Practice Test 3, M325K, 05/01/2014 **PRINTED** NAME:________________ EID:

No books, notes, calculators, or telephones are allowed.

Every problem is worth an equal number of points.

You must show your work; answers without substantiation do not count.

Answers must appear in the box provided!

No or the wrong answer in the answer box results in no credit!

This does not aim nor claim to be exhaustive! Use this as a guide of what to study and not of what not to study! Do not expect to find every test problem listed here! Sigh.

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a) How many binary operations $\mathcal{R} : A \to A$ are there on $A \stackrel{\text{def}}{=} \{a, b, c, d, e\}$?
b) How many of them are commutative?
c) have b for a neutral element?
d) have a neutral element?

Solution: Consult Lecture 17.

Mother has enough cookie dough left for two decent size cookies that can't be told apart, and also has some M&M's left over: a white one, a red one, a blue one, a yellow one, and a pink one. In how many ways can she put the M&M's on the cookies so that each cookie receives at least one M&M?

Answer: $S(5, 2) = 15$

During the first six weeks in his senior year Brad sends out at least one resume per day, but no more than 60 in all. Show that there is a stretch of consecutive days during which he sends out precisely 23 resumes.

Solution: Consult Herbert from Lecture 18

IV) The magician says "Select seven distinct integers between 1 and 24 and denote by S the set of numbers you got. For every non–void subset $A \subseteq S$ of yqour selection S compute the sum of the numbers in it and call it $s(A)$. In other words $s(A) \stackrel{\text{def}}{=} \sum_{i \in A} i$. You will find that there are two distinct subsets $A, A' \subseteq S$ with $s(A) = s(A')$." How does the magician know that?

Solution: Your set S of seven integers between 1 and 24 has 119 non–void subsets of size 5 or less. For any such subset $A \subset S$ we have $1 \leq s(A) \leq 20 + 21 + 22 + 23 + 24 = 110$. By the PHP there must be two subsets $A, A' \subset S$ of size 5 or less with the same sum: $S(A) = s(A')$.

(IV) On $A \stackrel{\text{def}}{=} \{0, 3, 5, 9, 15\}$ let $\preceq$ be the relation defined by $a \preceq b \iff a|b$.
(a) Draw the directed graph of $\preceq$.

(b) Find the incidence matrix of $\preceq$.

(c) Show that $\preceq$ is a partial order on A.

Solution:
Refl: $a|a \quad \forall\, a \in A \implies a \preceq a \quad \forall\, a \in A$
Trans: $a \preceq b \,\wedge\, b \preceq c \implies a|b \,\wedge\, b|c \implies a|c \implies a \preceq c.$
Antisym: $a \preceq b \,\wedge\, b \preceq a \implies a|b \,\wedge\, b|a \implies a = b.$

(d) Find the minimal and the maximal elements of $\preceq$. (e) Draw its Hasse diagram.

(fe) Toplogically sort $(A, \preceq)$ in two different ways and draw the resulting Hasse diagrams.

Let A, B be finite sets with $|A| = m$ and $|B| = n$.
(a) How many relations from A to B are there?
(b) How many functions $f : A \to B$ are there?
(c) How many injective functions $f : A \to B$ are there?
(d) How many surjective functions $f : A \to B$ are there?

Solution: Consult Lecture 16.

Answer: (a) $2^{m \times n}$ (b) n^m (c) none if $m > n$, $n!/(n-m)!$ otherwise (d)$n! \times S(m, n)$

Let A, B be sets and $f : A \to B$ a function. Show that f is injective if and only if $f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$ for any pair of subsets $A_1, A_2 \subseteq A$.

Solution: Consult Lecture 16

Let A, B be sets. In which cases is $A \times B = B \times A$? **Prove your assertion!**

Solution: Consult Lecture 8

Answer: In summary, $A \times B = B \times A$ **if and only if** OR OR .

V) Let $|A| = m$. (a) How many binary operations are there on A? (b) How many of them have a neutral element? (c) How many of them are commutative? (d) How many commutative ones have a neutral element?

Answer: (a) $m^{m \times m}$ (b) $m \times m^{(m-1)\times(m-1)}$ (c) $m^{m(m+1)/2}$ (d)$m \times m^{m(m-1)/2}$

Let $f, g : \mathbb{R} \to \mathbb{R}$, $f : x \mapsto ax + b$, $g : x \mapsto cx + d$, where a, b, c, d are constants in $\mathbb{R}$. What relationship(s) must these four constants satisfy in order that $f \circ g = g \circ f$?

Solution: $f \circ g(x) = f(g(x)) = f(cx + d) = a(cx + d) + b$. $g \circ f(x) = g(f(x)) = g(ax + b) = c(ax + b) + d$. $f \circ g = g \circ f \iff a(cx + d) + b = c(ax + b) + d \quad \forall\, x \in \mathbb{R} \iff ad + b = cb + d$.

How are reflexivity ***etc.*** of the relation $\mathcal{R} \subseteq A \times A$ reflected in the incidence matrix $M(\mathcal{R})$?

Solution: Consult Lecture 21.

Set $A := \{3, 6, 12, 18, 21\}$ and define the relation $\mathcal{R}$ on A by $a\mathcal{R}b \iff a|b$.
b) Show that $\mathcal{R}$ is a partial order.
c) Find the incidence matrix of $\mathcal{R}$.
d) Display the directed graph and the Hasse diagram of $\mathcal{R}$.
e) Topologically order $\mathcal{R}$.

Solution: Consult Lecture 16

$\mathbb{Z}$ define the relation $\simeq$ by $x \simeq y$ if x and y have the same remainder modulo 37. Show that $\simeq$ is an equivalence relation.
