Exclusion – Inclusion

Let U **be a set and** $A_1, A_2, \ldots, A_n$ **subsets of** U**. For** $1 \leq k \leq n$ **set**

$$S_k \stackrel{\text{def}}{=} \sum_{0<i_1<i_2<\cdots<i_k\leq n} \left|A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}\right|,$$

"the sum of the sizes of all k–fold intersections."

We found it plausible earlier that

$$|\bigcup_{i=1}^{n} A_i| = S_1 - S_2 + S_3 - S_4 \pm \cdots S_n, \qquad (P_0)$$

but never quite proved it. Observe that (P_0) **is equivalent with the**

Principle of Exclusion and Inclusion

$$\left|\bigcap_{i=1}^{n} \bar{A}_i\right| = |U| - S_1 + S_2 - S_3 \pm \cdots S_n \,. \qquad (P)$$

Proof: **We'll show that every element $x \in U$ contributes the same to the left and right side of (P). For instance, if x is in none of the A_i then x increases the left–hand side by 1; and the right–hand side as well, since it increases only $|U|$ and not any of the other numbers.**

For another example, suppose x is in precisely one of the A_i s. Then x contributes zero to the left–hand side, a 1 to $|U|$ on the right, and a -1 from $-S_1$, for a total of 0.

Now consider the (general) case that x belongs to exactly r of the A_i. Then x contributes a 0 to the left–hand side. On the right–hand side it increases

$$\begin{array}{rl} |U| & \textbf{by } 1 \ , \\ S_1 & \textbf{by } r \ , \\ S_2 & \textbf{by } \binom{r}{2} \ , \\ \vdots & \quad \vdots \\ S_k & \textbf{by } \binom{r}{k} \ , \\ \vdots & \quad \vdots \\ S_r & \textbf{by } \binom{r}{r} \ , \\ S_l & \textbf{by } 0 \textbf{ for } l > r \ . \end{array}$$

Its total contribution to the left is

$$1 - r + \binom{r}{2} \mp \cdots (-1)^r \binom{r}{r} = (1-1)^r = 0$$

as well. **QED**

Exercise: **Formulate this as a proof by induction.**

Example, Eulers ϕ–function: **For every natural number n denote by $\phi(n)$ the number of natural numbers $m \le n$ that are relatively prime to n. ϕ is Euler's ϕ–function.**

Here is a computation of ϕ:

Given $0 < n \in \mathbb{N}$, find its prime power decomposition

$$n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$$

and consider the sets $U \overset{\text{def}}{=} \{1, 2, \ldots, n\}$ and

$$A_i \overset{\text{def}}{=} \{m \in U : p_i | m\} \qquad i = 1, 2, \ldots, k$$

Then $\phi(n)$ is the size of the set

$$\{m \in U : \gcd(m, n)=1\} = \bigcap_{i=1}^{k} \bar{A}_i \,.$$

Now clearly $|U| = n$**,** $|A_i| = n/p_i$**,** $|A_i \cap A_j| = n/(p_i p_j)$ **for** $i \neq j$**, etc. By the Principle of Exclusion and Inclusion**

$$\begin{aligned}
\phi(n) &= n - \left[\frac{n}{p_1} + \cdots + \frac{n}{p_k}\right] \\
&+ \left[\frac{n}{p_1 p_2} + \cdots + \frac{n}{p_{k-1} p_k}\right] \\
&\vdots \qquad \vdots \\
&\pm \textstyle\sum_{1 \leq i_1 < i_2 < \cdots < i_r \leq k} \frac{n}{p_{i_1} \cdots p_{i_r}} \\
&\vdots \qquad\qquad \vdots \\
&\pm \frac{n}{p_1 p_2 \cdots p_k} \\
&= n \left[1 - \left(\frac{1}{p_1} + \cdots + \frac{1}{p_k}\right)\right. \\
&+ \left(\frac{1}{p_1 p_2} + \cdots + \frac{1}{p_{k-1} p_k}\right) \\
&\vdots \qquad\qquad \vdots \\
&\left.\pm \frac{1}{p_1 p_2 \cdots p_k}\right]
\end{aligned}$$

$$
\begin{aligned}
\phi(n) = \frac{n}{p_1 p_2 \cdots p_k} \Big[\; & p_1 p_2 \cdots p_k \\
& - \sum_i p_1 p_2 \cdots \check{p}_i \cdots p_k \\
& + \sum_{i<j} p_1 p_2 \cdots \check{p}_i \cdots \check{p}_j \cdots p_k \\
& \vdots \qquad\qquad\qquad \vdots \\
& \pm 1 \Big] \\
& = \frac{n}{p_1 p_2 \cdots p_k} \prod_{i=1}^{k} (p_i - 1) \\
& = n \prod_{i=1}^{k} \left(\frac{p_i - 1}{p_i} \right) \; . \\
& = n \prod_{i=1}^{k} \left(1 - \frac{1}{p_i} \right) \; .
\end{aligned}
$$

Example: **Find the number D_n of permutations of $\{1, 2, \ldots, n\}$ in which no number is in its proper place.**

Solution: **Let U be the set of all permutations of $\{1, 2, \ldots, n\}$ and A_i the set of permutations that leave the number i in place i. Using the PEI find**

$$D_n \approx n!e^{-1} \approx n!/3 \,.$$