Practice Final , M325K, 12/13/2014 **PRINTED** NAME:____________________EID:

No books, notes, calculators, or telephones are allowed.

Every problem is worth an equal number of points.

You must show your work; answers without substantiation do not count.

Answers must appear in the box provided!

No or the wrong answer in the answer box results in no credit!

This does not aim nor claim to be exhaustive! Use this as a guide of what to study and not of what not to study! Do not expect to find every test problem listed here!
The actual test will contain 7–10 problems very similar to those below.

Pick 8–10 problems from the earlier tests and practice tests, including two "story problems." Here are the earlier practice tests:

Practice Test 1, 325K **PRINTED** NAME:____________________EID:

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The actual test will contain 5–6 problems very similar to those below.

How many arrangements are there of the letters SOCIOLOGICAL?

Answer: $\binom{12}{3\ 2\ 2\ 2\ 1\ 1\ 1}$

In how many ways can you distribute 37 books on 4 shelves in such a way that no shelf is left empty? (The order of the books on any of the shelves does matter.)

Answer: $37! \times \binom{36}{3}$

How many five–card hands are there that contain at least one club?

Answer: $\binom{52}{5} - \binom{39}{5}$

How many one–pair poker hands are there? Etc.

Answer: $\binom{13}{1}\binom{4}{2}\binom{12}{3}\binom{4}{1}^3$ p. 28#8h

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How many two–pairs poker hands are there? Etc.

Answer: $\binom{13}{2}\binom{4}{2}^2\binom{11}{1}\binom{4}{1}$

A curious boy has to traverse 8 blocks going East and 6 blocks going North when walking from home to school. How many ways does he have to walk to school if he never walks South or West?

Answer: $\binom{14}{6} = \binom{14}{8}$

Determine the coefficient of x^7y^3 in the binomial expansion of $(2x+3y)^{10}$.

Answer: $2^7 3^3 \binom{10}{3}$

Determine the number of integer solutions of $x_1 + x_2 + \cdots + x_7 < 27$, $\mathbb{Z} \ni x_i \geq -2$.

Answer: $\#\{x_1 + x_2 + \cdots + x_7 < 27 \ , \ \mathbb{Z} \ni x_i \geq -2\} = \binom{40+7}{7}$

How many positive ($x_i \geq 0$) integer solutions are there to the pair of simultaneous equations

$$x_1 + \cdots + x_7 = 37 \ , \ x_1 + x_2 + x_3 = 6 \ .$$

Answer: $\binom{8}{2} \times \binom{34}{3}$

The boss has 5 secretaries. She wishes to givem them X–mas bonuses, but finds herself with 9 one–dollar bills and 5 five–dollar bills.
a) In how many ways can she distribute these bills among the secretaries?

b) In how many ways can she distribute them among the secretaries if every one of them must have at least one one–dollar bill?

Answer: a) $\binom{9+4}{4}\binom{5+4}{4}$ b) $\binom{4+4}{4}\binom{5+4}{4}$

Use truth tables or argue otherwise but cogently to show that $[(p \to q) \wedge (q \to r)] \to (p \to r)$ is a logical implication.

Consider the statement P:
"If 7 divides the product of two integers then it divides one of the factors."
[$7|m$ reads "seven divides m" and has negation $7 \not| m$.]
a) Write P using quantifiers.
$P \iff$

$$\forall\, m, n \in \mathbb{Z} \ \ 7|mn \implies 7|m \ \vee \ 7|n.$$

b) Compute (and simplify) the negation $\neg P$.
$\neg P \iff$

$$\exists\, m, n \in \mathbb{Z} \ \ 7|mn \ \wedge \ 7 \not| m \ \wedge \ 7 \not| n.$$

How many subsets of $\{1, 2, 3, \ldots, 15\}$ contain at least one odd integer?

Answer: $2^{15} - 2^7$

Prove that for $A, B, C, D \subseteq \mathcal{U}$

$$A \subseteq C \ \wedge \ B \subseteq D \quad \implies \quad A \cup B \ \subseteq \ C \cup D \, .$$

Write down a few laws of set theory.

How many permutations of the alphabet contain the patterns "BIG" or "ROUND" ?

Let $A, B \subset \mathcal{U}$. Show that

$$[A \times B = B \times A] \iff [(A = \emptyset) \vee (B = \emptyset) \vee (A = B)]$$

Consult Lecture 16

Of 52 monitored people by a TV rating company, 22 watch "Reality Shows," 21 watch News, and 21 watch Soaps; 10 watch both "Reality Shows" and News, 10 watch "Reality Shows" and Soaps, and 10 watch both News and Soaps; 7 of then watch all three types of shows.
How many viewers watch only "Reality Shows?"

Answer: A Venn diagram shows that there are 9 of these.

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If you scramble the letters of the word SOCIOLOGICAL and then put them down randomly in linear order, what is the probability you will spell SOCIOLOGICAL?

Answer: $\displaystyle \binom{12}{3\ 2\ 2\ 2\ 1\ 1\ 1}^{-1}$

What is the probability of getting a one–pair hand in 5 card poker?

Answer: $13 \cdot \binom{4}{2} \cdot \binom{12}{3} \cdot 4^3 \Big/ \binom{52}{5}$

A gambler rolls a fair die until a 6 comes up. What is the probability that he will need exactly 4 rolls?

Answer: $\displaystyle \left(\frac{5}{6}\right)^3 \left(\frac{1}{6}\right)$

2% of all drivers at 11:00 PM are drunk. A drunk driver has a probability of 10% of causing an accident, while a sober driver has a probability of 1%. (These figures are made of whole cloth; I don't know what the real figures are; it would be good to know.) What is the probability that a driver causing an accident was drunk?

30% of all drivers drive while yapping on their cell phones. A driver on a cell phone has 1% probability of causing an accident, while a driver paying attention to the road only has a 0.1% probabability. (These figures are made of whole cloth; I don't know what the real figures are; it would be good to know.) What is the probability that a driver causing an accident was on his/her cell?

Your friend was careful but still has a 4% chance of being pregnant (Event E). You buy her a pregancy test that has $\mathbb{P}[TP|E] = .90$ and $\mathbb{P}[TP|\overline{E}] = .10$ (TP is the event the test shows positive). The test is positive. What is the probability p your friend is actually pregnant?
[Give an answer of the form $p = n/d$, n, d integers.]

Answer: $p = \dfrac{3}{11}$

Prove that for every $n \in \{1, 2, 3...\}$ $\displaystyle \sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}.$

Define recursively $F_0 \stackrel{\text{def}}{=} 0$, $F_1 \stackrel{\text{def}}{=} 1$, and $F_{n+2} \stackrel{\text{def}}{=} F_n + F_{n+1}$ for $n \geq 0$.

Show that $\displaystyle \sum_{i=0}^{n} F_i = F_{n+2} - 1$ for $n = 1, 2, \ldots$.

How many different positive divisors does 2395800 have?

Answer: $2395800 = 2^3 3^2 5^2 11^3$ has $4 \times 3 \times 3 \times 4 = 144$ divisors

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Find the greatest common divisor of 462 and 3640 and write it as a linear combination of 462 and 3640.

$$\begin{aligned} 3640 = 7 \times 462 + 406 &\implies ((3640, 462)) = ((462, 406)) \\ 462 = 1 \times 406 + 56 &\implies ((462, 406)) = ((406, 56)) \\ 406 = 7 \times 56 + 14 &\implies ((406, 56)) = ((56, 14)) = ((14)) \,. \end{aligned}$$

Hence $gcd(3640, 462) = 14$ and

$$\begin{aligned} 14 =& 406 - 7 \times 56 = 406 - 7 \times (462 - 406) = -7 \times 462 + 8 \times 406 \\ =& -7 \times 462 + 8 \times (3640 - 7 \times 462) = 8 \times 3640 - 63 \times 462 \end{aligned}$$

Answer: $gcd(462, 3640) = 14 = 8 \times 3640 - 63 \times 462$

Show that $\sqrt{7}$ is irrational.

Solution: Adapt lecture 15, replacing 2 by 7.

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a) Show: if $p > 0$ is a prime number and $0 < k < p$ then

$$p \Big| \binom{p}{k} \,.$$

b) Show: if $p > 0$ is prime then $n^p \cong_p n$ for all $n \geq 0$.

Solution: This is true by inspection for $n = 0$. For $n > 0$, by induction

$$\begin{aligned} n^p - n = [(n-1)+1]^p - n &= \sum_{k=0}^{p} \binom{p}{k} (n-1)^k - n \\ &= \binom{p}{0} (n-1)^0 + \sum_{k=1}^{p-1} \binom{p}{k} (n-1)^k + \binom{p}{p} (n-1)^p - n \\ &= 1 + qp + (n-1)^p - n = qp + [(n-1)^p - (n-1)] \cong_p 0 \end{aligned}$$

IV) Find the positive generator of the ideal $((42, 56, 77))$.

Let a, b, n be strictly positive integers. Prove that $n \times gcd(a, b) = gcd(an, bn)$.

Set $g \stackrel{\text{def}}{=} gcd(a, b)$ and $\gamma \stackrel{\text{def}}{=} gcd(an, bn)$.
$g = xa + yb \implies ng \in ((an, bn)) \implies \gamma | (ng)$.
$\gamma = \xi an + \eta bn = n \times (\xi a + \eta b) = n \times (kg) \exists k \implies (ng) | \gamma$. Hence $\gamma = ng$.

III) The XYZ Company has a dinner for all its employees. A place setting costs \$45, except for the women, who get an extra little vase with a flower for an additional \$2. The company lays out a total of \$1613 for this dinner. How may male and female employees does XYX company have?

Solution:

First, $1 = 23 \cdot 45 - 22 \cdot 47\ ,$

and so $1613 = (1613 \cdot 23 - k \cdot 47) \times 45 + (k \cdot 45 - 1613 \cdot 22) \times 47\ , \qquad \forall\, k \in \mathbb{Z}\ .$

We want $k \in \mathbb{Z}$ so that both parentheses are positive:

from $\dfrac{1613 \cdot 23}{47} = 789.34 \geq k \geq 788.58 = \dfrac{1613 \cdot 22}{45}$

follows $k = 789$

and $1613 = 16 \times 45 + 19 \times 47\ .$

Therefore

Answer: XYZ company has 16 male and 19 female employees

The farmer and his son fill the dry stock pond, which holds 224 liters, by filling their buckets, which hold 21 and 17 liters, respectively, with water, carrying them to the tank and dumping the water. When they are done, they argue about who did more work. They are sure that neither made more than twice the number of runs from spout to tank than the other. How many times did the son (with the 17 liter bucket) and the father (with the 21 liter bucket) make the run?

Solution:

First, $1 = 5 \cdot 17 - 4 \cdot 21\ ,$

and so $224 = 224 \times 5 \cdot 17 - 224 \times 4 \cdot 21$

and $224 = (224 \times 5 - k \cdot 21) \times 17 + (k \cdot 17 - 224 \times 4) \times 21\ , \qquad \forall\, k \in \mathbb{Z}\ .$

We want $k \in \mathbb{Z}$ so that both parentheses are positive:

from $53.x = \dfrac{1120}{21} = \dfrac{224 \times 5}{21} \geq k \geq \dfrac{224 \times 4}{17} = \dfrac{896}{17} = 52.x$

follows $k = 53$

and
$$\begin{aligned}224 &= (1120 - 53 \cdot 21) \times 17 + (53 \cdot 17 - 896)\\ &= (1120 - 1113) \cdot 17 + (901 - 896) \times 21\\ &= 7 \cdot 17 + 5 \cdot 21\ .\end{aligned}$$

Therefore

Answer: Father makes 5 runs and the Son 7.

Let A, B be finite sets with $|A| = m$ and $|B| = n$.
(a) How many relations from A to B are there?
(b) How many functions $f : A \to B$ are there?
(c) How many injective functions $f : A \to B$ are there?
(d) How many surjective functions $f : A \to B$ are there?

Answer: (a) $2^{m \times n}$ (b) n^m (c) none if $m > n$, $n!/(n-m)!$ otherwise (d)$n! \times S(m, n)$

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a) How many binary operations $\mathcal{R} : A \to A$ are there on $A \overset{\text{def}}{=} \{a, b, c, d, e\}$?
b) How many of them are commutative?
c) have b for a neutral element?
d) have a neutral element?
Mother has enough cookie dough left for two decent size cookies that can't be told apart, and also has some M&M's left over: a white one, a red one, a blue one, a yellow one, and a pink one. In how many ways can she put the M&M's on the cookies so that each cookie receives at least one M&M?

Answer: $S(5, 2) = 15$

During the first six weeks in his senior year Brad sends out at least one resume per day, but no more than 60 in all. Show that there is a stretch of consecutive days during which he sends out precisely 23 resumes.

The magician says "Select seven distinct integers between 1 and 24 and denote by S the set of numbers you got. For every non–void subset $A \subseteq S$ of your selection S compute the sum of the numbers in it and call it $s(A)$. In other words $s(A) \overset{\text{def}}{=} \sum_{i \in A} i$. You will find that there are two distinct subsets $A, A' \subseteq S$ with $s(A) = s(A')$." How does the magician know that?

Solution: Your set S of seven integers between 1 and 2 has 119 non–void subsets of size 5 or less. For any such subset $A \subset S$ we have $1 \leq s(A) \leq 20 + 21 + 22 + 23 + 24 = 110$. By the PHP there must be two subsets $A, A' \subset S$ of size 5 or less with the same sum: $S(A) = s(A')$.

On $A \overset{\text{def}}{=} \{0, 3, 5, 9, 15\}$ let $\preceq$ be the relation defined by $a \preceq b \iff a|b$.
(a) Draw the directed graph of $\preceq$.

(b) Find the incidence matrix of $\preceq$.

(c) Show that $\preceq$ is a partial order on A.

(d) Draw its Hasse diagram.

(e) Toplogically sort $(A, \preceq)$ in two different ways and draw the resulting Hasse diagrams.

Let A, B be sets and $f : A \to B$ a function. Show that f is injective if and only if $f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$ for any pair of subsets $A_1, A_2 \subseteq A$.

Solution: Consult Lecture 16.

Let A, B be sets. In which cases is $A \times B = B \times A$? **Prove your assertion!**

Solution: Consult Lecture 16.

Answer: In summary, $A \times B = B \times A$ **if and only if** $A = \emptyset$ or $B = \emptyset$ or $A = B$.

Let $|A| = m$. (a) How many binary operations are there on A? (b) How many of them have a neutral element? (c) How many of them are commutative? (d) How many commutative ones have a neutral element?

Answer: (a) $m^{m \times m}$ (b) $m \times m^{(m-1)\times(m-1)}$ (c) $m^{m(m+1)/2}$ (d)$m \times m^{m(m-1)/2}$

Let $f, g : \mathbb{R} \to \mathbb{R}$, $f : x \mapsto ax + b$, $g : x \mapsto cx + d$, where a, b, c, d are constants in $\mathbb{R}$. What relationship(s) must these four constants satisfy in order that $f \circ g = g \circ f$?

How are reflexivity ***etc.*** of the relation $\mathcal{R} \subseteq A \times A$ reflected in the incidence matrix $M(\mathcal{R})$?

Set $A := \{3, 6, 12, 18, 21\}$ and define the relation $\mathcal{R}$ on A by $a\mathcal{R}b \iff a|b$.
b) Show that $\mathcal{R}$ is a partial order.
c) Find the incidence matrix of $\mathcal{R}$.
d) Display the directed graph and the Hasse diagram of $\mathcal{R}$.
e) Topologically order $\mathcal{R}$.

On $\mathbb{Z}$ define the relation $\simeq$ by $x \simeq y$ if x and y have the same remainder modulo 37. Show that $\simeq$ is an equivalence relation.

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State the multinomial formula for $(a+b+c)^{10}$:

$$(a+b+c)^{10} = \sum$$

$$(a+b+c)^{10} = \sum_{\substack{0 \le i,j,k \le 10 \\ i+j+k=10}} \binom{n}{i,j,k} a^i b^j c^k$$

(b) How many summands are there in this sum?

Answer: There are $\binom{10+(3-1)}{(3-1)} = \binom{12}{2} = 66$ summands.

Write the converse, inverse, and contrapositive of the implication "If the moon is made of yellow cheese then the pope is catholic." For this statement and its converse, inverse, and contrapositive determine its truth value.

Truth value of "If the moon is made of yellow cheese then the pope is catholic.":

converse:

truth value:

contrapositive:

truth value:

inverse:

truth value:

Let $\mathcal{R}_1$, $\mathcal{R}_2$ be relations on a set A. Prove or disprove the following

a) If $\mathcal{R}_1$ and $\mathcal{R}_2$ both are reflexive then $\mathcal{R}_1 \cup \mathcal{R}_2$ is reflexive. **True** False
The reason:

b) If $\mathcal{R}_1$ and $\mathcal{R}_2$ both are symmetric then $\mathcal{R}_1 \cap \mathcal{R}_2$ is symmetric. **True** False
The reason:

c) If $\mathcal{R}_1$ and $\mathcal{R}_2$ both are antisymmetric then $\mathcal{R}_1 \cap \mathcal{R}_2$ is antisymmetric. **True** False
The reason:

d) If $\mathcal{R}_1$ and $\mathcal{R}_2$ both are transitive then $\mathcal{R}_1 \cup \mathcal{R}_2$ is transitive. True **False**
The reason:

How many (a) reflexive, (b) symmetric, (c) antisymmetric, (d) reflexive and symmetric, (e) symmetric and antisymmetric, (f) reflexive, symmetric, and non–transitive, relations are there on a set of size n?

Answer: (a) (b) (c) (d) (e) (f) .

(a) Let $A \stackrel{\text{def}}{=} \{1,2,3,4\}$ and $\mathcal{R} \stackrel{\text{def}}{=} \{(1,1),(1,2),(2,3),(3,3),(3,4),(4,4)\}$.
Find the relation matrix and the directed graph of $\mathcal{R}$:

$$M(\mathcal{R}) = \left[\quad \right]$$. Graph:

(b) With A and $\mathcal{R}$ as in a), find two relation $\mathcal{S}, \mathcal{T}$ on A such that $\mathcal{S} \neq \mathcal{T}$ yet $\mathcal{R} \circ \mathcal{S} = \mathcal{R} \circ \mathcal{T} = \{(1,1),(1,2),(1,4)\}$.

On $A \stackrel{\text{def}}{=} \{1,2,4,6,10,12\}$ let $\preceq$ be the relation defined by $a \preceq b \iff a|b$.
(a) Draw the directed graph of $\preceq$ and its incidence matrix.

(b) Show that $\preceq$ is a partial order on A.

(c) Draw its Hasse diagram.

(d) Toplogically sort $(A, \preceq)$ in two different ways and draw the resulting Hasse diagrams.

How many equivalence relations are there on the set $A \stackrel{\text{def}}{=} \{1,2,4,6,10,12\}$?

As many as there are partitions. Consult lecture 23.

Answer: There are $\sum_{i=1}^{6} S(6,i)$ equivalence relations on A.

On $\mathbb{Z}$ define the relation $\simeq$ by $x \simeq y$ if a and b have the same remainder modulo 37. Show that $\simeq$ is an equivalence relation.