Equivalence Relations

We fix a set A together with an equivalence relation $\approx$ on it.

Definition: **Let $a \in A$.**

The $\approx$–Equivalence Class of a is the subset

$$[a] \stackrel{\text{def}}{=} \{b \in A : \; b \approx a\} \subseteq A \, .$$

Proposition: **Let $a, a' \in A$. Then either**

$$[a] = [a'] \quad \textbf{or} \quad [a] \cap [a'] = \emptyset \;\; :$$

Any two equivalence classes are either identical or disjoint.

Proof: **We'll prove the logically equivalent**

statement

$$[a] \cap [a'] \neq \emptyset \implies [a] = [a'] \,.$$

Assume $[a] \cap [a'] \neq \emptyset$**. Then** $[a]$ **and** $[a']$ **have an element in common, say**

$$b \in [a] \cap [a'] \,.$$

This says $b \approx a$ **and** $b \approx a'$**. Then**

$$\begin{aligned} x \in [a] &\implies x \approx a \\ &\implies x \approx b \quad [\textbf{as } a \approx b] \\ &\implies x \approx a' \quad [\textbf{as } b \approx a'] \end{aligned}$$

$$\implies x \in [a'] \,.$$

Conversely

$$\begin{aligned} x \in [a] &\implies x \approx a \\ &\implies x \approx b \quad [\textbf{as } a \approx b] \\ &\implies x \approx a' \quad [\textbf{as } b \approx a'] \end{aligned}$$

$$\implies x \in [a'] \,.$$

Hence $[a] = [a']$ **QED**

Examples: **Let A be this class and define $\approx$ by $a \approx b$ if a and b have the same hair color. What is your hair color, what is your equivalence class?**

Let $A = \mathbb{Z}$, fix an integer n and define $\approx$ by

$$a \approx b \iff a - b \in ((n)) \ ,$$
$$\textbf{i.e.} \quad a \approx b \iff n|(a-b) \ .$$

This means $a \approx b$ if a, b have the same remainder mod n, an equivalence relation we wrote earlier as

$$a \cong b \mod n \ .$$

The equivalence class of $k \in \mathbb{Z}$ is the infinite set

$$[k] = \{k + qn : q \in \mathbb{Z}\} \ ,$$

and the set of equivalence classes is the finite set

$$Z_n \stackrel{\text{def}}{=} \{[0], [1], \ldots, [n-1]\} \ .$$

Definition: **Let A be a set. A** Partition of A **is collection of non–void mutually disjoint subsets of A whose union is A.**

Example: **A set A with m elements has 1 partition $\mathcal{P}$ of size $|\mathcal{P}| = 1$ and 1 partition of size $|\mathcal{P}| = m$. It has no partitions with strictly more than m sets in it.**

Example: **A set A with m elements has exactly**

$S_{m,k}$ **partitions of size $|\mathcal{P}| = k$.**

Example: A set A with m elements has exactly

$$\sum_{k=1}^{m} S_{m,k} \text{ partitions.}$$

Example: The collection of equivalence classes with respect to an equivalence relation $\approx$ on A forms a partition of A, the **Partition $\mathcal{P}[\approx]$ associated with $\approx$.**

Example: Given a partition $\mathcal{P}$ on a set A, define an equivalence relation $\approx_{\mathcal{P}}$ on A by

$$a \approx_{\mathcal{P}} \iff a, b \text{ lie in the same set of } \mathcal{P}.$$

This is the **Equivalence Realtion Associated with $\mathcal{P}$.**

Exercise: Identify $\approx_{\mathcal{P}[\cong]}$ and $\mathcal{P}[\approx_{\mathcal{P}}]$.

Proposition: A finite set A admits as many equivalence relations as partitions. Therefore a set A of size $|A| = m$ admits exactly

$$\sum_{k=1}^{m} S_{m,k} \text{ equivalence relations.}$$