Practice Final, 427K, 12/16/2014 **PRINTED** NAME:________________________EID:

No books, notes, calculators, or telephones are allowed.

Every problem is worth an equal number of points.

You must show your work; answers without substantiation do not count.

Answers must appear in the box provided!

No or the wrong answer in the answer box results in no credit!

This does not aim nor claim to be exhaustive! Use this as a guide of what to study and not of what not to study! Do not expect to find every test problem listed here! Sigh. There will be 7–10 problems on the final.

A problem worked out in class or one of the problems assigned from Sections 1.1-2, 2.1-2.

Answer:

A problem worked out in class or one of the problems assigned from Sections 2.3-5.

Answer:

A problem worked out in class or one of the problems assigned from Sections 2.6-8, 3.1-2.

Answer:

A problem worked out in class or one of the problems assigned from Sections 3.1-3.2.

Answer:

A problem worked out in class or one of the problems assigned from Sections 3.3-3.6.

Answer:

What does the Wronskian do for you?

a) How is it defined: $W[y_1, y_2] \stackrel{\text{def}}{=}$

b) What does it say about a pair y_1, y_2 of solutions of a SOLODE?

c) State Abel's theorem.

State the existence and uniqueness theorem for the **general** FOODE.

State the existence and uniqueness theorem for the SOLODE.

Describe Euler's one-step approximation method - how do you get from one point to the next?

Describe the method of Integrating Factors.
What does it mean to say that $\phi(t)$ is a solution of $y' = f(t, y)$?

Find the general solution of the FOLODE $ty' + 2y = \sin t$, $t > 0$.

Solution: Write the equation in standard form:

$$y' + \frac{2}{t}y = \frac{\sin t}{t}$$

Then $p(t) = 2/t$, the integrating factor is

$$e^{\int p(t)\,dt} = e^{2\ln t} = t^2\ .$$

and turns the equation into

$$t^2y' + 2ty = (t^2y)' = t\sin t$$

and

$$t^2y(t) = \int t\sin t\;dt = c - t\cos t + \sin t\ .$$

Hence

Answer: $y = \dfrac{c - t\cos t + \sin t}{t^2}$

A corpse is discovered hanging on a hook in a meat locker that is being kept at $-3°C$. Its temperature is measured immediately and is found to be $27°C$. Three hours later its temperature is down to $7°C$. Assuming that at the time of death the body had the normal body temperature of $37°C$, when did death occur?

Let $T(t)$ denote the temperature at time t, where $t = 0$ at the time of discovery of the body. By Newton's law, the rate of change of T is proportional to the difference of T and the ambient temperature:

$$T'(t) = -k \times (T(t) - (-3)) \text{ or } (T(t) + 3)' = -k \times (T(t) + 3)$$

which gives $\quad T(t) + 3 = Ce^{-kt}$ or $T(t) = Ce^{-kt} - 3$.

The initial condition $T(0) = 27$ gives

$$C = 30 \quad \text{, so that } T(t) = 30e^{-kt} - 3 \text{ .}$$

Now $\quad T(3) = 7$ reads $30e^{-k3} = 10$ or $e^{-3k} = 1/3$,

whence $\quad -3k = \ln(1/3) = -\ln 3 \quad \text{and} \quad k = \dfrac{\ln 3}{3}$

and $\quad T(t) = 30e^{-\ln 3 \times t \big/ 3} - 3$.

Solving $\quad 37 = T(t_d) = 30e^{-\ln 3 \times t_d \big/ 3} - 3$

gives $\quad 40 = 30e^{-\ln 3 \times t_d \big/ 3}$,

$$4/3 = e^{-\ln 3 \times t_d \big/ 3} \text{ .}$$

$$\ln(4/3) = -\ln 3 \times t_d/3 \text{ .}$$

and $\quad t_d = 3\dfrac{\ln 3 - \ln 4}{\ln 3}$.

Answer: $k = -\dfrac{1}{3}\ln(1/3)$, time–of–death $= 3\frac{\ln(3/4)}{\ln(1/3)}$ hours before discovery

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A certain population $Y(t)$ satisfies the logistic equation $Y' = Y(1 - Y/1000)$. At time $t = 0$ its value $Y(0)$ is 30% of the carrying capacity.
(a) Find $Y(t)$
(b) At which time T does $Y(T)$ reach 60% of the carrying capacity?

Solution: The carrying capacity clearly is 1000. Set $y(t) \stackrel{\text{def}}{=} Y(t)/1000$. Then

$$y' = y(1-y) \qquad y(0) = .3\,.$$

Then separate the variables and use partial fractions:

$$dt = \frac{dy}{y(1-y)} = \Big(\frac{1}{y} + \frac{1}{1-y}\Big)dy$$

whence
$$t + c = \ln|y| - \ln|1-y| = \ln\left|\frac{y}{1-y}\right|$$

and
$$Ce^t = \frac{y}{1-y} \implies y = \frac{Ce^t}{1+Ce^t}\,.$$

The initial condition $y(0) = 300/1000 = .3$ gives

$$C = \frac{y(0)}{1-y(0)} = \frac{.3}{1-.3} = \frac{3}{7}$$

and (a):
$$y(t) = \frac{Ce^t}{1+Ce^t} = \frac{1}{1+(7/3)e^{-t}}\,.$$

The condition
$$Y(T) = .6 \times 1000 = 600$$

reads
$$.6 = \frac{1}{1+(7/3)\times e^{-T}}$$

or
$$e^{-T} = \frac{3\times(1-.6)}{7\times .6} = \frac{4}{7}$$

& resolves to (b):
$$T = \ln\frac{7}{4}\,.$$

Answer: (a): $Y(t) = \dfrac{1000}{1+(7/3)e^{-t}}$ (b): $T = \ln\dfrac{7}{4}$

Find an integrating factor μ and solve $y\,dx + (2x - ye^y)\,dy = 0$.

Answer: $\mu(y) = y$, $xy^2 - (y^2 - 2y + 2)e^y = c$, p. 95#21

The FOOIVP

$$y' = \frac{1}{1 + t^2 y^3} \,, \qquad y(1) = 1 \,,$$

cannot be handled with any of our four methods, so you decide to approximate $y(1.2)$ using Euler's method. What approximate value for $y(1.2)$ do you get?

Solution: The step size is $h = 0.2$. Clearly

$$y(1.2) \approx y(1) + 0.2 \times f(1,1) = 1 + 0.2 \times \frac{1}{1+1} = 1.1 \,.$$

Answer: $y(1.2) \approx 1.1$

!!!Read the whole problem before answering!!!
a) Describe the most general circumstance in which the Method of Variation of Parameters applies.

A fundamental System $\{y_1, y_2\}$ of a homogeneous SOLODE $L[y] = 0$ is known.

b) What is it intended to accomplish?

It lets you then compute a particular solution Y of the corresponding inhomogeneous SOLODE $L[y] = g$.

c) How does it work?

Try $Y = v_1 y_1 + v_2 y_2$, v_i functions. Writing out $L[Y] = g$ leads to $v_1' y_1 + v_2' y_2 = 0$ and $v_1' y_1' + v_2' y_2' = g$. Solve for v_1, v_2:

$$v_1'(t) = \frac{-y_2(t) g(t)}{W[y_1, y_2](t)} \quad \textit{and} \quad v_2'(t) = \frac{y_1(t) g(t)}{W[y_1, y_2](t)} \,.$$

Now get v_1, v_2 by integration.

!!!Read the whole problem before answering!!!
a) Describe the most general circumstance in which the Method of Reduction of Order applies.
b) What is it intended to accomplish?
c) How does it work?

Find a particular solution of $y'' + 4y = 3\csc t$. [Hint: $\int \csc t\, dt = \ln\left|\csc t + \cot t\right| + C$.]

The corresponding HSOLODE $y'' + 4y = 0$ has FS $\{\cos(2t), \sin(2t)\}$ with Wronskian $W = W[\cos(2t), \sin(2t)] = 2$. We use Variation of Parameters to find a particular solution $Y = v_1 y_1 + v_2 y_2$. This leads to

$$v_1'(t) = \frac{-y_2(t)g(t)}{W[y_1, y_2](t)} = \frac{-\sin(2t) \times 3\csc t}{2} = 3\cos t$$

and

$$\begin{aligned} v_2'(t) &= \frac{y_1(t)g(t)}{W[y_1, y_2](t)} = \frac{\cos(2t) \times 3\csc t}{2} \\ &= \frac{3(1 - 2\sin^2 t)}{2\sin t} = \frac{3}{2}\csc t - 3\sin t\ . \end{aligned}$$

Therefore

$$\begin{aligned} v_1 &= 3\int \cos t\, dt = 3\sin t\ , \\ v_2 &= \frac{3}{2}\int \csc t\, dt - 3\int \sin t\, dt = \frac{3}{2}\ln\left|\csc t + \cot t\right| - 3\cos t \end{aligned}$$

and

$$\begin{aligned} Y &= 3\cos(2t)\sin t + \frac{3}{2}\sin(2t)\ln\left|\csc t + \cot t\right| - 3\sin(2t)\cos t \\ &= \frac{3}{2}\sin(2t)\ \times \ln\left|\csc t + \cot t\right| - 3\sin t\ . \end{aligned}$$

Answer: $Y = \dfrac{3}{2}\sin(2t)\ln\left|\csc t + \cot t\right| - 3\sin t$

Use Undetermined Coefficients (!) to find the **general** solution of $y'' + 2y' + 5y = 3\cos(2t)$.

The corresponding CCHSOLODE has characteristic polynomial $r^2 + 2r + 5$ with roots $1 \pm 2i$ and FS $\{e^t \cos(2t), e^t \sin(2t)\}$. For a particular solution Y of $y'' + 2y' + 5y = 3\cos(2t)$ we try $Y = A\cos(2t) + B\sin(2t)$. Doing the algebra results in

$$A = \frac{3}{17} \quad \text{and} \quad B = \frac{12}{17}\,.$$

Answer: $y = \frac{3}{17}\cos 2t + \frac{12}{17}\sin 2t + c_1 e^{-t}\cos 2t + c_2 e^{-t}\sin 2t$, p. 178#2

Do one of: p190 # 1-21; p203 # 1-12.

Answer:

Do one of: p249 # 1-20, 21-28; p259 # 1-14, 16-21; p265 # 1-17.

Answer:

Do one of: p271 # 1-14; p278 # 1-22.

Answer:

Do one of: p284 # 1-16; p292 # 1-17; p312 # 1-24; p322 # 1-16.

Answer:

Find the regular singular points of $(1 - 2x^2)y'' + xy' + y = 0$.
For each one of them predict the convergence radius of its "mucked–up Euler solution."

Answer:

!!!Read the whole question before answering its parts!!!
What does the ratio test for power series do for you?

(a) When does it apply and what does it produce?

(b) What exactly does it say?

(c) Can you use it to determine the convergence radius for the cosine series $\cos x = 1 - x^2/2! + x^4/4! \pm \ldots$?

!!!Read the whole question before answering its parts!!!
Suppose the functions P, Q, R in the SOLODE $P(x)y'' + Q(x)y' + R(x)y = 0$ are analytic at the point x_0.
(a) What does it mean that x_0 is an ordinary point?
Answer: x_0 is an ordinary point if .

(b) If x_0 is an ordinary point, what do you do?

(c) What can you say about the convergence radius of the resulting power series?

Complete the following definitions concerning the SOLODE

$$P(x)y'' + Q(x)y' + R(x)y = 0 \tag{$*$}$$

with analytic coefficients P, Q, R:

(a) x_0 is an ordinary point for $(*)$ if ;

(b) x_0 is a singular point for $(*)$ if ;

(c) x_0 is a regular singular point for $(*)$ if

!!!Read the whole question before answering its parts!!!

Suppose the functions P, Q, R in the SOLODE $P(x)y'' + Q(x)y' + R(x)y = 0$ are analytic at the point x_0.

(a) What does it mean that x_0 is a regular singular point?

Answer: x_0 is a regular singular point if

(b) If x_0 is an regular singular point, what do you do?

(c) What can you say about the convergence radius of the resulting series?

Ve) Write down the general Euler equation and describe how you would treat it.

Some Laplace Transforms

$f(t)$	$F(s)$	$f(t)$	$F(s)$
$\sin t$	$1/(s^2+1)$	t^n	$n!/s^{n+1}$
$\cos t$	$s/(s^2+1)$	$u_c(t)f(t-c)$	$e^{-cs}F(s)$
$\operatorname{Sinh} t$	$1/(s^2-1)$	$\delta_c(t)$	e^{-cs}
$\operatorname{Cosh} t$	$s/(s^2-1)$	$f(at)$	$a^{-1}F(s/a)$
$f'(t)$	$sF(s)-f(0)$	$e^{ct}f(t)$	$F(s-c)$
$f''(t)$	$s^2F(s)-sf(0)-f'(0)$	$f*g(t)$	$F(s)G(s)$

To which signal functions f does the Laplace transform apply?

(b) Write down the definition: $\mathcal{L}\{f\}(s) \stackrel{\text{def}}{=}$, s

Compute the Laplace transform of some simple function.

Write down the definition of the convolution of two functions f, g.

How do convolution and Laplace transform interact?

Do one of: p312 # 1-24; p322 # 1-16; p337 # 1-16; p344 # 1-12, 17-22; p351 # 1, 3-10; p351 # 3-11, 13-18; p322 # 1-26.

For example, find the Laplace–inverse of $F(s) \stackrel{\text{def}}{=} \dfrac{e^{-s}}{s^2(s^2-1)}$.

Solution 1: Since $\dfrac{e^{-s}}{s^2(s^2-1)} = \left(\dfrac{e^{-s}}{s^2} \cdot \dfrac{1}{s^2-1}\right)$, it is the convolution

of $\quad \mathcal{L}^{-1}\left\{\dfrac{e^{-s}}{s^2}\right\}(t) = u_1(t)\mathcal{L}^{-1}\left\{\dfrac{1}{s^2}\right\}(t-1) = u_1(t)(t-1)$

with $\quad \mathcal{L}^{-1}\left\{\dfrac{1}{s^2-1}\right\} = \mathrm{Sinh} t$,

i.e., $\quad \mathcal{L}^{-1}\{F(s)\} = \displaystyle\int_0^t u_1(\tau)\,(\tau-1)\,\mathrm{Sinh}(t-\tau)\,d\tau$

as this vanishes when $t < 1$, continue for $t \geq 1$

$$= \int_1^t u_1(\tau)\,(\tau-1)\,\mathrm{Sinh}(t-\tau)\,d\tau = \int_1^t (\tau-1)\,\mathrm{Sinh}(t-\tau)\,d\tau$$

with $\sigma = \tau - 1$:
$$= \int_0^{t-1} \sigma\,\mathrm{Sinh}(t-1-\sigma)\,d\sigma$$

integration by parts:
$$= \left[-\sigma \mathrm{Cosh}(t-1-\sigma) - \mathrm{Sinh}(t-1-\sigma)\right]_0^{t-1}$$
$$= u_1(t)\left[\mathrm{Sinh}(t-1) - (t-1)\right] .$$

Solution 2: Since, by partial fractions, $\dfrac{1}{s^2(s^2-1)} = \dfrac{1}{(s^2-1)} - \dfrac{1}{s^2}$,

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s^2(s^2-1)}\right\}(t) &= u_1(t)\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2-1)}\right\}(t-1)\\ &= u_1(t)\left[\mathcal{L}^{-1}\left\{\frac{1}{(s^2-1)}\right\}(t-1) - \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}(t-1)\right]\\ &= u_1(t)\left[\mathrm{Sinh}(t-1) - (t-1)\right] .\end{aligned}$$

Answer: $\mathcal{L}^{-1}\{F(s)\}(t) = u_1(t)\left[\mathrm{Sinh}(t-1) - (t-1)\right]$

Do one of: p344 # 1-12, 17-22; p351 # 1, 3-10; p351 # 3-11, 13-18; p322 # 1-26.

Let the function $g : [0,\infty) \to \mathbb{R}$ be defined by [drawing it might help]

$$g(t) \stackrel{\text{def}}{=} \begin{cases} t & \text{for } 0 \le t \le 1, \\ 2-t & \text{for } 1 \le t \le 2, \\ 0 & \text{for } 2 \le t < \infty. \end{cases}$$

Use the Laplace transform to solve the IVP $y'' - y = g$, $y(0) = 1$, $y'(0) = 0$.

Solution (check the computations!): The usual algebra gives

$$Y(s) = \frac{sy(0) + y'(0)}{s^2-1} + \frac{G(s)}{s^2-1} = \frac{s}{s^2-1} + \frac{G(s)}{s^2-1} = Y_1(s) + Y_2(s)\ .$$

The Laplace inverse of the first summand $Y_1(s)$ is Cosh$t$, the Laplace inverse of $Y_2(s)$ is convolution of $g(t)$ with the solution $y_{aux}(t)$ of $y'' - y = 0$, $y(0) = 0$, $y'(0) = 1$; since that solution has Laplace transform $\dfrac{1}{s^2-1}$, we have $y_{aux}(t) = \text{Sinh}t$, and therefore $\mathcal{L}^{-1}Y_2(t) = (g * \text{Sinh})(t)$ $= \int_0^t g(\tau)\text{Sinh}(t-\tau)\,d\tau$. For the computation of this convolution we'll need the τ–antiderivative of $\tau\text{Sinh}(t-\tau)$; it is $-\tau\text{Cosh}(t-\tau) - \text{Sinh}(t-\tau)$. Therefore

$$I(t) \stackrel{\text{def}}{=} \big(y_{aux} * g\big)(t) = \int_0^t g(\tau)\text{Sinh}(t-\tau)\,d\tau$$

for $0 \le t \le 1$:

$$= \int_0^t \tau\text{Sinh}(t-\tau)\,d\tau = \Big[-\tau\text{Cosh}(t-\tau) - \text{Sinh}(t-\tau)\Big]_0^t$$
$$= -t + \text{Sinh}t = \text{Sinh}t - t$$

for $1 \le t \le 2$:

$$\begin{aligned} I(t) &= \text{Sinh}1 - 1 + \int_1^t g(\tau)\text{Sinh}(t-\tau)\,d\tau \\ &= \text{Sinh}1 - 1 + \int_1^t (2-\tau)\text{Sinh}(t-\tau)\,d\tau \\ &= \text{Sinh}1 - 1 - 2\text{Cosh}(t-\tau)\Big|_1^t - \Big[-\tau\text{Cosh}(t-\tau) - \text{Sinh}(t-\tau)\Big]_1^t \\ &= \text{Sinh}1 - 1 - 2 + 2\text{Cosh}(t-1) + \Big[\tau\text{Cosh}(t-\tau) + \text{Sinh}(t-\tau)\Big]_1^t \\ &= \text{Sinh}1 - 3 + 2\text{Cosh}(t-1) + t - \text{Cosh}(t-1) - \text{Sinh}(t-1) \\ &= \text{Sinh}1 - 3 + \text{Cosh}(t-1) + t - \text{Sinh}(t-1) \end{aligned}$$

for $2 \le t < \infty$:

$$\begin{aligned} I(t) &= \text{Sinh}1 - 3 + \text{Cosh}1 + 2 - \text{Sinh}1 + \int_2^t g(\tau)\text{Sinh}(t-\tau)\,d\tau \\ &= \text{Cosh}1 - 1 \end{aligned}$$

Alternatively, we can observe that $g(t) = t - 2u_1(t)\cdot(t-1) + u_2(t)(t-2)$, look up its Laplace transform in the table, multiply that with $1/(s^2-1)$, and compute the Laplace–inverse of that from the table.

Answer: $y(t) = \text{Cosh}t + \begin{cases} \text{Sinh}t - t & \text{for } 0 \le t \le 1; \\ t + \text{Cosh}(t-1) - \text{Sinh}(t-1) + \text{Sinh}1 - 3 & \text{for } 1 \le t \le 2; \\ \text{Cosh}1 - 1 & \text{for } 2 \le t < \infty. \end{cases}$

Apply the Laplace Transform to a CCSOLODE with Impulse Input.

A weight of mass 100g stretches a spring 10cm. If the mass is pulled down an additional 3cm and then is released, and if there is no air resistance, determine the displacement $u(t)$ of the mass at any time t.
[The gravitational constant in the metric system is roughly $10m/sec^2$. Remember that $100cm = 1m$]

Answer: $u(t) = 0.03\cos(10t)$ m p. 203#5

Find the general power series solution of $2y'' + (x+1)y' + 3y = 0$ about the point $x_0 = 2$.

Solve the SOLIVP $x^2y'' + 3xy' + 5y = 0$, $x > 0$, $y(1) = 1, y'(1) = -1$.

Solution:
Try $y = x^r$, compute $L[x^r] = [r(r-1) + 3r + 5]x^r$, get the indicial equation $r^2 + 2r + 5 = (r+1)^2 + 4 = 0$ with roots $r_{1,2} = -1 \pm 2i$. In the general solution $x^{-1}(c_1\cos(2\ln x) + c_2\sin(2\ln x)$ we must choose $c_1 = 1, c_2 = 0$ to accommodate the initial values:

Answer: $y = x^{-1}\cos(2\ln x)$ p. 278#16

Answer a theoretical question as in part I, problem V.

Show that $x^2y'' + xy' + (x-2)y = 0$ has a regular singular point at $x_0 = 0$. Find the indicial equation and its roots, the recurrence relation, and the series solutions for $x > 0$ corresponding to the two roots (three non–zero terms will suffice).

Solution: $P(0) = 0$: 0 is a singular point.

$$\lim_{x\to 0} x \cdot \frac{x}{x^2} = 1 \quad \text{and} \quad \lim_{x\to 0} x^2 \cdot \frac{x-2}{x^2} \ :$$

0 is regular. Indicial equation is $F(r) = r(r-1) + r - 2 = r^2 - 2 = 0$ with roots $r_{1,2} = \pm\sqrt{2}$.

Try $$y = \sum_{n=0}^{\infty} a_n x^{r+n}\,, \qquad a_0 = 1\,,$$

$$\Longrightarrow \qquad xy = \sum_{n=0}^{\infty} a_n x^{r+n+1} = \sum_{m=1}^{\infty} a_{m-1} x^{r+m}$$

and $$-2y = \sum_{n=0}^{\infty} -2a_n x^{r+n}\,.$$

Also $$y' = \sum_{n=0}^{\infty} a_n (r+n) x^{r+n-1} \quad \Longrightarrow \quad xy' = \sum_{n=0}^{\infty} a_n (r+n) x^{r+n}$$

and $$y'' = \sum_{n=0}^{\infty} a_n (r+n)(r+n-1) x^{r+n-2}$$

$$\Longrightarrow \qquad x^2y'' = \sum_{n=0}^{\infty} a_n (r+n)(r+n-1) x^{r+n}$$

Hence
$$\begin{aligned} L[y] &= \big[-2 + r + r(r-1)\big]x^r = [r^2-2]x^r \\ &\quad + \sum_{n=1}^{\infty} \big[a_{n-1} - 2a_n + a_n(r+n) + a_n(r+n)(r+n-1)\big] x^{r+n} \\ &= x^r\Big\{F(r) + \sum_{n=1}^{\infty} \big[a_{n-1} + a_n F(r+n)\big] x^n\Big\} \end{aligned}$$

$L[y] = 0$ produces again the indicial equation $F(r) = r^2 - 2 = 0$, and the recurrence relation

$$a_n = \frac{-a_{n-1}}{F(r+n)}\,, \qquad n = 1, 2, 3, \ldots$$

$$\begin{array}{ll} r = \sqrt{2} & r = -\sqrt{2} \\ a_1 = \dfrac{-a_0}{F(1+\sqrt{2})} = \dfrac{-1}{1+2\sqrt{2}} & a_1 = \dfrac{-1}{1-2\sqrt{2}} \\ a_2 = \dfrac{1/(1+2\sqrt{2})}{4+4\sqrt{2}} & a_2 = \dfrac{1/(1-2\sqrt{2})}{4-4\sqrt{2}} \end{array}$$

Answer: $y_1 = x^{\sqrt{2}}[1 - \dfrac{x}{1+2\sqrt{2}} + \dfrac{x^2/(1-2\sqrt{2})}{4+4\sqrt{2}} \ldots]$, $y_2 = x^{-\sqrt{2}}[\ldots]$ p. 284#6

Find the Laplace-inverse of $F(s) \stackrel{\text{def}}{=} \dfrac{2}{(s-3)(s^2-4s+5)}$.

$$\frac{2}{(s-3)(s^2-4s+5)} = \frac{\alpha}{s-3} + \frac{\beta s+\gamma}{s^2-4s+5}$$

$$\Longrightarrow \qquad 2 = \alpha(s^2-4s+5) + (\beta s+\gamma)(s-3)$$

$$= (\alpha+\beta)s^2 + (-4\alpha-3\beta+\gamma)s + 5\alpha - 3\gamma$$

$$\Longrightarrow \qquad \beta = -\alpha\ ,\ \gamma = \alpha\ ,\ \alpha = 1$$

whence

$$\frac{2}{(s-3)(s^2-4s+5)} = \frac{1}{s-3} - \frac{s}{(s-2)^2+1} + \frac{1}{(s-2)^2+1}$$

$$= \frac{1}{s-3} - \frac{s-2}{(s-2)^2+1} - \frac{1}{(s-2)^2+1}$$

and

$$\mathcal{L}^{-1}\{F\}(t) = e^{3t} - e^{2t}\sin t - e^{2t}\cos t$$

Solve $y'' + y = u_\pi$, $y(0) = 1$, $y'(0) = 0$.

Solution: A FS for the homogeneous SOLODE is $\{\cos, \sin\}$. The solution for the homogeneous IVP is $y_h(t) = \cos(t)$.
The solution for the homogeneous IVP with $y(0) = 0$ and $y'(0) = 1/1 = 1$ is $y_p(t) = \sin(t)$. The convolution

$$(\sin * u_\pi)(t) = \int_0^t \sin(t - \tau) u_\pi(\tau)\, d\tau$$

equals zero if $t \leq \pi$ and

$$\cos(t - \tau)\Big|_\pi^t = 1 - \cos(t - \pi)$$

for $t > \pi$. Hence $(\sin * u_\pi)(t) = u_\pi(t)(1 - \cos(t - \pi))$ and

$$y(t) = y_h(t) + (y_p * u_\pi)(t) = \cos(t) + u_\pi(t)(1 - \cos(t - \pi))\ .$$

Look at the old quizzes. I might put one similar to them on the test.

Describe the Euler method, the improved Euler method, and the Runge–Kutta method, including estimates of the local and global errors in terms of the step size, and the number of computations required.
(a) State Fourier's theorem.
Describe the Gibbs phenomenon.
Describe the method of separation of variables.
What are even (odd) functions?
How can you get a pure sine (cosine) series for a function $f : [0, L] \to \mathbb{R}$ from Fourier's theorem?

Do one of: p449 #1ab-12ab; p456 # 1-12; p461 # 1-12.

Do one of: p610 # 1-6; p575 # 1–21; p585 # 1-24; p592 # 1ab-6ab, 7a-12a; p 600 # 1-26.

Let $f : [-\pi, \pi] \to \mathbb{R}$ be the function defined by [drawing it might help]

$$f(x) = \begin{cases} -\pi - x & \text{for } -\pi \le x \le -\pi/2 \\ x & \text{for } -\pi/2 \le x \le \pi/2 \\ \pi - x & \text{for } \pi/2 \le x \le \pi \end{cases}$$

(a) Find the Fourier series $\widetilde{f}$ of f. (b) At which points x is $f(x) = \widetilde{f}(x)$? (Give reasons)

Solution: We first remark for later that $\int x \cdot \sin(nx)\, dx = \dfrac{-x \cdot \cos(nx)}{n} + \dfrac{\sin(nx)}{n^2}$.
f is odd, so $a_n = 0$ for $n = 0, 1, 2, 3 \ldots$. Also,

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot\sin(nx)\, dx = \frac{2}{\pi}\int_0^{\pi} f(x)\cdot\sin(nx)\, dx \\
&= \frac{2}{\pi}\int_0^{\pi/2} x\cdot\sin(nx)\, dx + \frac{2}{\pi}\int_{\pi/2}^{\pi} (\pi - x)\cdot\sin(nx)\, dx \\
&= \frac{2}{\pi}\cdot\left[\frac{-x\cdot\cos(nx)}{n} + \frac{\sin(nx)}{n^2}\right]_0^{\pi/2} + \frac{2}{\pi}\cdot\left[\frac{-\pi}{n}\cos(nx) + \frac{x\cdot\cos(nx)}{n} - \frac{\sin(nx)}{n^2}\right]_{\pi/2}^{\pi} \\
&= \frac{2}{\pi}\cdot\left[\frac{-\pi/2\cdot\cos(n\pi/2)}{n} + \frac{\sin(n\pi/2)}{n^2}\right] \\
&\quad + \frac{2}{\pi}\cdot\left[\frac{-\pi}{n}\big(\cos(n\pi) - \cos(n\pi/2)\big) + \frac{\pi\cdot\cos(n\pi) - \pi/2\cdot\cos(n\pi/2)}{n} - \frac{\sin(n\pi) - \sin(n\pi/2)}{n^2}\right] \\
&= \frac{4\sin(n\pi/2)}{\pi n^2}\ .
\end{aligned}$$

Note that $b_n = 0$ when n is even, as should be the case.
Therefore

Answer: (a) $\widetilde{f}(x) = \displaystyle\sum_{n=1}^{\infty} \frac{4\sin(n\pi/2)}{\pi n^2}\, \sin(nx)$

and (b) at all points, because f is continuous.

Do one of p620 # 1-8, 9-13; p632 # 1-8; p645 # 1-5; p632 # 1-8

Answer:

Solve the heat conduction problem $u_t = 7u_{xx}$ in an insulated rod of length π whose ends are maintained at $0°$ Celsius at all times and whose initial temperature $u(x,0)$ is given by $u(x,0) = f(x) \quad \forall\, x \in [0,2\pi]$, where

$$f(x) \stackrel{\text{def}}{=} \begin{cases} x & \text{for } 0 \le x \le \pi/2, \\ \pi - x & \text{for } \pi/2 \le x \le \pi. \end{cases}$$

Solution: Separation of variables shows that $u(t,x)$ can be found of the form

$$u(t,x) = \sum_{n=1}^{\infty} b_n \; e^{-7n^2 t} \; \sin(nx) \; ,$$

provided the constants b_n are chosen so that

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} b_n \; \sin(nx) \; .$$

Fourier's theorem says that this representation holds, if we choose

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \; \sin(nx) \; dx \; .$$

These integrals were computed in another problem:

$$b_n = \frac{4\sin(n\pi/2)}{\pi n^2} \; .$$

Consequently

Answer: $u(t,x) = \displaystyle\sum_{n=1}^{\infty} \frac{4\sin(n\pi/2)}{\pi n^2} \cdot e^{-7n^2 t} \cdot \sin(nx)$

Do one of: p610 # 1-14; p620 # 1-14; p632 # 1-8.

Do one of: p 7 # 1, 3, 5, 11, 13, 15, 17, 19, 21; p 15 # 1–7; p 24 all; p 39 # 1–30; p 47 # 1–20, 30–38.

Answer:

Do one of: p 59 # 1–14. p 99 # 1–22; 25–31; p 88 # 1–4, 16; p107 # 1–4.

Answer:

Do one of: p 75 # 1–12; p142 # 1–25; p151 # 1–14.

Answer:

Do one of: p158 # 15–21; p164 # 1–25; p172 # 1–15, 20, 22–30.

Answer:

Do one of: p184 # 1–26.

Answer:

Find the regular singular points of $(1-2x^2)y'' + xy' + y = 0$.
For each one of them predict the convergence radius of its "mucked–up Euler solution."

Answer:

!!!Read the whole question before answering its parts!!!
What does the ratio test do for you?

(a) When does it apply and what does it produce?

(b) How does it work?

(c) Can you use it to determine the convergence radius for the cosine series $\cos x = 1 - x^2/2! + x^4/4! \pm \ldots$?

Write down the definition of the Laplace transform of a function $f(t)$.

To which differential equations does the Laplace transform apply?
How is the Laplace transform used to solve differential equations?

Write down the definition of the convolution of two functions f, g.
How do convolution and Laplace transform interact?

Do one of: p311 # 1-24; p320 # 1-16; p329 # 1-17; p337 # 1-16;

Do one of: p344 # 1-12, 17-22; p351 # 1, 3-10; p351 # 3-11, 13-18; p320 # 1-26.

Apply the Laplace Transform to a CCSOLODE with Impulse Input.

(a) State Fourier's theorem.
(b) Describe the Gibbs phenomenon.
(c) Describe the method of separation of variables.
(d) What are even (odd) functions?
(e) How can you get a pure sine (cosine) series for a function $f : [0, L] \to \mathbb{R}$ from Fourier's theorem?

Do one of: p451 1ab-12ab; p458 # 1-12; p463 # 1-12.

Do one of: p618 # 1-6; p593 # 1-24; p608 # 1ab-6ab, 7a-12a; p600 # 1-26; p610 # 7-12

Let $f : [-\pi, \pi] \to \mathbb{R}$ be the function defined by [drawing it might help]

$$f(t) = \begin{cases} -\pi - x & \text{for } -\pi \le x \le -\pi/2 \\ x & \text{for } -\pi/2 \le x \le \pi/2 \\ \pi - x & \text{for } \pi/2 \le x \le \pi \end{cases}$$

(a) Find the Fourier series $\widetilde{f}$ of f. (b) At which points x is $f(x) = \widetilde{f}(x)$? (Give reasons)

Solution: We first remark for later that $\int x \cdot \sin(nx)\, dx = \dfrac{-x \cdot \cos(nx)}{n} + \dfrac{\sin(nx)}{n^2}$.
f is odd, so $a_n = 0$ for $n = 0, 1, 2, 3 \ldots$. Also,

$$\begin{aligned}
b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cdot \sin(nx)\, dx = \frac{2}{\pi} \int_0^{\pi} f(x) \cdot \sin(nx)\, dx \\
&= \frac{2}{\pi} \int_0^{\pi/2} x \cdot \sin(nx)\, dx + \frac{2}{\pi} \int_{\pi/2}^{\pi} (\pi - x) \cdot \sin(nx)\, dx \\
&= \frac{2}{\pi} \cdot \Big[\frac{-x \cdot \cos(nx)}{n} + \frac{\sin(nx)}{n^2} \Big]_0^{\pi/2} + \frac{2}{\pi} \cdot \Big[\frac{-\pi}{n} \cos(nx) + \frac{x \cdot \cos(nx)}{n} - \frac{\sin(nx)}{n^2} \Big]_{\pi/2}^{\pi} \\
&= \frac{2}{\pi} \cdot \Big[\frac{-\pi/2 \cdot \cos(n\pi/2)}{n} + \frac{\sin(n\pi/2)}{n^2} \Big] \\
&+ \frac{2}{\pi} \cdot \Big[\frac{-\pi}{n} \big(\cos(n\pi) - \cos(n\pi/2) \big) + \frac{\pi \cdot \cos(n\pi) - \pi/2 \cdot \cos(n\pi/2)}{n} - \frac{\sin(n\pi) - \sin(n\pi/2)}{n^2} \Big] \\
&= \frac{4 \sin(n\pi/2)}{\pi n^2} .
\end{aligned}$$

Note that $b_n = 0$ when n is even, as should be the case.
Therefore

Answer: (a) $\displaystyle \widetilde{f}(x) = \sum_{n=1}^{\infty} \frac{4 \sin(n\pi/2)}{\pi n^2} \sin(nx)$

and (b) at all points, because f is continuous.

I) (a) Apply the Laplace Transform to a CCSOLODE with Impulse Input.
I') (a) State Fourier's theorem.
(b) Describe the Gibbs phenomenon.
(c) Describe the method of separation of variables.
(d) What are even (odd) functions?
(e) How can you get a pure sine (cosine) series for a function $f : [0, L] \to \mathbb{R}$ from Fourier's theorem?

Do one of: p344 # 1-12, 17-22; p351 # 1, 3-10; p351 # 3-11, 13-18.

Answer:

Do one of: p449 #1ab-12ab; p456 # 1-12; p461 # 1-12.

Answer:

Let $f : [-\pi, \pi] \to \mathbb{R}$ be the function defined by [drawing it might help]

$$f(t) = \begin{cases} -\pi - x & \text{for } -\pi \leq x \leq -\pi/2 \\ x & \text{for } -\pi/2 \leq x \leq \pi/2 \\ \pi - x & \text{for } \pi/2 \leq x \leq \pi \end{cases}$$

(a) Find the Fourier series $\widetilde{f}$ of f. (b) At which points x is $f(x) = \widetilde{f}(x)$? (Give reasons)

We first remark for later that $\int x \cdot \sin(nx)\, dx = \dfrac{-x \cdot \cos(nx)}{n} + \dfrac{\sin(nx)}{n^2}$. f is odd, so $a_n = 0$ for $n = 0, 1, 2, 3 \ldots$. Also,

$$\begin{aligned}
b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cdot \sin(nx)\, dx = \frac{2}{\pi} \int_0^{\pi} f(x) \cdot \sin(nx)\, dx \\
&= \frac{2}{\pi} \int_0^{\pi/2} x \cdot \sin(nx)\, dx + \frac{2}{\pi} \int_{\pi/2}^{\pi} (\pi - x) \cdot \sin(nx)\, dx \\
&= \frac{2}{\pi} \cdot \Big[\frac{-x \cdot \cos(nx)}{n} + \frac{\sin(nx)}{n^2}\Big]_0^{\pi/2} + \frac{2}{\pi} \cdot \Big[\frac{-\pi}{n} \cos(nx) + \frac{x \cdot \cos(nx)}{n} - \frac{\sin(nx)}{n^2}\Big]_{\pi/2}^{\pi} \\
&= \frac{2}{\pi} \cdot \Big[\frac{-\pi/2 \cdot \cos(n\pi/2)}{n} + \frac{\sin(n\pi/2)}{n^2}\Big] \\
&+ \frac{2}{\pi} \cdot \Big[\frac{-\pi}{n} \big(\cos(n\pi) - \cos(n\pi/2)\big) + \frac{\pi \cdot \cos(n\pi) - \pi/2 \cdot \cos(n\pi/2)}{n} - \frac{\sin(n\pi) - \sin(n\pi/2)}{n^2}\Big] \\
&= \frac{4 \sin(n\pi/2)}{\pi n^2}\, .
\end{aligned}$$

Note that $b_n = 0$ when n is even, as should be the case.
Therefore

Answer: (a) $\widetilde{f}(x) = \displaystyle\sum_{n=1}^{\infty} \frac{4 \sin(n\pi/2)}{\pi n^2} \sin(nx)$

and (b) at all points, because f is continuous.

Do one of: p585 # 1-24; p529 # 1-12; p 600 # 1-26.

Answer:

Solve the heat conduction problem $u_t = 7u_{xx}$ in an insulated rod of length 2π whose ends are maintained at $0°$ Celsius at all times and whose initial temperature $u(0,x)$ is given by $u(0,x) = f(x) \quad \forall\, x \in [0, 2\pi]$, where

$$f(x) \stackrel{\text{def}}{=} \begin{cases} x & \text{for } 0 \le x \le \pi/2, \\ \pi - x & \text{for } \pi/2 \le x \le \pi. \end{cases}$$

Solution: Separation of variables shows that $u(t,x)$ can be found of the form

$$u(t,x) = \sum_{n=1}^{\infty} b_n \; e^{-7n^2 t} \; \sin(nx) \; ,$$

provided the constants b_n are chosen so that

$$u(0,x) = f(x) = \sum_{n=1}^{\infty} b_n \; \sin(nx) \; .$$

Fourier's theorem says that this representation holds, if we choose

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \; \sin(nx) \; dx \; .$$

These integrals were computed in problem IV:

$$b_n = \frac{4 \sin(n\pi/2)}{\pi n^2} \; .$$

Consequently

Answer: $u(t,x) = \sum_{n=1}^{\infty} \frac{4 \sin(n\pi/2)}{\pi n^2} \cdot e^{-7n^2 t} \cdot \sin(nx)$

Do a similar wave equation problem: let

$$f(x) \stackrel{\text{def}}{=} \begin{cases} x & \text{for } 0 \le x \le \pi/2, \\ \pi - x & \text{for } \pi/2 \le x \le \pi. \end{cases}$$

Solve the wave equation $u_{tt} = 81 u_{xx}$ for a string of length π with initial conditions $u(0,x) = f(x)$ and $u_t(0,x) = 0$.

Solution:

$$u(t,x) = \frac{\check{f}(x+9t) + \check{f}(x-9t)}{2} \ ,$$

where $\check{f}$ is the 2π–periodic extension of the odd extension of f.

Do a similar Laplace equation problem: Let

$$f(x) \stackrel{\text{def}}{=} \begin{cases} x & \text{for } 0 \le x \le \pi/2, \\ \pi - x & \text{for } \pi/2 \le x \le \pi. \end{cases}$$

Then solve the Laplace equation on a square sheet of side π with the boundary conditions $u(0,y) = u(x,0) = u(\pi,y) = 0$ and $u(x,\pi) = f(x)$.

Solution: In lecture 23 we saw that the solution is

$$u(x,y) = \sum_{n=1}^{\infty} b_n \sin(nx) \mathrm{Sinh}(ny)$$

with

$$b_n = \frac{1}{\mathrm{Sinh}(n\pi)} \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx)\, dx$$

by earlier problem

$$= \frac{4 \sin(n\pi/2)}{\pi n^2 \mathrm{Sinh}(n\pi)} \ .$$

Do one of: p610 # 1-14; p620 # 1-14; p632 # 1-8.

Add here the practice test 3.