Partial Orders on Finite Sets

Fix henceforth a non–void finite partially ordered set $(A, \preceq)$. We pronounce $a \preceq b$ as "a precedes b."

Definitions: **An element $m \in A$ is minimal (for the order $\preceq$) if there is no other element $a \in A$ preceding it:**

$$\nexists a \in A \ni m \neq a \preceq m \, .$$

An element $M \in A$ is maximal if it precedes no other element of A:

$$\nexists a \in A \ni M \preceq a \neq M \, .$$

An element $l \in A$ is a least element (for the

order $\preceq$) if it precedes all other elements:

$$l \preceq a \qquad \forall a \in A\,.$$

An element $g \in A$ is a greatest element (for the order $\preceq$) if all other elements precede it:

$$a \preceq g \qquad \forall a \in A\,.$$

Observations: Assume now A is finite.
If $(A, \preceq)$ has a least element it is unique.
If $(A, \preceq)$ has a greatest element it is unique.
$(A, \preceq)$ has a minimal element.
$(A, \preceq)$ has a maximal element.
If $(A, \preceq)$ has only one minimal element then that element is a least element.
If $(A, \preceq)$ has only one maximal element then that element is a greatest element.

Proof: Suppose m is the one and only minimal element. If m were not least then there would be an element $a_1 \in A$ different from m that it does not precede. a_1 not being minimal, there is another element $a_2 \in A$ with $a_2 \preceq a_1$. Continue on and arrive at an infinite sequence

$$a_1 \succeq a_2 \succeq a_3 \succeq \cdots$$

of distinct elements of A, all different from m (why?), a contradiction.

Proposition: Let $A = \{a_1, a_2, \ldots, a_n\}$ be a set equipped with a partial order $\preceq$ and let $M = M[\preceq]$ be the incidence matrix of $\preceq$.

a) The element $a_i \in A$ is least iff $M_{i,j} = 1$ for

all $j = 1, 2, \ldots, n$, i.e., iff the i^{th} row of M contains only 1s.

b) The element $a_i \in A$ is minimal iff $M_{j,i} = 0$ for all $j \neq i$, i.e., iff the i^{th} column of M contains only one 1, the one on the diagonal.

c) The element $a_i \in A$ is greatest iff $M_{j,i} = 1$ for all $j = 1, 2, \ldots, n$, i.e., iff the i^{th} column of M contains only 1s.

b) The element $a_i \in A$ is maximal iff $M_{i,j} = 0$ for all $j \neq i$, i.e., iff the i^{th} row of M contains only one 1, the one on the diagonal.

Example: **Let $A \stackrel{\text{def}}{=} \{2, 3, 4, 5, 6, 7, 8, 9\}$ and define $\preceq$ by**

$$a \preceq b \iff a|b \, .$$

This is a partial order on A with incidence

matrix

	2	3	4	5	6	7	8	9
2	1	0	1	0	1	0	1	0
3	0	1	0	0	1	0	0	1
4	0	0	1	0	0	0	1	0
5	0	0	0	1	0	0	0	0
6	0	0	0	0	1	0	0	0
7	0	0	0	0	0	1	0	0
8	0	0	0	0	0	0	1	0
9	0	0	0	0	0	0	0	1

It has minimal elements $2, 3, 5, 7$**.**

It has maximal elements $5, 6, 7, 8, 9$**.**

Exercise: Suppose M **is an** $n \times n$**–matrix of** 0**s and** 1**s that has the following properties:** $I \leq M$**,** $M \wedge M^T = I$**, and** $M \diamond M \leq M$**. Prove that** M **contains a row and a column that have only one** 1 **in them.**

Definition: Let $\mathcal{R}$ be a relation on A, and let $A_0 \subseteq A$. The **Induced Relation**

$$\mathcal{R}_{|A_0} : A_0 \to A_0$$

is defined by

$$\mathcal{R}_{|A_0} \stackrel{\text{def}}{=} \{(a,b) \in A_0 \times A_0 : a\mathcal{R}b\} \ .$$

Lemma: If $\mathcal{R}$ is reflexive, transitive, symmetric, or antisymmetric, then so is $\mathcal{R}_{|A_0}$.

Lemma: If $M = M[\mathcal{R}]$ is the incidence matrix of $\mathcal{R}$ then the incidence matrix of $\mathcal{R}_{|A_0 \times A_0}$ is obtained from M by deleting all rows and columns belonging to elements of $A \setminus A_0$.

Example: Let $A \stackrel{\text{def}}{=} \{2,3,4,5,6,7,8,9\}$ and define $\preceq$ by

$$a \preceq b \iff a|b \ .$$

This is a partial order on A with incidence matrix

	2	3	4	5	6	7	8	9
2	1	0	1	0	1	0	1	0
3	0	1	0	0	1	0	0	1
4	0	0	1	0	0	0	1	0
5	0	0	0	1	0	0	0	0
6	0	0	0	0	1	0	0	0
7	0	0	0	0	0	1	0	0
8	0	0	0	0	0	0	1	0
9	0	0	0	0	0	0	0	1

Let us remove from A the set $\{2, 3, 5, 7\}$ of its minimal elements, leaving the set $A_0 \stackrel{\text{def}}{=} \{4, 6, 8, 9\}$. The induced partial order

has incidence matrix

	2	3	4	5	6	7	8	9
2	1	0	1	0	1	0	1	0
3	0	1	0	0	1	0	0	1
4	0	0	1	0	0	0	1	0
5	0	0	0	1	0	0	0	0
6	0	0	0	0	1	0	0	0
7	0	0	0	0	0	1	0	0
8	0	0	0	0	0	0	1	0
9	0	0	0	0	0	0	0	1

Hasse Diagrams

Let $(A, \preceq)$ be a finite poset. There is the digraph of $\preceq$, but that may clutter up a piece of paper awfully if A is large. Here is another way to depict $\preceq$, the Hasse diagram of $(A, \preceq)$. Here is how to make it.

0) Get a lined piece of paper.

1a) Identify the minimal elements of $(A, \preceq)$ and write them down on the last line of the paper.

1b) Let A_1 denote the remainder, A with the minimal elements for $\preceq$ removed, and let $\preceq_1$ be the order $\preceq$ induces on it.

2a) Identify the minimal elements of $(A_1, \preceq_1$ $)$ and write them down on the penultimate line of the paper.

2b) Let A_2 denote the remainder, A_1 with the minimal elements for $\preceq_1$ removed, and let $\preceq_2$ be the order $\preceq_1$ induces on it.
3a) Identify the minimal elements of $(A_2, \preceq_2)$ and write them down on the last free line of the paper.
3b) Let A_3 denote the remainder, A_2 with the minimal elements for $\preceq_2$ removed, and let $\preceq_3$ be the order $\preceq_2$ induces on it.

Keep going until A is exhausted.

1c) Whenever an element a on the last line $\preceq$–precedes an element b of the penultimate line, draw a line connecting a and b.
2c) Whenever an element a on the penul-

timate line $\preceq$–precedes an element b of the line above, draw a line connecting a and b.
3c) Whenever an element a on the third line from below $\preceq$–precedes an element b of the line above, draw a line connecting a and b.

Keep going until you must stop.

The resulting picture is the Hasse diagram of $(A, \preceq)$. It evidently contains all the information about $(A, \preceq)$.

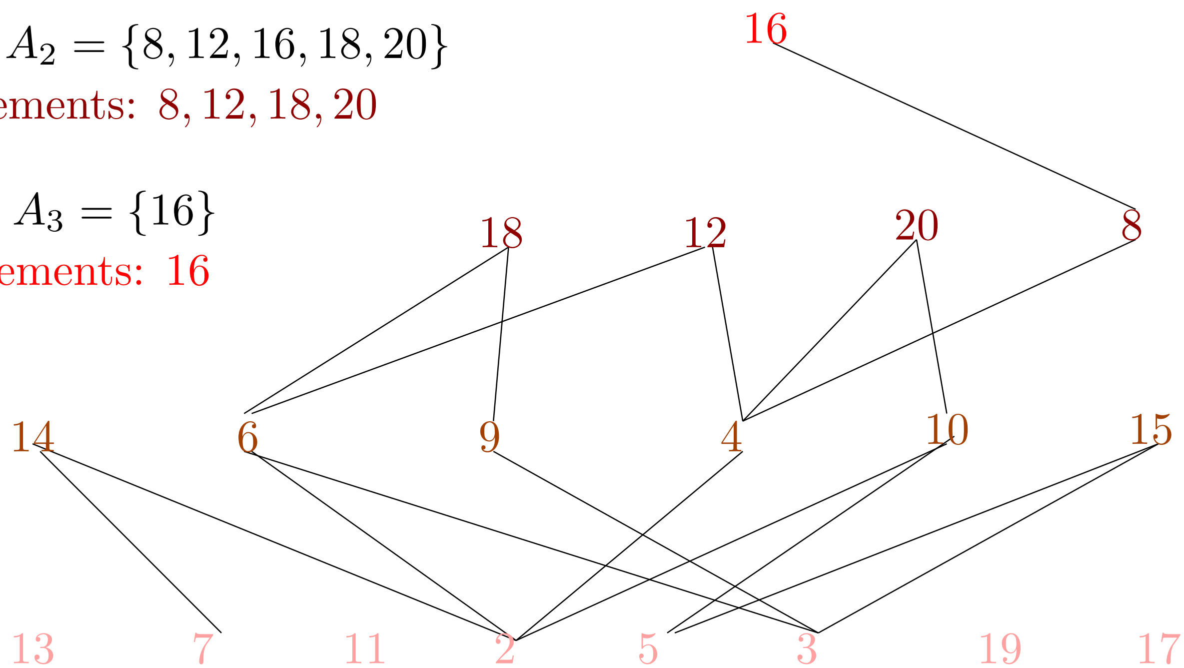
Hasse diagram of the poset
$\{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20\}$
eqipped with the order $a \preceq b \iff a|b$

Minimal elements: $2, 3, 5, 7, 11, 13, 17, 19$

Remainder $A_1 = \{4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20\}$
Minimal elements: $4, 6, 9, 10, 14, 15$

Remainder $A_2 = \{8, 12, 16, 18, 20\}$
Minimal elements: $8, 12, 18, 20$

Remainder $A_3 = \{16\}$
Minimal elements: 16



Topological Sorting

Definition: **Let A be a set and $\preceq, \ll$ two partial orders on A. We say that $\ll$ refines $\preceq$ or $\ll$ respects $\preceq$ if $\preceq \subseteq \ll$, i.e. if**

$$a \preceq b \implies a \ll b \qquad \forall a, b \in A \,.$$

Given a finite poset $(A, \preceq)$, the problem arises to find a total order $\ll$ that respects $\preceq$. Viz. a task chart. This can always be done (in many ways), by the

Topological Sorting Algorithm:
1) Construct a Hasse diagram for $\preceq$.
2) Then define $\ll$ by this prescription: for any two elements $a, b \in A$ say $a \ll b$
if 2a) a and b are listed in the same line and a is listed to the left of b
or 2b) a is listed on a line lower than b.
[Instead one may perform an arbitrary permutation of every line before applying the algorithm 2a)-2b).]

Hasse diagram of the poset
$\{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20\}$
eqipped with the order $a \preceq b \iff a|b$

Minimal elements: $2, 3, 5, 7, 11, 13, 17, 19$

Remainder $A_1 = \{4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20\}$
Minimal elements: $4, 6, 9, 10, 14, 15$

Remainder $A_2 = \{8, 12, 16, 18, 20\}$
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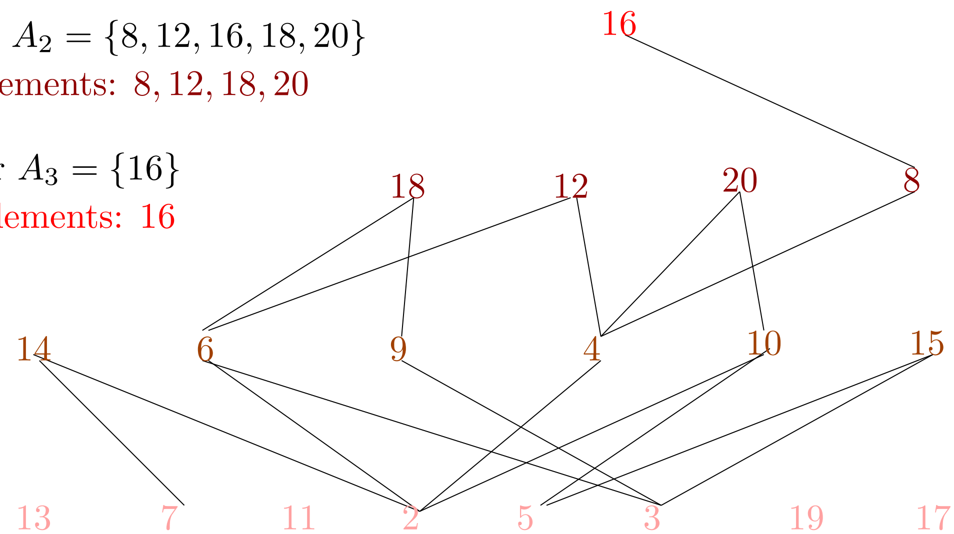
Remainder $A_3 = \{16\}$
Minimal elements: 16



Exercise: Describe the instructions you would give a computer to produce a total order $\ll$ refining a partial order that is given by an incidence matrix M. What will the incidence matrix of your total order $\ll$ look like?

Observation: The topological sorting algorithms described above do not necessarily produce every total order refining $\preceq$. For instance, in the preceding example, where $A = \{2, \ldots, 20\}$ and $a \preceq b \iff a|b$, there is the natural order $\leq$ of numbers. It is a total order refining $\preceq$, and $4 \leq 19$. In any total order $\ll$ produced by any of the topological sorting algorithms described above we have $19 \ll 4$.

However $\ldots$.