b,c,d,e,f\}$



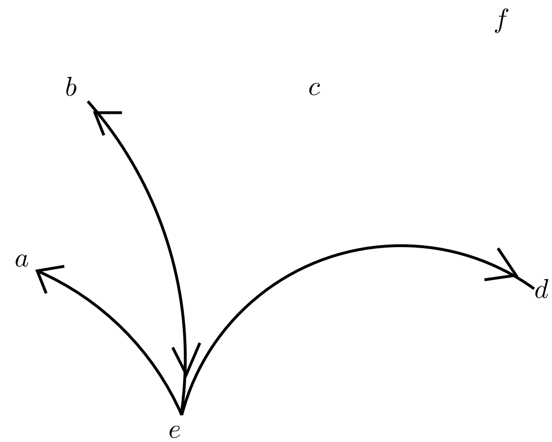
Directed Graph of the **reflexive** relation

$\{(a, a), (b, b), (c, c), (d, d), (e, e), (f, f), (e, b), (e, a), (e, d), (b, e)\}$

on $\{a, b, c, d, e, f\}$



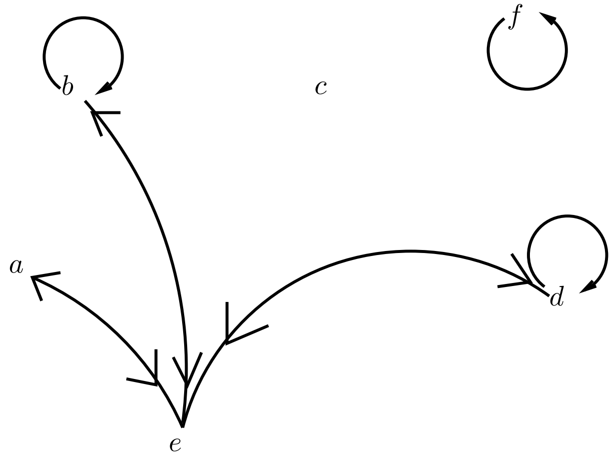

or



Graph of the **Symmetric** relation

$\{(b,b),(f,f),(d,d),(a,e),(e,a),(e,b),(b,e),(e,d),(d,e)\}$

on $\{a,b,c,d,e,f\}$

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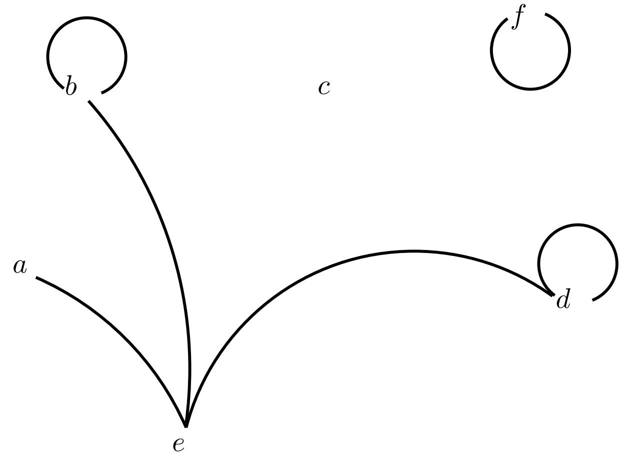
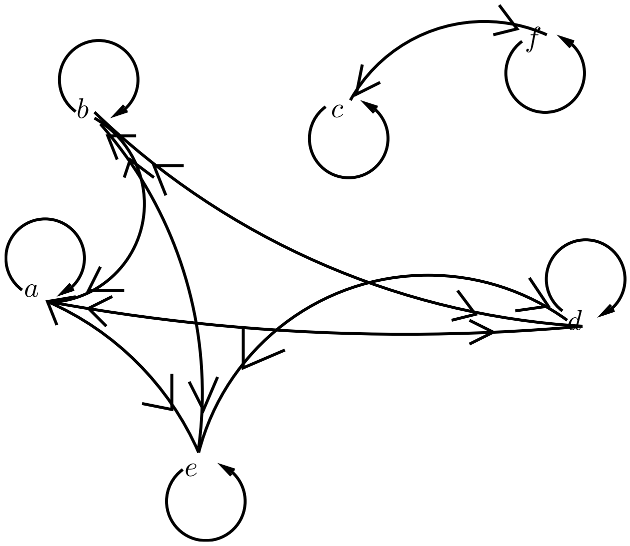


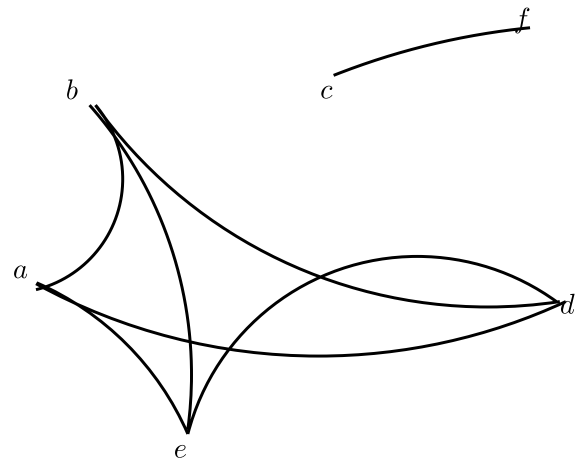
Graph of the **Eqivalence** relation

$\{(a,a),\cdots,(f,f),(a,b),(b,a),(a,d),(d,a),(e,d),(d,e),(d,b),(b,d),(c,f),(f,c)\}$

on $\{a,b,c,d,e,f\}$




or



# Incidence Matrices

**We'll restrict attention to relations $\mathcal{R} : A \to B$, where both $A$ and $B$ are finite, say $|A| = m$ and $|B| = n$.**

Definition: **Let $\mathcal{R}$ be a relation from $A = \{a_1, a_2, \ldots, a_m\}$ to $B = \{b_1, b_2, \ldots, b_n\}$**

**The Incidence Matrix of such a relation $\mathcal{R} : A \to B$ is the $m \times n$–matrix $M = M[\mathcal{R}]$ ($m$ rows and $n$ columns) whose element in the $i^{\text{th}}$ row and $j^{\text{th}}$ column is**

$$M_{i,j} = \begin{cases} 1 & \text{if } (a_i, b_j) \in \mathcal{R} \\ 0 & \text{if } (a_i, b_j) \notin \mathcal{R} \end{cases} .$$

**Example:** **Let** $A = \{a, b, c\}$**,** $B = \{1, 2\}$**, and**

$$\mathcal{R} = \{(a,1), (a,2), (c,2), (b,1)\} \ .$$

**Then**

$$M[\mathcal{R}] = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \ .$$

**Example:** **The incidence matrix of the equality relation** $=_A$ **on a set** $A$ **with** $n$ **elements is**

$$I_n \stackrel{\text{def}}{=} \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \ .$$

**Mnemonics:** **The incidence matrix of a relation** $\mathcal{R} : A \to B$ **has as many rows as** $A$ **has elements and as many columns as** $B$ **has elements.**

**Theorem:** **Let** $A = \{a_1, a_2, \ldots, a_m\}$**,** $B = \{b_1, b_2, \ldots, b_n\}$**, and** $C = \{c_1, c_2, \ldots, c_k\}$ **be finite sets and** $\mathcal{R} : A \to B$ **and** $\mathcal{S} : B \to C$ **relations. Then the incidence matrix of the composition** $\mathcal{R} \circ \mathcal{S}$ **("**$\mathcal{R}$ **before** $\mathcal{S}$**") is**

$$M[\mathcal{R} \circ \mathcal{S}] = M[\mathcal{R}] \diamond M[\mathcal{S}] \, ,$$

**where this product of an** $m \times n$**–matrix** $M$ **with an** $n \times k$**–matrix** $N$ **is defined by**

$$\begin{aligned} \big(M \diamond N\big)_{i,j} &= \bigvee\nolimits_{\nu=1}^{n} M_{i,\nu} \wedge N_{\nu,j} \\ &\stackrel{\text{def}}{=} \sup\nolimits_{\nu=1}^{n} M_{i,\nu} \wedge N_{\nu,j} \end{aligned}$$

**for** $1 \le i \le m$ **and** $1 \le j \le k$**.**

**Proof:** **Fix $i$ and $j$ as above.**

$$
\begin{aligned}
& M_{i,j}[\mathcal{R}\circ\mathcal{S}] = 1 \\
\iff\; & (a_i, c_j) \in \mathcal{R}\circ\mathcal{S} \\
\iff\; & \exists b_\nu \in B \;\ni\; (a_i, b_\nu) \in \mathcal{R} \wedge (b_\nu, c_j) \in \mathcal{S} \\
\iff\; & \exists \nu \in \{1,\ldots,n\} \ni M_{i,\nu}[\mathcal{R}]{=}1 \wedge M_{\nu,j}[\mathcal{S}]{=}1 \\
\iff\; & \exists \nu \in \{1,\ldots,n\} \ni \big(M_{i,\nu}[\mathcal{R}]{\wedge}M_{\nu,j}[\mathcal{S}]\big) = 1 \\
\iff\; & \bigvee_{\nu=1}^{n} M_{i,\nu}[\mathcal{R}]{\wedge}M_{\nu,j}[\mathcal{S}] = 1
\end{aligned}
$$

**Example:** $A{=}\{a,b,c\}$ $B{=}\{1,2,3\}$ $C{=}\{x,y,z,u\}$.

$$
\underbrace{\begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}}_{M[\mathcal{R}]} \diamond \underbrace{\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{M[\mathcal{S}]} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{M[\mathcal{R}\circ\mathcal{S}]}
$$

# Other Operations on Incidence Matrices

**Definition:** **Let** $M, N$ **be** $m \times n$**–matrices. We say** $M$ **is smaller than** $N$ **and write**

$$M \leq N \text{ if } M_{i,j} \leq N_{i,j} \qquad \forall i, j \ .$$

**We say** $M$ **is larger than** $N$ **and write**

$$M \geq N \text{ if } M_{i,j} \geq N_{i,j} \qquad \forall i, j \ .$$

**We call the minimum of** $M$ **and** $N$ **the matrix** $M \wedge N$ **with entries**

$$(M \wedge N)_{i,j} \stackrel{\text{def}}{=} M_{i,j} \wedge N_{i,j} \qquad \forall i, j \ .$$

**We call the maximum of** $M$ **and** $N$ **the matrix** $M \vee N$ **with entries**

$$(M \vee N)_{i,j} \stackrel{\text{def}}{=} M_{i,j} \vee N_{i,j} \qquad \forall i, j \ .$$

**The Transpose of an** $m \times n$**–matrix** $M$ **is**

**the** $n \times m$**–matrix** $M^T$ **with entries**

$$\left(M^T\right)_{i,j} \stackrel{\text{def}}{=} M_{j,i} \qquad \forall i, j \ .$$

**Examples:** **Let** $\mathcal{R}, \mathcal{S} : A \to B$ **be relations from a finite set to another. Then**

$$\mathcal{R} \subseteq \mathcal{S} \iff M[\mathcal{R}] \leq M[\mathcal{S}] \ .$$

**Examples:** **Let** $\mathcal{R} : A \to A$ **be a relation on a set** $A$ **with** $n$ **elements, and** $M \stackrel{\text{def}}{=} M[\mathcal{R}]$ **its incidence matrix. Then**

**a)** $\mathcal{R}$ **is reflexive iff** $I_n \leq M$**.**
**b)** $\mathcal{R}$ **is transitive iff** $M \diamond M \leq M$**.**
**c)** $\mathcal{R}$ **is symmetric iff** $M = M^T$**.**
**d)** $\mathcal{R}$ **is antisymmetric iff** $M \wedge M^T \leq I_n$**.**

**Example:** **If** $|A| = m$**, how many relations**

**on $A$ are there that are**
**reflexive? symmetric? antisymmetric?**
**both symmetric and antisymmetric?**
**both reflexive and symmetric?**
**both reflexive and antisymmetric?**
**transitive?**

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**Answer:** **as many as there are matrices with the corresponding features.**