## 

No books, notes, calculators, or telephones are allowed. Every problem is worth an equal number of points. You must show your work; answers without substantiation do not count. Answers must appear in the box provided!

No or the wrong answer in the answer box results in no credit!

This does not aim nor claim to be exhaustive! Use this as a guide of what to study and not of what not to study! Do not expect to find every test problem listed here! Sigh.

Solve  $y'' + y = u_{\pi}$ , y(0) = 1, y'(0) = 0.

Solution: A FS for the homogeneous SOLODE is  $\{\cos, \sin\}$ . The solution for the homogeneous IVP is  $y_h(t) = \cos(t)$ .

The solution for the homogeneous IVP with y(0) = 0 and y'(0) = 1/1 = 1 is  $y_p(t) = \sin(t)$ . The convolution

$$(\sin * u_\pi)(t) = \int_0^t \sin(t - \tau) u_\pi(\tau) \ d\tau$$

equals zero if  $t \leq \pi$  and

$$\cos(t-\tau)\Big|_{\pi}^{t} = 1 - \cos(t-\pi)$$

for  $t > \pi$ . Hence  $(\sin * u_{\pi})(t) = u_{\pi}(t)(1 - \cos(t - \pi))$  and

$$y(t) = y_h(t) + (y_p * u_\pi)(t) = \cos(t) + u_\pi(t)(1 - \cos(t - \pi)) .$$

Let the function  $g:[0,\infty)\to\mathbb{R}$  be defined by [drawing it might help]

$$g(t) \stackrel{\text{\tiny def}}{=} \begin{cases} t & \text{for } 0 \leq t \leq 1, \\ 2-t & \text{for } 1 \leq t \leq 2, \\ 0 & \text{for } 2 \leq t < \infty \end{cases}$$

Use the Laplace transform to solve the IVP y'' - y = g, y(0) = 1, y'(0) = 0.

Solution (check the computations!): The usual algebra gives

$$Y(s) = \frac{sy(0) + y'(0)}{s^2 - 1} + \frac{G(s)}{s^2 - 1} = \frac{s}{s^2 - 1} + \frac{G(s)}{s^2 - 1} = Y_1(s) + Y_2(s) .$$

The Laplace inverse of the first summand  $Y_1(s)$  is Cosht, the Laplace inverse of  $Y_2(s)$  is convolution of g(t) with the solution  $y_{aux}(t)$  of y'' - y = 0, y(0) = 0, y'(0) = 1; since that solution has Laplace transform  $\frac{1}{s^2 - 1}$ , we have  $y_{aux}(t) = \operatorname{Sinht}$ , and therefore  $\mathcal{L}^{-1}Y_2(t) = (g * \operatorname{Sinh})(t)$  $= \int_0^t g(\tau) \operatorname{Sinh}(t - \tau) d\tau$ . For the computation of this convolution we'll need the  $\tau$ -antiderivative of  $\tau \operatorname{Sinh}(t - \tau)$ ; it is  $-\tau \operatorname{Cosh}(t - \tau) - \operatorname{Sinh}(t - \tau)$ . Therefore, with  $I(t) \stackrel{\text{def}}{=} (y_{aux} * g)(t)$ ,

$$\begin{split} I(t) &= \int_{0}^{t} g(\tau) \mathrm{Sinh}(t-\tau) \, d\tau \\ &= \int_{0}^{t\wedge 1} \tau \mathrm{Sinh}(t-\tau) \, d\tau + \int_{t\wedge 1}^{t\wedge 2} (2-\tau) \mathrm{Sinh}(t-\tau) \, d\tau \\ &= \left[ -\tau \mathrm{Cosh}(t-\tau) - \mathrm{Sinh}(t-\tau) \right]_{0}^{t\wedge 1} \\ &+ \left[ -2 \mathrm{Cosh}(t-\tau) + \tau \mathrm{Cosh}(t-\tau) + \mathrm{Sinh}(t-\tau) \right]_{t\wedge 1}^{t\wedge 2} \\ &= -2 \mathrm{Sinh}(t-(t\wedge 1)) + \mathrm{Sinh}(t) + \mathrm{Sinh}(t-(t\wedge 2)) \\ &+ (t\wedge 2-2) \mathrm{Cosh}(t-(t\wedge 2)) + 2(1-t\wedge 1) \mathrm{Cosh}(t-(t\wedge 1)) \end{split}$$

for  $0 \le t \le 1$ :  $I(t) = \operatorname{Sinh} t - t$ 

for  $1 \le t \le 2$ :  $I(t) = t - 2 + \operatorname{Sinh}(t) - 2\operatorname{Sinh}(t - 1)$ for  $2 \le t < \infty$ :  $I(t) = \operatorname{Sinh}(t) + \operatorname{Sinh}(t - 2) - 2\operatorname{Sinh}(t - 1)$ 

Alternatively, we can observe that  $g(t) = t - 2u_1(t) \cdot (t-1) + u_2(t)(t-2)$ , look up its Laplace transform in the table, multiply that with  $1/(s^2 - 1)$ , and compute the Laplace-inverse of that from the table.

|  | $\int \sinh t - t$   | for $0 \le t \le 1$ ;    |
|--|--|--------------------------|
| Answer: $y(t) = \operatorname{Cosh} t + \langle$ | $t - 2 + \operatorname{Sinh}(t) - 2\operatorname{Sinh}(t - 1)$ | for $1 \le t \le 2$ ;    |
|  | $\sinh(t) + \sinh(t-2) - 2\sinh(t-1)$                          | for $2 \le t < \infty$ . |

Look at the old quizzes. I might put one similar to them on the test.

Describe the Euler method, the improved Euler method, and the Runge–Kutta method, including estimates of the local and global errors in terms of the step size, and the number of computations required.

State Fourier's theorem. Describe the Gibbs phenomenon. Describe the method of separation of variables. What are even (odd) functions? How can you get a pure sine (cosine) series for a function  $f : [0, L] \rightarrow \mathbb{R}$  from Fourier's theorem? Do one of: p449 #1ab-12ab; p456 # 1-12; p461 # 1-12. Do one of: p610 # 1-6; p575 # 1–21; p585 # 1-24; p592 # 1ab-6ab, 7a-12a; p 600 # 1-26.

Let  $f: [-\pi, \pi] \to \mathbb{R}$  be the function defined by [drawing it might help]

$$f(x) = \begin{cases} -\pi - x & \text{for } -\pi \le x \le -\pi/2\\ x & \text{for } -\pi/2 \le x \le \pi/2\\ \pi - x & \text{for } \pi/2 \le x \le \pi \end{cases}$$

(a) Find the Fourier series  $\tilde{f}$  of f. (b) At which points x is  $f(x) = \tilde{f}(x)$ ? (Give reasons)

Solution: We first remark for later that  $\int x \cdot \sin(nx) dx = \frac{-x \cdot \cos(nx)}{n} + \frac{\sin(nx)}{n^2}$ .  $f \text{ is odd, so } a_n = 0 \text{ for } n = 0, 1, 2, 3 \dots$  Also,

$$\begin{split} b_n &= \frac{1}{\pi} \int_{-\pi} f(x) \cdot \sin(nx) \, dx = \frac{2}{\pi} \int_0^{\pi} f(x) \cdot \sin(nx) \, dx \\ &= \frac{2}{\pi} \int_0^{\pi/2} x \cdot \sin(nx) \, dx + \frac{2}{\pi} \int_{\pi/2}^{\pi} (\pi - x) \cdot \sin(nx) \, dx \\ &= \frac{2}{\pi} \cdot \left[ \frac{-x \cdot \cos(nx)}{n} + \frac{\sin(nx)}{n^2} \right]_0^{\pi/2} + \frac{2}{\pi} \cdot \left[ \frac{-\pi}{n} \cos(nx) + \frac{x \cdot \cos(nx)}{n} - \frac{\sin(nx)}{n^2} \right]_{\pi/2}^{\pi} \\ &= \frac{2}{\pi} \cdot \left[ \frac{-\pi/2 \cdot \cos(n\pi/2)}{n} + \frac{\sin(n\pi/2)}{n^2} \right] \\ &+ \frac{2}{\pi} \cdot \left[ \frac{-\pi}{n} \left( \cos(n\pi) - \cos(n\pi/2) \right) + \frac{\pi \cdot \cos(n\pi) - \pi/2 \cdot \cos(n\pi/2)}{n} - \frac{\sin(n\pi) - \sin(n\pi/2)}{n^2} \right] \\ &= \frac{4 \sin(n\pi/2)}{\pi n^2} \, . \end{split}$$

Note that  $b_n = 0$  when n is even, as should be the case. Therefore

Answer: (a)  $\widetilde{f}(x) = \sum_{n=1}^{\infty} \frac{4\sin(n\pi/2)}{\pi n^2} \sin(nx)$ and (b) at all points, because f is continuous.

Do one of p620 # 1-8, 9-13; p632 # 1-8; p645 # 1-5; p632 # 1-8 Solve the heat conduction problem  $u_t = 7u_{xx}$  in an insulated rod of length  $\pi$  whose ends are maintained at 0° Celsius at all times and whose initial temperature u(x, 0) is given by  $u(x, 0) = f(x) \quad \forall x \in [0, 2\pi]$ , where

$$f(x) \stackrel{\text{def}}{=} \begin{cases} x & \text{for } 0 \le x \le \pi/2, \\ \pi - x & \text{for } \pi/2 \le x \le \pi. \end{cases}$$

Solution: Separation of variables shows that u(x,t) can be found of the form

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-7n^2 t} \sin(nx) ,$$

provided the constants  $b_n$  are chosen so that

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$$
.

Fourier's theorem says that this representation holds, if we choose

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \, \sin(nx) \, dx$$

These integrals were computed in another problem:

$$b_n = \frac{4\sin(n\pi/2)}{\pi n^2}$$

Consequently

Answer: 
$$u(x,t) = \sum_{n=1}^{\infty} \frac{4\sin(n\pi/2)}{\pi n^2} \cdot e^{-7n^2t} \cdot \sin(nx)$$

Do a similar wave equation problem: let

$$f(x) \stackrel{\text{def}}{=} \begin{cases} x & \text{for } 0 \le x \le \pi/2, \\ \pi - x & \text{for } \pi/2 \le x \le \pi. \end{cases}$$

Solve the wave equation  $u_{tt} = 81u_{xx}$  for a string of length  $\pi$  with initial conditions u(0, x) = f(x)and  $u_t(0, x) = 0$ .

Solution:

$$u(t,x) = \frac{\check{f}(x+9t) + \check{f}(x-9t)}{2}$$

where  $\check{f}$  is the  $2\pi$ -periodic extension of the odd extension of f.

Do a similar Laplace equation problem: Let

$$f(x) \stackrel{\text{def}}{=} \begin{cases} x & \text{for } 0 \le x \le \pi/2, \\ \pi - x & \text{for } \pi/2 \le x \le \pi. \end{cases}$$

Then solve the Laplace equation on a square sheet of side  $\pi$  with the boundary conditions  $u(0, y) = u(x, 0) = u(\pi, y) = 0$  and  $u(x, \pi) = f(x)$ .

Solution: In lecture 23 we saw that the solution is

$$u(x,y) = \sum_{n=1}^{\infty} b_n \sin(nx) \operatorname{Sinh}(ny)$$
$$b_n = \frac{1}{\operatorname{Sinh}(n\pi)} \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx) \, dx$$
$$= \frac{4 \sin(n\pi/2)}{\pi n^2 \operatorname{Sinh}(n\pi)} \, .$$

by earlier problem

with

Solve the damped wave equation

 $\begin{array}{ll} u_{tt}+\gamma u_t=\alpha^2 u_{xx}\\ \text{with side conditions} & u(t,0)=u(t,L)=0 \ ,\\ \text{and initial conditions} & u(0,x)=f(x) \ , \ u_t(0,x)=g(x) \ . \end{array}$ 

Solution 1: The method of separating variables applies: With u(t, x) = T(t)X(x), the PDE turns into  $T''(t)X(x) + \gamma T'(t)X(x) = \alpha^2 T(t)X''(x)$   $\implies \qquad \frac{T''(t) + \gamma T'(t)}{\alpha^2 T^t} = \frac{X''(x)}{X(x)} = \sigma$ , X(0) = X(L) = 0.  $\implies \qquad X_n(x) = \sin(\lambda_n x) \qquad \sigma = -\lambda_n^2$ , where  $\lambda_n \stackrel{\text{def}}{=} \frac{n\pi}{L}$ . and  $T''(t) + \gamma T'(t) + \alpha^2 \lambda_n^2 T(t) = 0$ .

This constant coefficient HSOLODE for T(t) has characteristic polynomial  $r^2 + \gamma r + \alpha^2 \lambda_n^2$  with roots

$$r = \frac{-\gamma \pm \sqrt{\gamma^2 - 4\alpha^2 \lambda_n^2}}{2}$$

We pick a fundamental set  $\{y_n, \overline{y}_n\}$  such that  $y_n(0) = 1$ ,  $y'_n(0) = 0$  and  $\overline{y}_n(0) = 0$ ,  $\overline{y}'_n(0) = 1$ .

In the underdamped case that  $\alpha \lambda_1 > \gamma/2$ we have  $r_n = -\gamma/2 \pm i\mu_n$ , where  $\mu_n \stackrel{\text{def}}{=} \sqrt{\alpha^2 \lambda_n^2 - \gamma^2/4}$ ,

$$y_n(t) = e^{-\gamma t/2} \left( \cos(\mu_n t) - \frac{\gamma}{2\mu_n} \sin(\mu_n t) \right)$$
  
$$\overline{y}_n(t) = e^{-\gamma t/2} \frac{1}{\mu_n} \sin(\mu_n t) .$$

and

In general

$$u_n(t,x) = \left[a_n y_n(t) + b_n \overline{y}_n(t)\right] \times \sin(\lambda_n x)$$

solves the three green conditions and has its variables separated, and

$$u(t,x) = \sum_{n=1}^{\infty} u_n(t,x) = \sum_{n=1}^{\infty} \left[ a_n y_n(t) + b_n \overline{y}_n(t) \right] \times \sin(\lambda_n x)$$

still solves the three green conditions even though its variables are not separated. The initial conditions  $\sim$ 

$$u(0,x) = \sum_{n=1}^{\infty} a_n \sin(\lambda_n x) = f(x) \quad \text{and} \quad u_t(0,x) = \sum_{n=1}^{\infty} b_n \sin(\lambda_n x) = g(x)$$

are accommodated by

$$a_n = \frac{2}{L} \int_0^L f(x) \sin(\lambda_n x) dx$$
 and  $b_n = \frac{2}{L} \int_0^L g(x) \sin(\lambda_n x) dx$