Appendix B

Answers to Selected Problems

Answers to most of the problems can be found in Appendix C, which is available on the web via `http://www.ma.utexas.edu/users/cup/Answers`.

2.1.6 Define $T_{n+1} = \inf\{t > T_n : X_{t+} - X_t \neq 0\} \wedge t$, where t is an instant past which X vanishes. The T_n are stopping times (why?), and X is a linear combination of $X_0 \cdot [\![0]\!]$ and the intervals $(\!(T_n, T_{n+1}]\!]$.

2.3.6 For $N = 1$ this amounts to $\zeta_1 \leq L_q \zeta_1$, which is evidently true. Assuming that the inequality holds for $N-1$, estimate

$$\begin{aligned}
\zeta_N^2 &= \zeta_{N-1}^2 + (\zeta_N - \zeta_{N-1})(\zeta_N + \zeta_{N-1}) \\
&\leq L_q^2 \sum_{n=1}^{N} (\zeta_n - \zeta_{n-1})^2 + (\zeta_N - \zeta_{N-1})(\zeta_N + \zeta_{N-1}) \\
&\quad - L_q^2 (\zeta_N - \zeta_{N-1})(\zeta_N - \zeta_{N-1}) \\
&= L_q^2 \sum_{n=1}^{N} (\zeta_n - \zeta_{n-1})^2 + (\zeta_N - \zeta_{N-1})\big((1 - L_q^2)\zeta_N + (L_q^2 + 1)\zeta_{N-1}\big) .
\end{aligned}$$

Now with $L_q^2 = (q+1)/(q-1)$ we have $1 - L_q^2 = -2/(q-1)$ and $L_q^2 + 1 = 2q/(q-1)$, and therefore

$$(1 - L_q^2)\zeta_N + (L_q^2 + 1)\zeta_{N-1} = \frac{2}{q-1}\big(- \zeta_N + q\zeta_{N-1}\big) \leq 0 .$$

2.5.17 Replacing F_n by $F_n - p$, we may assume that $p = 0$. Then $S_n \stackrel{\text{def}}{=} \sum_{\nu=1}^{n} F_\nu$ has expectation zero. Let $\mathcal{F}_n$ denote the σ-algebra generated by $F_1, F_2, \ldots, F_{n-1}$. Clearly S_n is a right-continuous square integrable martingale on the filtration $\{\mathcal{F}_n\}$. More precisely, the fact that $\mathbb{E}[F_\nu | \mathcal{F}_\mu] = 0$ for $\mu < \nu$ implies that

$$\mathbb{E}\Big[S_n^2\Big] = \mathbb{E}\Big[\big(\textstyle\sum_{\nu=1}^{n} F_\nu\big)^2\Big] = \mathbb{E}\Big[\textstyle\sum_{\nu=1}^{n} F_\nu^2\Big] + 2\textstyle\sum_{1 \leq \mu < \nu \leq n} \mathbb{E}\big[F_\mu \mathbb{E}[F_\nu | \mathcal{F}_\mu]\big] \leq n\sigma^2 .$$

Therefore $Z_n \stackrel{\text{def}}{=} S_n / n$ has $\|Z_n\|_{L^2} \leq \sigma\sqrt{1/n} \xrightarrow[n\to\infty]{} 0$. (Using Chebyscheff's inequality here we may now deduce the weak law of large numbers.) The strong law says that $Z_n \xrightarrow[n\to\infty]{} 0$ almost surely and evidently follows if we can show that Z_n converges almost surely to something as $n \to \infty$. To see that it does we write

$$Z_\nu - Z_{\nu-1} = \frac{F_\nu}{\nu} - \frac{1}{\nu(\nu-1)} \sum_{1 \leq i < \nu} F_i , \qquad \nu = 2, 3, \ldots ,$$

and set $$\widetilde{Z}_n \overset{\text{def}}{=} \sum_{1\le\nu\le n} \frac{F_\nu}{\nu}\,, \qquad n = 1, 2, 3, \ldots\,,$$

and $\widehat{Z}_1 = 0$, $$\widehat{Z}_n \overset{\text{def}}{=} \sum_{1<\nu\le n} \frac{S_{\nu-1}}{\nu(\nu-1)}\,, \qquad n = 2, 3, \ldots\,.$$

Then $$Z_n = \widetilde{Z}_n - \widehat{Z}_n\,,$$

and it suffices to show that both $\widetilde{Z}_{\textbf{.}}$ and $\widehat{Z}_{\textbf{.}}$ converge almost surely. Now $\widetilde{Z}_{\textbf{.}}$ is an L^2-bounded martingale and so has a limit almost surely. Namely, because of the cancellation as above,

$$\mathbb{E}\Big[\widetilde{Z}_n^2\Big] \le \sum_{1\le\nu\le n} \frac{\mathbb{E}[F_\nu^2]}{\nu^2} < \sum_{\nu<\infty} \frac{\sigma^2}{\nu^2} < \infty\,.$$

As to $\widehat{Z}$, since $$\lceil\!\lceil \widehat{Z} \rceil\!\rceil_\infty \overset{\text{def}}{=} \sum_{2\le\nu<\infty} \frac{|S_{\nu-1}|}{\nu(\nu-1)}$$

has $$\mathbb{E}[\lceil\!\lceil \widehat{Z} \rceil\!\rceil_\infty] \le \sum_{2\le\nu<\infty} \frac{\|S_{\nu-1}\|_{L^2}}{\nu(\nu-1)} \le \sum_{2\le\nu<\infty} \frac{\sigma\sqrt{\nu-1}}{\nu(\nu-1)} < \infty\,,$$

the sum defining $\widehat{Z}_\infty \overset{\text{def}}{=} \lim \widehat{Z}_n$ almost surely converges, even *absolutely.*

3.8.10 (i) Both $(Y,Z) \to [Y,Z](\varpi)$ and $(Y,Z) \to {}^c[Y,Z](\varpi)$ are inner products with associated lengths $S[Z](\varpi), \sigma[Z](\varpi)$. The claims are Minkowski's inequalities and follow from a standard argument.

(ii) Take $U = V = [\![0,T]\!]$ in theorem 3.8.9 and apply Hölder's inequality.

3.8.18 (i) Choose numbers $\alpha_i > 0$ satisfying inequality (3.7.12):

$$\textstyle\sum_{i=1}^\infty \lceil\!\lceil i\alpha_i \cdot X^i \rceil\!\rceil_{\mathcal{I}^0} < \infty$$

(X^i is X stopped at i). With each α_i goes a $\delta_i > 0$ so that $|x - x'| \le \delta_i$ implies $|F(x) - F(x')| \le \alpha_i$. Set $T_0^i = 0$ and $T_{k+1}^i \overset{\text{def}}{=} \inf\{t > T_k^i : |Z_t - Z_{T_k^i}| \ge \alpha_i\}$. Then set

$${}^i Y_t^\eta \overset{\text{def}}{=} \textstyle\sum_{k,\nu} F_\nu^\eta(X_{T_k^i})(X^\nu_{T_{k+1}^i\wedge t} - X^\nu_{T_k^i\wedge t})\,, \qquad \eta = 1, \ldots, d\,.$$

According to theorem 3.7.26, nearly everywhere ${}^i Y_{\textbf{.}} \xrightarrow[i\to\infty]{} \big(F(X)*X\big)_{\textbf{.}}$ uniformly on bounded time-intervals. Let us toss out the nearly empty set where the limit does not exist and elsewhere select this limit for $F(X)*X$. Now let ω, ω' be such that $X_{\textbf{.}}(\omega)$ and $X_{\textbf{.}}(\omega')$ describe the same arc via $t \mapsto t'$. The times T_k^i may well differ at ω and ω', in fact $T_k^i(\omega') = \big(T_k^i(\omega)\big)'$, but the values $X_{T_k^i}(\omega)$ and $X_{T_k^i}(\omega')$ clearly agree. Therefore ${}^i Y_t(\omega) = {}^i Y_{t'}(\omega')$ for all $n \in \mathbb{N}$ and all $t \ge 0$. In the limit $\big(F(X)*X\big)_t(\omega) = \big(F(X)*X\big)_{t'}(\omega')$.

(ii) We start with the case $d = 1$. Apply (i) with $f(x) = 2x$ and toss out in addition the nearly empty set of $\omega \in \Omega$ where $[W,W]_t(\omega) \ne t$ for some t. Now if $W_{\textbf{.}}(\omega)$ and $W_{\textbf{.}}(\omega')$ describe the same arc via $t \mapsto t'$, then $[W,W]_{\textbf{.}}(\omega) = W_{\textbf{.}}^2(\omega) - 2(W_{\textbf{.}}*W)_{\textbf{.}}(\omega)$ and $[W,W]_{\textbf{.}}(\omega') = W_{\textbf{.}}^2(\omega') - 2(W_{\textbf{.}}*W)_{\textbf{.}}(\omega')$ describe the same arc also via $t \mapsto t'$. This reads $t = t'$.

In the case $d > 1$ apply the foregoing to the components W^η of $\boldsymbol{W}$ separately.

3.9.10 (i) Set $M_t \overset{\text{def}}{=} \langle\xi|X_{T+t} - X_T\rangle$ and assume without loss of generality that $M_0 \le a$. This continuous local martingale has a continuous strictly increasing square function $[M,M]_t = \xi_\mu\xi_\nu\big([X^\mu,X^\nu]_{T+t} - [X^\mu,X^\nu]_T\big) \xrightarrow[t\to\infty]{} \infty$. The time transformation $T^\lambda = T^{\lambda+} \stackrel{\text{def}}{=} \inf\{t : [M,M]_t \ge \lambda\}$ of exercise 3.9.8 turns M into a Wiener process. The zero-one law of exercise 1.3.47 on page 41 shows that $T = T^\pm$ almost surely.

(ii) In the previous argument replace T by S. The continuous time transformation $T^\lambda = T^{\lambda+} \stackrel{\text{def}}{=} \inf\{t : [M,M]_t \ge \lambda\}$ turns $M_t \stackrel{\text{def}}{=} \langle \xi | X_{S+t} - X_S \rangle$ into a standard Wiener process, which exceeds any number, including $a \pm \langle \xi | X_S \rangle$, in finite time (exercise 1.3.48 on page 41). Therefore the stopping times T and $T^\pm$ are almost surely finite. Consider the set $\mathcal{X}^-$ of paths $[S(\omega), \infty) \ni t \mapsto X_t(\omega)$ that have $T(\omega) > \tau$. Each of them stays on the same side of H as $X_S(\omega)$ for $t \in [S(\omega), \tau]$, in fact by continuity it stays a strictly positive distance away from H during this interval. Any other path $X_.(\omega')$ sufficiently close to a path in $\mathcal{X}^-$ will not enter H during $[S(\omega'), \tau]$ either and thus will have $T(\omega') > \tau$: the set $\mathcal{X}^-$ is open.

Next consider the set $\mathcal{X}^+$ of paths $[S(\omega), \infty) \ni t \mapsto X_t(\omega)$ that have $T^+(\omega) < \tau$. If $X_.(\omega) \in \mathcal{X}^+$, then there exists a $\sigma < \tau$ with $\langle \xi | X_\sigma \rangle > a$: $X_S(\omega)$ and $X_\sigma(\omega)$ lie on different sides of H. Clearly if ω' is such that the path $X_.(\omega')$ is sufficiently close to that of ω, then $X_\sigma(\omega')$ will also lie on the other side of H: $\mathcal{X}^+$ is open as well. After removal of the nearly empty set $[T \ne T^+]$ we are in the situation that the sets $\{X_.(\omega) : T(\omega) \gtrless \tau\}$ are open for all τ: T depends continuously on the path.

3.10.6 (i) Let $K^1, K^2 \subset \boldsymbol{H}$ be disjoint compact sets. There are $\phi^i_n \in C_{00}[\boldsymbol{H}]$ with $\phi^i_n \downarrow_n K^i$ pointwise and such that ϕ^1_1 and ϕ^2_1 have disjoint support. Then $\phi^i_n * \beta \to K^i * \beta$ as L^2-integrators and therefore uniformly on bounded time-intervals (theorem 2.3.6) and in law. Also, clearly $K^1 * \beta$ and $K^2 * \beta$ are independent, inasmuch as $\phi^1_m * \beta$ and $\phi^2_n * \beta$ are. In the next step exhaust disjoint relatively compact Borel subsets B^1, B^2 of $\boldsymbol{H}$ by compact subsets with respect to the Daniell means $B \mapsto \| B \times [0,n] \|^*_{\beta\text{–}2}$, $n \in \mathbb{N}$, which are $\mathcal{K}$-capacities (proposition 3.6.5), to see that $B^1 * \beta$ and $B^2 * \beta$ are independent Wiener processes. Clearly $\nu(B) \stackrel{\text{def}}{=} \mathbb{E}[(B*\beta)^2_1]$ defines an additive measure on the Borels of $\boldsymbol{H}$. It is σ-continuous, beacause it is majorized by $B \mapsto [\| B \times [0,1] \|^*_{\beta\text{–}2}]^2$, even inner regular.

3.10.9 Let I denote the image in $L^2 \stackrel{\text{def}}{=} L^2(\mathcal{F}^0_\infty[\beta], \mathbb{P})$ of $\mathfrak{L}^1[\beta\text{–}2]$ under the map $\check{X} \mapsto \int_0^\infty \check{X}\, d\beta$, and $\overline{I}$ the algebraic sum $I \oplus \mathbb{R}$. By theorem 3.10.6 (ii), I and then $\overline{I}$ are closed in L^2, and their complexifications $I_\mathbb{C}, \overline{I}_\mathbb{C}$ are closed in $L^2_\mathbb{C}$. By the Dominated Convergence Theorem for the L^2-mean both vector spaces are closed under pointwise limits of bounded sequences.

For $h \in \mathcal{E}[\check{\boldsymbol{H}}] \stackrel{\text{def}}{=} \mathcal{E}[\boldsymbol{H}] \otimes C_{00}(\mathbb{R}_+)$ set $M^h \stackrel{\text{def}}{=} h * \beta$, which is a martingale with square bracket $[M^h, M^h]_t = \int_0^t h^2(\eta, s)\ \nu(d\eta) ds$. Then let $G^h = \exp\big(iM^h + [M^h, M^h]/2\big)$ be the Doléans–Dade exponential of iM^h. Clearly $G^h_\infty = 1 + \int_0^\infty iG^h\, dM^h$ belongs to $\overline{I}_\mathbb{C}$, and so does its scalar multiple $\exp\big(iM^h_\infty\big)$. The latter form a multiplicative class $\mathcal{M}$ contained in $\overline{I}_\mathbb{C}$ and generating $\mathcal{F}^0_\infty[\beta]$ (page 409). By exercise A.3.5 the vector space $\overline{I}_\mathbb{C}$ contains all bounded $\mathcal{F}^0_\infty[\beta]$-measurable functions. As it is mean closed, it contains all of $L^2_\mathbb{C}$. Thus $\overline{I} = L^2$.

4.3.10 Use exercise 3.7.19, proposition 3.7.33, and exercise 3.7.9.

4.5.8 Inequality (4.5.9) and the homogeneity argument following it show that for any bounded previsible $X \ge 0$

$$\mu^{\langle\sigma\rangle}(X) \le \big(\mu^{\langle\rho\rangle}(X)\big)^{\frac{\tau-\sigma}{\tau-\rho}} \cdot \big(\mu^{\langle\tau\rangle}(X)\big)^{\frac{\sigma-\rho}{\tau-\rho}} \le \big(\mu^{\langle\rho\rangle} \vee \mu^{\langle\tau\rangle}\big)(X)\ .$$

4.5.9 Since $z \mapsto (z+\zeta)^p - z^p$ increases, inequality (4.5.16) can with the help of theorem 2.3.6 be continued as

$$\| Z_\infty \|^p_{L^p} \le \big(C^\star_p \,\lceil\!\lceil Z \rceil\!\rceil_{\mathcal{I}^p} + \zeta[Z]\big)^p - C^{\star p}_p \,\lceil\!\lceil Z \rceil\!\rceil^p_{\mathcal{I}^p}\ .$$

Replace Z by $X{*}Z$ and take the supremum over X with $|X| \le 1$ to obtain

$$\lceil\!\lceil Z \rceil\!\rceil_{\mathcal{I}^p}^p \le \big(C_p^\star \lceil\!\lceil Z \rceil\!\rceil_{\mathcal{I}^p} + \zeta[Z]\big)^p - C_p^{\star p} \lceil\!\lceil Z \rceil\!\rceil_{\mathcal{I}^p}^p \ ,$$

or $$(1 + C_p^{\star p})^{1/p} \lceil\!\lceil Z \rceil\!\rceil_{\mathcal{I}^p} \le C_p^\star \lceil\!\lceil Z \rceil\!\rceil_{\mathcal{I}^p} + \zeta[Z]$$

and $$\lceil\!\lceil Z \rceil\!\rceil_{\mathcal{I}^p} \le c_p^\diamond \zeta[Z]$$

with $$c_p^\diamond \le \Big((1 + C_p^{\star p})^{1/p} - C_p^*\Big)^{-1} \le 0.6 \cdot 4^p \ .$$

4.5.21 Let $T = \inf\{t : \ A_t \ge a\}$ and $S = \inf\{t : Y_t \ge y\}$. Both are stopping times, and T is predictable (theorem 3.5.13); there is an increasing sequence T_n of finite stopping times announcing T. On the set $[Y_\infty > y, A_\infty < a]$, S is finite, $Y_S \ge y$, and the T_n increase without bound. Therefore

$$[Y_\infty > y, A_\infty < a] \le \frac{1}{y} \cdot \sup_n Y_{S \wedge T_n}$$

and so $$\mathbb{P}[Y_\infty > y, A_\infty < a] \le \frac{1}{y} \cdot \sup_n \mathbb{E}[Y_{S \wedge T_n}]$$

$$\le \frac{1}{y} \cdot \sup_n \mathbb{E}[A_{S \wedge T_n}] \le \frac{1}{y} \cdot \mathbb{E}[A_\infty \wedge a] \ .$$

Applying this to sequences $y_n \downarrow y$ and $a_n \uparrow a$ yields inequality (4.5.30). This then implies $\mathbb{P}[Y = \infty, A \le a] = 0$ for all $a < \infty$; then $\mathbb{P}[Y = \infty, A < \infty] = 0$, which is (4.5.31).

4.5.24 Use the characterizations 4.5.12, 4.5.13, and 4.5.14. Consider, for instance, the case of $\boldsymbol{Z}^{\langle q \rangle}$. Let ${}^{q\prime}\!\boldsymbol{X}, {}^{q\prime}\!\check{H}$ be the quantities of exercise 4.5.14 and its answer for $'\boldsymbol{Z}$. Then $\check{H}(\boldsymbol{y}, s) \overset{\text{def}}{=} {}^{q\prime}\!\check{H} \circ C(\boldsymbol{y}, s) = {}^{q\prime}\!\check{H}(C\boldsymbol{y}, s) = \langle {}^{q\prime}\!\boldsymbol{X}_s | C\boldsymbol{y} \rangle = \langle C^T\, {}^{q\prime}\!\boldsymbol{X}_s | \boldsymbol{y} \rangle$, where $C^T : \ell^\infty('d) \to \ell^\infty(d)$ denotes the (again contractive) transpose of C. By exercise 4.5.14, the Doléans–Dade measure $'\mu$ of $|\check{H}|^q {*} \jmath_{\boldsymbol{Z}}$ is majorized by that of $\Lambda^{\langle q \rangle}[\boldsymbol{Z}]$. But $'\mu$ is the Doléans–Dade measure of $\Lambda^{\langle q \rangle}['\boldsymbol{Z}]$! Indeed, the compensator of $|\check{H}|^q {*} \jmath_{\boldsymbol{Z}} = |{}^{q\prime}\!\check{H} \circ C|^q {*} \jmath_{\boldsymbol{Z}} = |{}^{q\prime}\!\check{H}|^q {*} C[\jmath_{\boldsymbol{Z}}] = |{}^{q\prime}\!\check{H}|^q {*} \jmath_{'\boldsymbol{Z}}$ is $\Lambda^{\langle q \rangle}['\boldsymbol{Z}]$. The other cases are similar but easier.

5.2.2 Let $S < T^\mu$ on $[T^\mu > 0]$. From inequality (4.5.1)

$$\big\| |\boldsymbol{\Delta}{*}\boldsymbol{Z}|_S^\star \big\|_{L^p} \le C_p^\diamond \cdot \max_{\rho = 1, p^\diamond} \Big\| \Big(\int_0^S |\boldsymbol{\Delta}|^\rho \, d\Lambda \Big)^{1/\rho} \Big\|_{L^p}$$

$$\le C_p^\diamond \cdot \max_{\rho = 1, p^\diamond} \Big\| \Big(\int_0^\mu \delta^\rho \, d\lambda \Big)^{1/\rho} \Big\|_{L^p} = \delta \cdot C_p^\diamond \max_{\rho = 1, p^\diamond} \mu^{1/\rho} \ .$$

Letting S run through a sequence announcing T^μ, multiplying the resulting inequality $\big\| |\boldsymbol{\Delta}{*}\boldsymbol{Z}|_{T^\mu -}^\star \big\|_{L^p} \le \delta \cdot C_p^\diamond \max_{\rho = 1, p^\diamond} \mu^{1/\rho}$ by $e^{-M\mu}$, and taking the supremum over $\mu > 0$ produces the claim after a little calculus.

5.2.18 (i) Since $e^{pmW_t - p^2 m^2 t/2} = \mathscr{E}_t[pmW]$ is a martingale of expectation one

we have $$\big|\mathscr{E}_t[mW]\big|^p = e^{pmW_t - pm^2 t/2} = \mathscr{E}_t[pmW] \cdot e^{(p^2 - p)m^2 t/2} \ ,$$

$$\mathbb{E}\big[\big|\mathscr{E}_t[mW]\big|^p\big] = e^{(p^2 - p)m^2 t/2} \ , \quad \text{and} \quad \big\|\mathscr{E}_t[mW]\big\|_{L^p} = e^{m^2(p-1)t/2} \ .$$

Next, from $e^{|x|} \le e^x + e^{-x}$ we get

$$e^{|mW_t|} \le e^{mW_t} + e^{-mW_t} = e^{m^2 t/2} \times \big(\mathscr{E}_t[mW] + \mathscr{E}_t[-mW]\big) \ ,$$

$$e^{|mW|_t^\star} \le e^{m^2t/2} \times \left(\mathscr{E}_t^\star[mW] + \mathscr{E}_t^\star[-mW]\right) ,$$

and $$\left\| e^{|mW|_t^\star} \right\|_{L^p} \le e^{m^2t/2} \times \left(\left\|\mathscr{E}_t^\star[mW]\right\|_{L^p} + \left\|\mathscr{E}_t^\star[-mW]\right\|_{L^p}\right)$$

by theorem 2.5.19: $$\le e^{m^2t/2} \times 2p' \cdot e^{m^2(p-1)t/2} = 2p' \cdot e^{m^2pt/2} .$$

(ii) We do this with $|\ |$ denoting the ℓ^1-norm on $\mathbb{R}^d$. First,

$$\left\| e^{|m\boldsymbol{Z}^\star|_t} \right\|_{L^p} = e^{|m|t} \times \prod_\eta \left\| e^{m|W^{\eta\star}|_t} \right\|_{L^p}$$

by independence of the W^η: $$\le e^{|m|t} \times \left(2p' \cdot e^{m^2pt/2}\right)^{d-1}$$

$$= (2p')^{d-1} \times e^{(|m|+(d-1)m^2p/2)\cdot t} .$$

Thus $$\left\| e^{|m\boldsymbol{Z}^\star|_t} \right\|_{L^p} \le A_{p,d} \times e^{M_{d,m,p}\cdot t} . \tag{1}$$

Next, $$\left\| |\boldsymbol{Z}^\star|_t^r \right\|_{L^p} = \left\| |\boldsymbol{Z}^\star|_t \right\|_{L^{rp}}^r \le \left(t + \sum_\eta \left\| |W^{\eta\star}|_t \right\|_{L^{rp}} \right)^r$$

$$= \left(t + (d{-}1) \cdot \left\| |W|_t^\star \right\|_{L^{rp}} \right)^r$$

by theorem 2.5.19: $$\le \left(t + (d{-}1)(rp)' \cdot \left\| |W|_t \right\|_{L^{rp}} \right)^r$$

$$\le 2^{r'} \left(t^r + (d{-}1)^r (rp)'^r \cdot \left\| |W|_t \times \right\|_{L^{rp}}^r \right)$$

by exercise A.3.47 with $\sigma = \sqrt{t}$: $$= 2^{r'} \left(t^r + (d{-}1)^r (rp)'^r \Gamma_{p,r} \cdot t^{r/2} \right) .$$

Thus $$\left\| |\boldsymbol{Z}^\star|_t^r \right\|_{L^p} \le B_r t^r + B_{d,r,p} t^{r/2} . \tag{2}$$

Applying Hölder's inequality to (1) and (2), we get

$$\left\| |\boldsymbol{Z}^\star|_t^r \cdot e^{|m\boldsymbol{Z}^\star|_t} \right\|_{L^p} \le \left(B_r t^r + B_{d,r,2p} t^{r/2} \right) \times \left(A_{2p,d} e^{M_{d,m,p}t} \right)$$

$$= t^{r/2} A_{2p,d} \left(B_{d,r,2p} + B_r t^{r/2} \right) \times e^{M_{d,m,2p}t} \ :$$

we get, for suitable $B' = B'_{d,p,r}$, $M' = M'_{d,m,p,r}$, the desired inequality

$$\left\| |\boldsymbol{Z}^\star|_t^r \cdot e^{|m\boldsymbol{Z}^\star|_t} \right\|_{L^p} \le B' \cdot t^{r/2} \, e^{M't} .$$

5.3.1 $\mathfrak{S}_{p,M}^{\star n}$ is naturally equipped with the collection $\mathfrak{N}^\circ$ of seminorms $\| \ \|_{p^\circ,M^\circ}$, where $2 \le p^\circ < p$ and $M^\circ > M$. $\mathfrak{N}^\circ$ forms an increasing family with pointwise limit $\| \ \|_{p,M}$. For $0 \le \sigma \le 1$ set $u^\sigma \overset{\text{def}}{=} u + \sigma(v{-}u)$ and $X^\sigma \overset{\text{def}}{=} X + \sigma(Y{-}X)$. Write F for F_η, etc. Then the remainder $F[v,Y] - F[u,X] - D_1F[u,X]\cdot(v{-}u) - D_2F[u,X]\cdot(Y{-}X)$ becomes, as in example A.2.48,

$$RF[u,X;v,Y] = \int_0^1 \left(Df(u^\sigma, X^\sigma) - Df(u,X) \right) \cdot \begin{pmatrix} v{-}u \\ Y{-}X \end{pmatrix} \, d\sigma .$$

With $R^\sigma f \stackrel{\text{def}}{=} Df(u^\sigma, X^\sigma) - Df(u, X)$, $1/p^\circ = 1/p + 1/r$, and $\|R\|_{pp^\circ}$ denoting the operator norm of a linear operator $R : \ell^p(k{+}n) \to \ell^{p^\circ}(n)$ we get

$$|RF[u, X; v, Y]^\star_{T^\lambda-}|_{p^\circ} \le C \cdot r_\lambda \cdot \left(|v{-}u| + ||Y{-}X|^\star_{T^\lambda-}|_p\right) ,$$

where
$$r_\lambda \stackrel{\text{def}}{=} \sup_{t<T^\lambda} \sup_{0\le\sigma\le 1} \|R^\sigma f\|_{pp^\circ} ,$$

and where C is a suitable constant depending only on k, n, p, p°. Now r_λ is a bounded random variable and converges to zero in probability as $|v - u| + \|Y{-}X\|^\star_{p,M} \to 0$. (Use that the T^λ of definition (5.2.4) on page 283 are bounded; then X^σ_t ranges over a relatively compact set as $[0 \le \sigma \le 1]$ and $0 \le t \le T^\lambda$.) In other words, the uniformly bounded increasing functions $\lambda \mapsto \|r_\lambda\|_r$ converge pointwise – and thus uniformly on compacta – to zero as $|v - u| + \|Y{-}X\|^\star_{p,M} \to 0$. Therefore the first factor on the right in

$$\|RF[u, X; v, Y]\|_{p^\circ,M^\circ} \le C \sup_\lambda e^{(M-M^\circ)\lambda} \|r_\lambda\|_r \cdot \left(|v{-}u| + \|Y{-}X\|^\star_{p,M}\right)$$

converges to zero as $|v{-}u| + \|Y{-}X\|^\star_{p,M} \to 0$, which is to say $\|RF[u, X; v, Y]\|_{p^\circ,M^\circ} = o\big(|v - u| + \|Y{-}X\|^\star_{p,M}\big)$.

5.4.19 (ii) For fixed μ and δ set $k \stackrel{\text{def}}{=} \lceil \mu/\delta \rceil$, and $\lambda_i \stackrel{\text{def}}{=} i\delta$ and $T_i \stackrel{\text{def}}{=} T^{\lambda_i}$ for $i = 0, 1, \ldots, k$. Then $\lambda_{k-1} < \mu \le \lambda_k$. Let $\Delta^\star_i$ denote the maximal function of the difference of the global solution at T_i, which is $X_{T_i} = \Xi[C, \boldsymbol{Z}]_{T_i}$, from its Ξ'-approximate X'_{T_i}. Consider an $s \in [T^{\lambda_i}, T^{\lambda_{i+1}}]$.

Since
$$\begin{aligned} X'_s - X_s &= \Xi'[X'_{T_i}, \boldsymbol{Z}_s{-}\boldsymbol{Z}_{T_i}] - \Xi'[X_{T_i}, \boldsymbol{Z}_s{-}\boldsymbol{Z}_{T_i}] \\ &\quad + \Xi'[X_{T_i}, \boldsymbol{Z}_s{-}\boldsymbol{Z}_{T_i}] - \Xi[X_{T_i}, \boldsymbol{Z}_s{-}\boldsymbol{Z}_{T_i}] , \end{aligned}$$

5.4.17 gives
$$\|\Delta^\star_{i+1}\|_{L^p} \le \|\Delta^\star_i\|_{L^p} \times e^{L'\delta} + \left(\||X|^\star_{T_i}{+}1\|_{L^p}\right) \times (\underline{M}\delta)^r e^{\underline{M}\delta} ,$$

which implies
$$|\Delta^\star_k| \le (|X|^\star_{T_k}{+}1) \times (\underline{M}\delta)^r e^{\underline{M}\delta} \cdot \textstyle\sum_{0\le i<k} e^{iL'\delta}$$

for $X_\bullet \in \mathfrak{S}_{p,M}$, as $\lambda_k = k\delta$:
$$\le (\|X_\bullet\|^\star_M e^{M\lambda_k} + 1) \times (\underline{M}\delta)^r e^{\underline{M}\delta} \cdot \frac{e^{L'k\delta} - 1}{e^{L'\delta} - 1}$$

by (5.2.23), as $\delta \to 0$:
$$\le \frac{2}{1{-}\gamma} \left(\|{}^0C\|^\star_M + 1\right) e^{M\lambda_k} \times (\underline{M}\delta)^r e^{\underline{M}\delta} \cdot k\, e^{L'\lambda_k}$$

since $k = \lambda_k/\delta$:
$$\le \mathit{const}\, (\|C\|^\star_M + 1)\underline{M}^r e^{\underline{M}\delta} \delta^{r-1} \times \lambda_k{\cdot}e^{(M+L')\lambda_k}$$

$$\le \overline{B}{\cdot}(\|C\|_{L^p}{+}1) \times \delta^{r-1}{\cdot}e^{\overline{M}\lambda_k}$$

for suitable $\overline{B} = \overline{B}[f; \Xi']$ and $\overline{M} = \overline{M}[f; \Xi'] > M{+}L'$.

5.5.7 Let $\mathbb{P}, \overline{\mathbb{P}} \in \mathfrak{P}$. Thanks to the uniform ellipticity (5.5.17) there exist bounded functions h^η so that $f_0 = -\sum_\eta f_\eta \cdot h^\eta$. Then $M \stackrel{\text{def}}{=} \boldsymbol{h}{*}\boldsymbol{W}$ is a martingale under both $\mathbb{P}$ and $\overline{\mathbb{P}}$, and by exercise 3.9.12 so is the Doléans–Dade exponential G' of M. The Girsanov theorem 3.9.19 asserts that $\boldsymbol{W}' \stackrel{\text{def}}{=} \boldsymbol{W}_t + \int_0^\bullet \boldsymbol{h}_s\, ds$ is a Wiener process under the probabilities $\mathbb{P}'$ and $\overline{\mathbb{P}}'$ that on every $\mathcal{F}_t$ agree with $G'_t\mathbb{P}$ and $G'_t\overline{\mathbb{P}}$, respectively. Now X satisfies equation (5.5.20) with $\boldsymbol{W}'$ replacing $\boldsymbol{W}$, and therefore by assumption $\mathbb{P}' = \overline{\mathbb{P}}'$. This clearly implies $\mathbb{P} = \overline{\mathbb{P}}$.

5.5.13 Let $s \le t \le t'$. Then by Itô's formula

$$\begin{aligned} u(t'-t, X_t) &= u(t'-s, X_s) - \int_s^t \dot{u}(t'-\sigma, X_\sigma)\, d\sigma \\ &\quad + \int_s^t u_{;\nu}(t'-\sigma, X_\sigma)\, dX_\sigma^{x\nu} + \int_s^t \mathcal{A}u(t'-\sigma, X_\sigma)\, d\sigma \\ &= u(t'-s, X_s) + \int_s^t u_{;\nu}(t'-\sigma, X_\sigma)\, dX_\sigma^{x\nu}\ . \end{aligned}$$

Taking the conditional expectation under $\mathcal{F}_s$ exhibits $t \mapsto u(t'-t, X_t)$ as a bounded local martingale.

5.5.14 It suffices to consider equation (5.5.20). Recall that $\mathfrak{P}$ is the collection of all probabilities on $\mathscr{C}^n$ under which the process $\boldsymbol{W}_t$ of (5.5.19) is a standard Wiener process. Let $\mathbb{P}, \overline{\mathbb{P}} \in \mathfrak{P}$. From $\phi_0, \phi_1, \ldots, \phi_k \in C_b^\infty(\mathbb{R}^n)$ and $0 = t_0 < t_1 < \cdots < t_k$ make the function $\Phi \stackrel{\text{def}}{=} \phi_0(X_0^x) \cdot \phi_1(X_{t_1}^x) \cdots \phi_k(X_{t_k}^x)$ on the path space $\mathscr{C}^n$. Their collection forms a multiplicative class that separates the points of $\mathscr{C}^n$. Since path space is polish and consequently every probability on it is tight (proposition A.6.2), or simply because the functions Φ generate the Borel σ-algebra on path space, $\mathbb{P} = \overline{\mathbb{P}}$ will follow if we can show that $\mathbb{E} = \overline{\mathbb{E}}$ on the functions Φ (proposition A.3.12). This we do by induction in k. The case $k = 0$ is trivial. Note that the equality $\mathbb{E}[\Phi] = \overline{\mathbb{E}}[\Phi]$ on Φ made from smooth functions ϕ_i persists on functions Φ made from continuous, even bounded Baire, functions ϕ_i, by the usual sequential closure argument. To propel ourselves from k to $k+1$ let u denote a solution to the initial value problem $\dot{u} = \mathcal{A}u$ with $u(0, x) = \phi_{k+1}(x)$ and write $\Phi \stackrel{\text{def}}{=} \phi_0 \circ X_0^x \cdot \phi_1 \circ X_{t_1}^x \cdots \phi_k \circ X_{t_k}^x$. Then, with $t = t' = t_{k+1}$ and $s = t_k$, exercise 5.5.13 produces

$$\begin{aligned} \mathbb{E}\Big[\phi_{k+1}(X_{t_{k+1}}^x)|\mathcal{F}_{t_k}\Big] &= \mathbb{E}\Big[u(0, X_{t_{k+1}}^x)|\mathcal{F}_{t_k}\Big] \\ &= \mathbb{E}\big[u\big(t_{k+1}-t_k, X_{t_k}^x\big)\big] \\ \text{and so} \qquad \mathbb{E}\big[\Phi \cdot \phi_{k+1}(X_{t_{k+1}}^x)\big] &= \mathbb{E}\big[\Phi \cdot u\big(t_{k+1}-t_k, X_{t_k}^x\big)\big] \\ &= \overline{\mathbb{E}}\big[\Phi \cdot u\big(t_{k+1}-t_k, X_{t_k}^x\big)\big] \qquad (*) \\ \text{by the same token:} \qquad &= \overline{\mathbb{E}}\big[\Phi \cdot \phi_{k+1}(X_{t_{k+1}}^x)\big]\ . \end{aligned}$$

At $(*)$ we used the fact that the argument of the expectation is a k-fold product of the same form as Φ, so that the induction hypothesis kicks in.

A.2.23 Let U_n be the set of points x in F that have a neighborhood $V_n(x)$ whose intersection with E has ρ-diameter strictly less than $1/n$. The U_n clearly are open and contain E. Their intersection is E. Indeed, if $x \in \bigcap U_n$, then the sets $E \cap V_n(x)$ form a Cauchy filter basis in E whose limit must be x.

A.3.5 The family of all complex finite linear combinations of functions in $\mathcal{M} \cup \{1\}$ is a complex algebra $\mathcal{A}$ of bounded functions in $\mathcal{V}$ that is closed under complex conjugation; the σ-algebra it generates is again $\mathcal{M}^\Sigma$. The real-valued functions in $\mathcal{A}$ form a real algebra $\mathcal{A}_0$ of bounded functions that again generates $\mathcal{M}^\Sigma$. It is a multiplicative class contained in the bounded monotone class $\mathcal{V}_0$ of real-valued functions in $\mathcal{V}$. Now apply theorem A.3.4 suitably.

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