4

Control of Integral and Integrator

4.1 Change of Measure — Factorization

Let Z be a global $L^p(\mathbb{P})$-integrator and $0 \le p < q < \infty$. There is a probability $\mathbb{P}'$ equivalent with $\mathbb{P}$ such that Z is a global $L^q(\mathbb{P}')$-integrator; moreover, there is sufficient control over the change of measure from $\mathbb{P}$ to $\mathbb{P}'$ to turn estimates with respect to $\mathbb{P}'$ into estimates with respect to the original and presumably intrinsically relevant probability $\mathbb{P}$. In fact, all of this remains true for a whole vector $\boldsymbol{Z}$ of L^p-integrators. This is of great practical interest, since it is so much easier to compute and estimate in Hilbert space $L^2(\mathbb{P}')$, say, than in $L^p(\mathbb{P})$, which is not even locally convex when $0 \le p < 1$.

When $q \le 2$ or when $\boldsymbol{Z}$ is previsible, the universal constants that govern the change of measure are independent of the length of $\boldsymbol{Z}$, and that fact permits an easy extension of all of this to *random measures* (see corollary 4.1.14). These facts are the goal of the present section.

A Simple Case

Here is a result that goes some way in this direction and is rather easily established (pages 188–190). It is due to Dellacherie [18] and the author [6], and, in conjunction with the Doob–Meyer decomposition of section 4.3 and the Girsanov–Meyer lemma 3.9.11, it suffices to show that an L^0-integrator is in fact a semimartingale (proposition 4.4.1).

Proposition 4.1.1 *Let Z be a global L^0-integrator on $(\Omega, \mathcal{F}_., \mathbb{P})$. There exists a probability $\mathbb{P}'$ equivalent with $\mathbb{P}$ on $\mathcal{F}_\infty$ such that Z is a global $L^1(\mathbb{P}')$-integrator. Furthermore, $g \stackrel{\text{def}}{=} d\mathbb{P}/d\mathbb{P}'$ is bounded away from zero, and there exist universal constants $D[\lceil\!\lceil \boldsymbol{Z} \rceil\!\rceil_{[.]}]$ and $E_{[\alpha]} = E_{[\alpha]}[\lceil\!\lceil \boldsymbol{Z} \rceil\!\rceil_{[.]}]$ depending only on $\alpha \in (0,1)$ and the modulus of continuity $\lceil\!\lceil \boldsymbol{Z} \rceil\!\rceil_{[.]}$, so that*

$$\lceil\!\lceil Z \rceil\!\rceil_{\mathcal{I}^1[\mathbb{P}']} \le D[\lceil\!\lceil \boldsymbol{Z} \rceil\!\rceil_{[.]}] \quad \text{and} \quad \|g\|_{[\alpha;\mathbb{P}]} \le E_{[\alpha]}[\lceil\!\lceil \boldsymbol{Z} \rceil\!\rceil_{[.]}] \,, \tag{4.1.1}$$

which implies $$\|f\|_{[\alpha;\mathbb{P}]} \le \left(\frac{2E_{[\alpha/2]}}{\alpha}\right)^{1/r} \cdot \|f\|_{L^r(\mathbb{P}')} \tag{4.1.2}$$

for any $\alpha \in (0,1)$, $r \in (0,\infty)$, and $f \in \mathcal{F}_\infty$.

A remark about the utility of inequality (4.1.2) is in order. To fix ideas assume that f is a function computed from Z, for instance, the value at some time T of the solution of a stochastic differential equation driven by Z. First, it is rather easier to establish the existence and possibly uniqueness of the solution computing in the Banach space $L^1(\mathbb{P}')$ than in $L^0(\mathbb{P})$ – but generally still not as easy as in Hilbert space $L^2(\mathbb{P}')$. Second, it is generally very much easier to estimate the size of f in $L^r(\mathbb{P}')$ for $r > 1$, where Hölder and Minkowski inequalities are available, than in the non-locally convex space $L^0(\mathbb{P})$. Yet it is the original measure $\mathbb{P}$, which presumably models a physical or economical system and reflects the "true" probability of events, with respect to which one wants to obtain a relevant estimate of the size of f. Inequality (4.1.2) does that.

Apart from elevating the exponent from 0 to merely 1, there is another shortcoming of proposition 4.1.1. While it is quite easy to extend it to cover several integrators simultaneously, the constants of inequality (4.1.1) and (4.1.2) will increase linearly with their number. This prevents an application to a random measure, which can be viewed as an infinity of infinitesimal integrators (page 173). The most general theorem, which overcomes these problems and is in some sense best possible, is theorem 4.1.2 below.

Proof of Proposition 4.1.1. This result follows from part (ii) of theorem 4.1.2, whose detailed proof takes 20 pages. The reader not daunted by the prospect of wading through them might still wish to read the following short proof of proposition 4.1.1, since it shares the strategy and major elements with the proof of theorem 4.1.2 and yields in its less general setup better constants.

The first step is the following claim: *For every α in $(0,1)$ there exist a measurable function $k_\alpha : \Omega \to [0,1]$ and a constant ζ_α*

with $$0 \le k_\alpha \le 1\ ,\ \mathbb{E}\big[k_\alpha\big] \ge 1-\alpha\ ,$$

and $$\mathbb{E}\Big[\Big|k_\alpha \cdot \int X\,dZ\Big|\Big] \le \zeta_\alpha \tag{4.1.3}$$

for all X in the unit ball $\mathcal{E}_1 \stackrel{\text{def}}{=} \{X \in \mathcal{E} : \|X\|_{\mathcal{E}} \le 1\}$.
To see this fix an α in $(0,1)$ and set $T \stackrel{\text{def}}{=} \inf\{t : |Z_t| > \lceil Z \rceil_{[\alpha/2]}\}$.

Now $$\mathbb{P}\Big[\Big|\int [\![0,T]\!]\,dZ\Big| \ge \lceil Z \rceil_{[\alpha/2]}\Big] \le \alpha/2$$

means that $$\mathbb{P}\Big[\big|Z_T\big| \ge \lceil Z \rceil_{[\alpha/2]}\Big] \le \alpha/2$$

and produces $$\mathbb{P}[T<\infty] \le \mathbb{P}\Big[\big|Z_T\big| \ge \lceil Z \rceil_{[\alpha/2]}\Big] \le \alpha/2\ .$$

The complement $G \stackrel{\text{def}}{=} [T=\infty] = [Z^\star_\infty \le \lceil Z \rceil_{[\alpha/2]}]$ thus has $\mathbb{P}[G] \ge 1-\alpha/2$. Consider now the collection K of measurable functions k with $0 \le k \le G$ and $\mathbb{E}[k] \ge 1-\alpha$. K is clearly a convex and weak*-compact subset of $L^\infty(\mathbb{P})$

(see A.2.32). As it contains G, it is not void. For every $X \in \mathcal{E}_1$ define a function h_X on K by

$$h_X(k) \stackrel{\text{def}}{=} ¦ Z ¦_{[\alpha/2]} - \mathbb{E}\Big[\int X\,dZ \cdot k\Big] , \qquad k \in K .$$

Since, on G, $\int X\ dZ$ is a finite linear combination of bounded random variables, h_X is well-defined and real-valued. Every one of the functions h_X is evidently linear and continuous on K, and is non-negative at some point of K, to wit, at the set

$$k_X \stackrel{\text{def}}{=} G \cap \Big[\Big|\int X\,dZ\Big| \le ¦Z¦_{[\alpha/2]}\Big] .$$

Indeed,

$$\begin{aligned} h_X(k_X) &= ¦Z¦_{[\alpha/2]} - \mathbb{E}\left[\int X\,dZ \cdot G \cap \left[\left|\int X\,dZ\right| \le ¦Z¦_{[\alpha/2]}\right]\right] \\ &\ge ¦Z¦_{[\alpha/2]} - \mathbb{E}\left[\left|\int X\,dZ\right| \cdot \left[\left|\int X\,dZ\right| \le ¦Z¦_{[\alpha/2]}\right]\right] \ge 0 ; \end{aligned}$$

and since

$$\begin{aligned} \mathbb{E}[k_X] &= \mathbb{P}\left[G \cap \left[\left|\int X\ dZ\right| \le ¦Z¦_{[\alpha/2]}\right]\right] \\ &\ge 1 - \alpha/2 - \mathbb{P}\left[\left|\int X\ dZ\right| > ¦Z¦_{[\alpha/2]}\right] \ge 1 - \alpha , \end{aligned}$$

k_X belongs to K. The collection $\mathcal{H} \stackrel{\text{def}}{=} \{h_X : X \in \mathcal{E}_1\}$ is easily seen to be convex; indeed, $sh_X + (1-s)h_Y = h_{sX+(1-s)Y}$ for $0 \le s \le 1$. Thus Ky–Fan's minimax theorem A.2.34 applies and provides a common point $k_\alpha \in K$ at which every one of these functions is non-negative. This says that

$$\mathbb{E}\Big[k_\alpha \cdot \int X\ dZ\Big] \le ¦ Z ¦_{[\alpha/2]} \qquad \forall\, X \in \mathcal{E}_1 .$$

Note the lack of the absolute-sign under the expectation, which distinguishes this from (4.1.3). Since $|Z|$ is $k_\alpha{\cdot}\mathbb{P}$-a.s. bounded by $¦Z¦_{[\alpha/2]}$, though, part (i) of lemma 2.5.27 on page 80 applies, with subprobability $k_\alpha{\cdot}\mathbb{P}$, and produces

$$\mathbb{E}\Big[k_\alpha \cdot \Big|\int X\,dZ\Big|\Big] \le \sqrt{2}\,\Big(¦ Z ¦_{[\alpha/2]} + ¦ Z ¦_{[\alpha/2]}\Big) \le 3\, ¦ Z ¦_{[\alpha/2]}$$

for all $X \in \mathcal{E}_1$, which is the desired inequality (4.1.3), with $\zeta_\alpha = 3\,¦Z¦_{[\alpha/2]}$.

Now to the construction of $\mathbb{P}' = g'\mathbb{P}$. First we pick $\alpha \mapsto \zeta_\alpha \ge 1$ and decreasing on $(0,1)$, for instance $\zeta_\alpha \stackrel{\text{def}}{=} 1 \vee 3¦Z¦_{[\alpha/2]}$ or

$$\zeta_\alpha = \zeta^+_\alpha \stackrel{\text{def}}{=} 3¦ Z ¦_{[\alpha_1 \wedge \alpha/2]} , \qquad (\zeta^+)$$

where $\alpha_1 > 0$ has been picked so that $¦Z¦_{[\alpha_1]} \ge 1/3$ — if no such α_1 existed then Z would already be an $L^p(\mathbb{P})$-integrator for all $p < \infty$.
Since $\mathbb{P}[k_\alpha = 0] = 1 - \mathbb{P}[k_\alpha > 0] \le 1 - \mathbb{E}[k_\alpha] \le \alpha$ for $0 < \alpha < 1$, the bounded

function

$$g' \overset{\text{def}}{=} \gamma' \cdot \sum_{n=1}^{\infty} 2^{-n} \cdot \frac{k_{2^{-n}}}{\zeta_{2^{-n}}} \tag{4.1.4}$$

is $\mathbb{P}$-a.s. strictly positive and bounded, and with the proper choice of

$$\gamma' \in \big(\zeta_{1/2}, 4\zeta_{1/2}\big)$$

it can be made to have $\mathbb{P}$-expectation one. The measure $\mathbb{P}' \overset{\text{def}}{=} g' \cdot \mathbb{P}$ is then a probability equivalent with $\mathbb{P}$. Let $\mathbb{E}'$ denote the expectation with respect to $\mathbb{P}'$. Inequality (4.1.3) implies that for every $X \in \mathcal{E}_1$

$$\mathbb{E}'\Big[\Big|\int X\, dZ\Big|\Big] \le \gamma' < 4\zeta_{1/2} \ .$$

That is to say, Z is a global $L^1(\mathbb{P}')$-integrator of size $\lceil Z \rceil_{\mathcal{I}^1[\mathbb{P}']} < 4\zeta_{1/2}$. Towards the estimate (4.1.1) note that for any $\alpha, \lambda \in (0,1)$

$$\mathbb{P}[k_\alpha \le \lambda] = \mathbb{P}[1 - k_\alpha \ge 1 - \lambda] \le \frac{\mathbb{E}[1 - k_\alpha]}{1 - \lambda} \le \frac{\alpha}{1 - \lambda} \ ,$$

and thus $\quad \mathbb{P}[g \ge C] = \mathbb{P}[g' \le 1/C] = \mathbb{P}\Big[\sum_{n=1}^{\infty} \frac{2^{-n}\, k_{2^{-n}}}{\zeta_{2^{-n}}} \le \frac{1}{C\,\gamma'}\Big]$

for every single $n \in \mathbb{N}$: $\quad \le \mathbb{P}\Big[k_{2^{-n}} \le \frac{2^n\, \zeta_{2^{-n}}}{C\,\gamma'}\Big] \le \mathbb{P}\Big[k_{2^{-n}} \le \frac{2^n\, \zeta_{2^{-n}}}{C\,\zeta_{1/2}}\Big]$

if $C\zeta_{1/2} > 2^n \zeta_{2^{-n}}$: $\quad \le 2^{-n} \Big/ \Big(1 - \frac{2^n\, \zeta_{2^{-n}}}{C\,\zeta_{1/2}}\Big) \ .$

Given $\alpha \in (0,1)$, we choose n so that $\alpha/4 < 2^{-n} \le \alpha/2$ and set

$$C \overset{\text{def}}{=} \frac{8\zeta_{\alpha/4}}{\alpha\zeta_{1/2}} \ge \frac{2^{n+1}\zeta_{2^{-n}}}{\zeta_{1/2}} \ .$$

Then $\quad \frac{2^n\, \zeta_{2^{-n}}}{C\,\zeta_{1/2}} \le 1/2$ and so $\mathbb{P}[g \ge C] \le 2^{-n+1} \le \alpha$,

which says $\quad \|g\|_{[\alpha;\mathbb{P}]} \le \frac{8\zeta_{\alpha/4}}{\alpha\zeta_{1/2}} \le 8\zeta_{\alpha/4}\big/\,\alpha$ and proves (4.1.1) .

For the choice $\zeta = \zeta^+$ this gives the estimates

$$D^{(4.1.1)}[\lceil \boldsymbol{Z} \rceil_{[\cdot]}] \le 12 \lceil Z \rceil_{[\alpha_1 \wedge 1/4]}$$

and

$$E^{(4.1.1)}_{[\alpha]}[\lceil \boldsymbol{Z} \rceil_{[\cdot]}] \le 24 \lceil Z \rceil_{[\alpha_1 \wedge \alpha/8]}\big/\alpha \ .$$

The last inequality (4.1.2) follows from a simple application of exercise A.8.17 to inequality (4.1.1). ∎

The Main Factorization Theorem

Theorem 4.1.2 *(i) Let $0 < p < q < \infty$ and $\boldsymbol{Z}$ a d-tuple of global $L^p(\mathbb{P})$-integrators. There exists a probability $\mathbb{P}'$ equivalent with $\mathbb{P}$ on $\mathcal{F}_\infty$ with respect to which $\boldsymbol{Z}$ is a global L^q-integrator; furthermore $d\mathbb{P}'/d\mathbb{P}$ is bounded, and there exist universal constants $D = D_{p,q,d}$ and $E = E_{p,q}$ depending only on the subscripted quantities*

such that $$\lceil\!\lceil \boldsymbol{Z} \rceil\!\rceil_{\mathcal{I}^q[\mathbb{P}']} \le D_{p,q,d} \cdot \lceil\!\lceil \boldsymbol{Z} \rceil\!\rceil_{\mathcal{I}^p[\mathbb{P}]} , \tag{4.1.5}$$

and such that the Radon–Nikodym derivative $g \overset{\text{def}}{=} d\mathbb{P}/d\mathbb{P}'$ satisfies

$$\|g\|_{L^{p/(q-p)}(\mathbb{P})} \le E_{p,q} \tag{4.1.6}$$

– this inequality has the consequence that for any $r > 0$ and $f \in \mathcal{F}_\infty$

$$\|f\|_{L^r(\mathbb{P})} \le E_{p,q}^{p/qr} \cdot \|f\|_{L^{rq/p}(\mathbb{P}')} . \tag{4.1.7}$$

If $0 < p < q \le 2$ or if $\boldsymbol{Z}$ is previsible, then D does not depend on d.

(ii) Let $p = 0 < q < \infty$, and let $\boldsymbol{Z}$ be a d-tuple of global $L^0(\mathbb{P})$-integrators with modulus of continuity[1] $\lceil\!\lceil \boldsymbol{Z} \rceil\!\rceil_{[.]}$. *There exists a probability $\mathbb{P}' = \mathbb{P}/g$ equivalent with $\mathbb{P}$ on $\mathcal{F}_\infty$, with respect to which $\boldsymbol{Z}$ is a global L^q-integrator; furthermore, $g^{-1} = d\mathbb{P}'/d\mathbb{P}$ is bounded and there exist universal constants $D = D_{q,d}[\lceil\!\lceil \boldsymbol{Z} \rceil\!\rceil_{[.]}]$ and $E = E_{[\alpha],q}[\lceil\!\lceil \boldsymbol{Z} \rceil\!\rceil_{[.]}]$, depending only on $q, d, \alpha \in (0,1)$ and the modulus of continuity $\lceil\!\lceil \boldsymbol{Z} \rceil\!\rceil_{[.]}$,*

such that $$\lceil\!\lceil \boldsymbol{Z} \rceil\!\rceil_{\mathcal{I}^q[\mathbb{P}']} \le D_{q,d}[\lceil\!\lceil \boldsymbol{Z} \rceil\!\rceil_{[.]}] , \tag{4.1.8}$$

and $$\|g\|_{[\alpha]} \le E_{[\alpha],q}[\lceil\!\lceil \boldsymbol{Z} \rceil\!\rceil_{[.]}] \quad \forall \alpha \in (0,1) \tag{4.1.9}$$

– this implies $$\|f\|_{[\alpha+\beta;\mathbb{P}]} \le \Big(E_{[\alpha],q}[\lceil\!\lceil \boldsymbol{Z} \rceil\!\rceil_{[.]}]/\beta\Big)^{1/r} \cdot \|f\|_{L^r(\mathbb{P}')} \tag{4.1.10}$$

for any $f \in \mathcal{F}_\infty$, $r > 0$, and $\alpha, \beta \in (0,1)$. Again, in the range $q \le 2$ or when $\boldsymbol{Z}$ is previsible the constant D does not depend on d.

Estimates independent of the length d of $\boldsymbol{Z}$ are used in the control of random measures – see corollary 4.1.14 and theorem 4.5.25. The proof of theorem 4.1.2 varies with the range of p and of $q > p$, and will provide various estimates[2] for the constants D and E. The implication (4.1.6) $\Longrightarrow$ (4.1.7) results from a straightforward application of Hölder's inequality and is left to the reader:

Exercise 4.1.3 (i) Let μ be a positive σ-additive measure and $0 < p < q < \infty$. The condition $1/g \le C$ has the effect that $\|f\|_{L^r(\mu/g)} \le C^{1/r}\|f\|_{L^r(\mu)}$ for all

[1] $\lceil\!\lceil \boldsymbol{Z} \rceil\!\rceil_{[\alpha]} \overset{\text{def}}{=} \sup\{\|\int \boldsymbol{X}\, d\boldsymbol{Z}\|_{[\alpha;\mathbb{P}]} : \boldsymbol{X} \in \mathcal{E}_1^d\}$ for $0 < \alpha < 1$; see page 56.

[2] See inequalities (4.1.6), (4.1.34), (4.1.35), (4.1.40), and (4.1.41).

measurable functions f and all $r > 0$. The condition $\|g\|_{L^{p/(q-p)}(\mu)} \le c$ has the effect that for all measurable functions f that vanish on $[g = 0]$ and all $r > 0$

$$\|f\|_{L^r(\mu)} \le c^{p/(qr)} \cdot \|f\|_{L^{rq/p}(d\mu/g)} .$$

(ii) In the same vein prove that (4.1.9) implies (4.1.10).

(iii) $\mathfrak{L}^1[Z\text{–}q;\mathbb{P}'] \subset \mathfrak{L}^1[Z\text{–}p;\mathbb{P}]$, the injection being continuous.

The remainder of this section, which ends on page 209, is devoted to a detailed proof of this theorem. For both parts (i) and (ii) we shall employ several times the following

Criterion 4.1.4 (Rosenthal) *Let E be a normed linear space with norm $\| \ \|_E$, μ a positive σ-finite measure, $0 < p < q < \infty$, and $\mathcal{I} : E \to L^p(\mu)$ a linear map. For any constant $C > 0$ the following are equivalent:*

(i) There exists a measurable function $g \ge 0$ with $\|g\|_{L^{p/(q-p)}(\mu)} \le 1$ such that for all $x \in E$

$$\left(\int |\mathcal{I}x|^q \,\frac{d\mu}{g}\right)^{1/q} \le C \cdot \|x\|_E . \tag{4.1.11}$$

(ii) For any finite collection $\{x_1, \ldots, x_n\} \subset E$

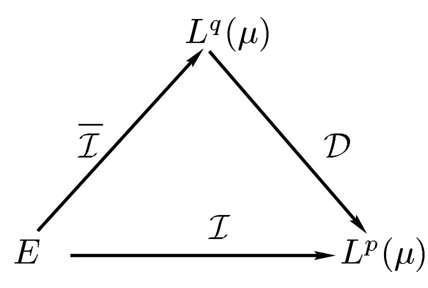
$$\left\| \left(\sum_{\nu=1}^{n} |\mathcal{I}x_\nu|^q\right)^{1/q} \right\|_{L^p(\mu)} \le C \cdot \left(\sum_{\nu=1}^{n} \|x_\nu\|_E^q\right)^{1/q} . \tag{4.1.12}$$

(iii) For every measure space $(T, \mathcal{T}, \tau{\ge}0)$ and q-integrable $f : T \to E$

$$\Big\| \|\mathcal{I}f\|_{L^q(\tau)} \Big\|_{L^p(\mu)} \le C \cdot \Big\| \|f\|_E \Big\|_{L^q(\tau)} . \tag{4.1.13}$$

The smallest constant C satisfying any and then all of (4.1.11), (4.1.12), and (4.1.13) is the ***p–q-factorization constant*** of $\mathcal{I}$ and will be denoted by

$$\eta_{p,q}(\mathcal{I}) .$$



It may well be infinite, of course. Its name comes from the following way of looking at (i): the map $\mathcal{I}$ has been "factored as" $\mathcal{I} = \mathcal{D} \circ \overline{\mathcal{I}}$, where $\overline{\mathcal{I}} : E \to L^q(\mu)$ is defined by $\overline{\mathcal{I}}(x) = \mathcal{I}(x) \cdot g^{-1/q}$ and $\mathcal{D} : L^q(\mu) \to L^p(\mu)$ is the "diagonal map" $f \mapsto f \cdot g^{1/q}$. The number $\eta_{p,q}(\mathcal{I})$ is simply the operator (quasi)norm of $\overline{\mathcal{I}}$ – the operator (quasi)norm of $\mathcal{D}$ is $\|g\|^{1/q}_{L^{p/(q-p)}(\mu)} \le 1$. Thus, if $\eta_{p,q}(\mathcal{I})$ is finite, we also say that $\mathcal{I}$ ***factorizes through L^q***.

We are of course primarily interested in the case when $\mathcal{I}$ is the stochastic integral $\boldsymbol{X} \mapsto \int \boldsymbol{X}\, d\boldsymbol{Z}$, and the question arises whether μ/g is a probability when μ is. It won't be automatically but can be made into one:

Exercise 4.1.5 Assume in criterion 4.1.4 that μ is a *probability* $\mathbb{P}$ and $\eta_{p,q}(\mathcal{I}) < \infty$. Then there is a *probability* $\mathbb{P}'$ equivalent with $\mathbb{P}$ such that $\mathcal{I}$ is continuous as a map into $L^q(\mathbb{P}')$:

$$\|\mathcal{I}\|_{q;\mathbb{P}'} \le \eta_{p,q}(\mathcal{I}) ,$$

and such that $g' \stackrel{\text{def}}{=} d\mathbb{P}'/d\mathbb{P}$ is bounded and $g \stackrel{\text{def}}{=} d\mathbb{P}/d\mathbb{P}' = g'^{-1}$ satisfies

$$\|g\|_{L^{p/(q-p)}(\mathbb{P})} \le 2^{(p\vee(q-p))/p} , \tag{4.1.14}$$

and therefore $$\|f\|_{L^r(\mathbb{P})} \le 2^{(p\vee(q-p))/rq}\|f\|_{L^{rq/p}(\mathbb{P}')} \le 2^{1/r}\|f\|_{L^{rq/p}(\mathbb{P}')} \tag{4.1.15}$$

for all measurable functions f and exponents $r > 0$.

Exercise 4.1.6 (i) $\eta_{p,q}(\mathcal{I})$ depends isotonically on q.
(ii) For any two maps $\mathcal{I}, \mathcal{I}' : E \to L^p(\mu)$ and $0 < p < q < \infty$ we have

$$\eta_{p,q}(\mathcal{I}+\mathcal{I}') \le 2^{0\vee(1-q)/q} \cdot 2^{0\vee(1-p)/p} \times \big[\eta_{p,q}(\mathcal{I}) + \eta_{p,q}(\mathcal{I}')\big] .$$

Proof of Criterion 4.1.4. If (i) holds, then $\int |\mathcal{I}x|^q \frac{d\mu}{g} \le C^q \cdot \|x\|_E^q$ for all $x \in E$, and consequently for any finite subcollection $\{x_1, \dots, x_n\}$ of E,

$$\sum_{\nu=1}^n \int |\mathcal{I}x_\nu|^q \,\frac{d\mu}{g} \le C^q \cdot \sum_{\nu=1}^n \|x_\nu\|_E^q$$

and $$\Big\|\Big(\sum_{\nu=1}^n |\mathcal{I}x_\nu|^q\Big)^{1/q}\Big\|_{L^q(d\mu/g)} \le C \cdot \Big(\sum_{\nu=1}^n \|x_\nu\|_E^q\Big)^{1/q} .$$

Inequality (4.1.11) implies that $\mathcal{I}x$ vanishes μ-almost surely on $[g = 0]$, so exercise 4.1.3 applies with $r = p$ and $c = 1$, giving

$$\Big\|\Big(\sum_{\nu=1}^n |\mathcal{I}x_\nu|^q\Big)^{1/q}\Big\|_{L^p(\mu)} \le \Big\|\Big(\sum_{\nu=1}^n |\mathcal{I}x_\nu|^q\Big)^{1/q}\Big\|_{L^q(d\mu/g)} .$$

This together with the previous inequality results in (4.1.12).

The reverse implication (ii)$\Rightarrow$(i) is a bit more difficult to prove. To start with, consider the following collection of measurable functions:

$$K = \Big\{k \ge 0 :\ \|k\|_{L^{q/(q-p)}(\mu)} \le 1\Big\} .$$

Since $1 < q/(q-p) < \infty$, this convex set is weakly compact – see the proof of theorem A.2.25 on page 379 (iv) in the Answers. Next let us define a host $\mathcal{H}$ of numerical functions on K, one for every finite collection $\{x_1, \dots, x_n\} \subset E$, by

$$k \mapsto h_{x_1,\dots,x_n}(k) \stackrel{\text{def}}{=} C^q \cdot \sum_{\nu=1}^n \|x_\nu\|_E^q - \int^* \sum_{\nu=1}^n |\mathcal{I}x_\nu|^q \cdot \frac{1}{k^{q/p}}\, d\mu . \tag{$*$}$$

The idea is to show that there is a point $k \in K$ at which every one of these functions is non-negative. Given that, we set $g = k^{q/p}$ and are done: $\|k\|_{L^{q/(q-p)}(\mu)} \le 1$ translates into $\|g\|_{L^{p/(q-p)}(\mu)} \le 1$, and $h_x(k) \ge 0$ is inequality (4.1.11). To prove the existence of the common point k of positivity we start with a few observations.

a) An $h = h_{x_1,\ldots,x_n} \in \mathcal{H}$ may take the value $-\infty$ on K, but never $+\infty$.
b) Every function $h \in \mathcal{H}$ is concave – simply observe the minus sign in front of the integral in $(*)$ and note that $k \mapsto 1/k^{q/p}$ is convex.
c) Every function $h = h_{x_1,\ldots,x_n} \in \mathcal{H}$ is upper semicontinuous (see page 376) in the weak topology $\sigma(L^{q/(q-p)}, L^{q/p})$. To see this note that the subset $[h_{x_1,\ldots,x_n} \geq r]$ of K is convex, so it is weakly closed if and only if it is norm-closed (theorem A.2.25 (iii)). In other words, it suffices to show that $h_{x_1,\ldots,x_n}$ is upper semicontinuous in the norm topology of $L^{q/(q-p)}$ or, equivalently, that

$$k \mapsto \int \sum_{\nu=1}^{n} |\mathcal{I}x_\nu|^q \cdot \frac{1}{k^{q/p}} \, d\mu$$

is lower semicontinuous in the norm topology of $L^{q/(q-p)}$. Now

$$\int |\mathcal{I}x_\nu|^q \cdot k^{-q/p} \, d\mu = \sup_{\epsilon > 0} \int \left(\epsilon^{-1} \wedge |\mathcal{I}x_\nu|^q\right) \cdot (\epsilon \vee |k|)^{-q/p} \, d\mu \, ,$$

and the map that sends k to the integral on the right is norm-continuous on $L^{q/(q-p)}$, as a straightforward application of the Dominated Convergence Theorem shows. The characterization of semicontinuity in A.2.19 gives c).
d) For every one of the functions $h = h_{x_1,\ldots,x_n} \in \mathcal{H}$ there is a point $k_{x_1,\ldots,x_n} \in K$ (depending on h!) at which it is non-negative. Indeed,

$$k_{x_1,\ldots,x_n} = \Big(\int \Big(\sum_{1 \leq \nu \leq n} |\mathcal{I}x_\nu|^q\Big)^{p/q} d\mu\Big)^{(p-q)/q} \cdot \Big(\sum_{1 \leq \nu \leq n} |\mathcal{I}x_\nu|^q\Big)^{p(q-p)/q^2}$$

meets the description: raising this function to the power $q/(q-p)$ and integrating gives 1; hence $k_{x_1,\ldots,x_n}$ belongs to K. Next,

$$k_{x_1,\ldots,x_n}^{-q/p} = \Big(\int \Big(\sum_{1 \leq \nu \leq n} |\mathcal{I}x_\nu|^q\Big)^{p/q} d\mu\Big)^{(q-p)/p} \cdot \Big(\sum_{1 \leq \nu \leq n} |\mathcal{I}x_\nu|^q\Big)^{(p-q)/q} ;$$

thus

$$\sum_{1 \leq \nu \leq n} |\mathcal{I}x_\nu|^q \cdot k_{x_1,\ldots,x_n}^{-q/p} = \Big(\int \Big(\sum_{1 \leq \nu \leq n} |\mathcal{I}x_\nu|^q\Big)^{p/q} d\mu\Big)^{(q-p)/p} \cdot \Big(\sum_{1 \leq \nu \leq n} |\mathcal{I}x_\nu|^q\Big)^{p/q} ,$$

and therefore

$$h_{x_1,\ldots,x_n}(k_{x_1,\ldots,x_n}) = C^q \cdot \sum_{1 \leq \nu \leq n} \|x_\nu\|_E^q - \Big(\int \Big(\sum_{1 \leq \nu \leq n} |\mathcal{I}x_\nu|^q\Big)^{p/q} d\mu\Big)^{q/p} .$$

Thanks to inequality (4.1.12), this number is non-negative.
e) Finally, observe that the collection $\mathcal{H}$ of concave upper semicontinuous functions defined in $(*)$ is convex. Indeed, for $\lambda, \lambda' \geq 0$ with sum $\lambda + \lambda' = 1$,

$$\lambda \cdot h_{x_1,\ldots,x_n} + \lambda' \cdot h_{x'_1,\ldots,x'_{n'}} = h_{\lambda^{1/q}x_1,\ldots,\lambda^{1/q}x_n,\lambda'^{1/q}x'_1,\ldots,\lambda'^{1/q}x'_{n'}} .$$

Ky–Fan's minimax theorem A.2.34 now guarantees the existence of the desired common point of positivity for all of the functions in $\mathcal{H}$.

The equivalence of (ii) with (iii) is left as an easy excercise. ▮

Proof for $p > 0$

Proof of Theorem 4.1.2 (i) for $0 < p < q \le 2$. We have to show that $\eta_{p,q}(\mathcal{I})$ is finite when $\mathcal{I} : \mathcal{E}^d \to L^p(\mathbb{P})$ is the stochastic integral $\boldsymbol{X} \mapsto \int \boldsymbol{X}\, d\boldsymbol{Z}$, in fact, that $\eta_{p,q}(\mathcal{I}) \le D_{p,q,d} \cdot ⦙\boldsymbol{Z}⦙_{\mathcal{I}^p[\mathbb{P}]}$ with $D_{p,q,d}$ finite. Note that the domain $\mathcal{E}^d$ of the stochastic integral is the set of step functions over an algebra of sets. Therefore the following deep theorem from Banach space theory applies and provides in conjunction with exercises 4.1.6 (i) and 4.1.5 for $0 < p < q \le 2$ the estimates

$$D_{p,q,d}^{(4.1.5)} < 3 \cdot 8^{1/p} \quad \text{and} \quad E_{p,q}^{(4.1.6)} \le 2^{(p \vee (q-p))/p} . \tag{4.1.16}$$

Theorem 4.1.7 *Let $\boldsymbol{B}$ be a set, $\mathcal{A}$ an algebra of subsets of $\boldsymbol{B}$, and let $\mathcal{E}$ be the collection of step functions over $\mathcal{A}$. $\mathcal{E}$ is naturally equipped with the sup-norm $\|x\|_{\mathcal{E}} \stackrel{\text{def}}{=} \sup\{|x(\varpi)| : \varpi \in \boldsymbol{B}\}$, $x \in \mathcal{E}$.*

Let μ be a σ-finite measure on some other space, let $0 < p < 2$, and let $\mathcal{I} : \mathcal{E} \to L^p(\mu)$ be a continuous linear map of size

$$\|\mathcal{I}\|_p \stackrel{\text{def}}{=} \sup\Big\{ \|\mathcal{I}x\|_{L^p(\mu)} : \|x\|_{\mathcal{E}} \le 1 \Big\} .$$

There exist a constant C_p and a measurable function $g \ge 0$ with

$$\|g\|_{L^{p/(2-p)}(\mu)} \le 1$$

such that

$$\Big(\int |\mathcal{I}x|^2 \,\frac{d\mu}{g}\Big)^{1/2} \le C_p \cdot \|\mathcal{I}\|_p \cdot \|x\|_{\mathcal{E}}$$

for all $x \in \mathcal{E}$. The universal constant C_p can be estimated in terms of the Khintchine constants of theorem A.8.26:

$$\begin{aligned} C_p &\le \Big(\big(2^{1/3} + 2^{-2/3}\big)\, 2^{0 \vee (1-p)/p}\, K_p^{(\text{A.8.5})}\, K_1^{(\text{A.8.5})}\Big)^{3/2} \\ &\le \big(2\sqrt{2}\big)^{1/p + 1 \vee 1/p} < 2^{3(2+p)/2p} < 3 \cdot 8^{1/p} . \end{aligned} \tag{4.1.17}$$

Exercise 4.1.8 The theorem persists if K is a compact space and $\mathcal{I}$ is a continuous linear map from $C(K)$ to $L^p(\mu)$, or if $\mathcal{E}$ is an algebra of bounded functions containing the constants and $\mathcal{I} : \mathcal{E} \to L^p(\mu)$ is a bounded linear map.

Theorem 4.1.7 was first proved by Rosenthal [96] in the range $1 \le p < q \le 2$ and was extended to the range $0 < p < q \le 2$ by Maurey [67] and Schwartz [68]. The remainder of this subsection is devoted to its proof. The next two lemmas, in fact the whole drift of the following argument, are from Pisier's paper [85]. We start by addressing the special case of theorem 4.1.7 in which $\mathcal{E}$

is $\ell^\infty(k)$, i.e., $\mathbb{R}^k$ equipped with the sup-norm. Note that $\ell^\infty(k)$ meets the description of $\mathcal{E}$ in theorem 4.1.7: with $\boldsymbol{B} = \{1, \ldots, k\}$, $\ell^\infty(k)$ consists exactly of the step functions over the algebra $\mathcal{A}$ of all subsets of $\boldsymbol{B}$. This case is prototypical; once theorem 4.1.7 is established for it, the general version is not far away (page 202).

If $\mathcal{I} : \ell^\infty(k) \to L^p(\mu)$ is continuous, then $\eta_{p,2}(\mathcal{I})$ is rather readily seen to be finite. In fact, a straightforward computation, whose details are left to the reader, shows that whenever the domain of $\mathcal{I} : E \to L^p(\mu)$ is k-dimensional ($k < \infty$),

$$\Big\| \Big(\sum_\nu |\mathcal{I} x_\nu|^2 \Big)^{1/2} \Big\|_{L^p(\mu)} \le k^{1/p+1/2} \|\mathcal{I}\|_p \cdot \Big(\sum_\nu \|x_\nu\|^2_{\ell^\infty(k)} \Big)^{\frac{1}{2}} ,$$

which reads
$$\eta_{p,2}(\mathcal{I}) \le k^{1/p+1/2} \cdot \|\mathcal{I}\|_p < \infty . \tag{4.1.18}$$

Thus there is factorization in the sense of criterion 4.1.4 if $\mathcal{I} : \ell^\infty(k) \to L^p(\mu)$ is continuous. In order to parlay this result into theorem 4.1.7 in all its generality, an estimate of $\eta_{p,2}(\mathcal{I})$ better than (4.1.18) is needed, namely, one that is independent of the dimension k. The proof below of such an estimate uses the Bernoulli random variables ϵ_ν that were employed in the proof of Khintchine's inequality (A.8.4) and, in fact, uses this very inequality twice. Let us recall their definition:

We fix a natural number n – it is the number of vectors $x_\nu \in \ell^\infty(k)$ that appear in inequality (4.1.12) – and denote by T^n the n-fold product of two-point sets $\{1, -1\}$. Its elements are n-tuples $t = (t_1, t_2, \ldots, t_n)$ with $t_\nu = \pm 1$. $\epsilon_\nu : t \mapsto t_\nu$ is the ν^{th} coordinate function. The natural measure on T^n is the product τ of uniform measure on $\{1, -1\}$, so that $\tau(\{t\}) = 2^{-n}$ for $t \in T^n$. T^n is a compact abelian group and τ is its normalized Haar measure. There will be occasion to employ convolution on this group.

The $\epsilon_\nu, \nu = 1 \ldots n$, are independent and form an orthonormal set in $L^2(\tau)$, which is far from being a basis: since the σ-algebra $\mathcal{T}^n$ on T^n is generated by 2^n atoms, the dimension of $L^2(\tau)$ is 2^n. Here is a convenient extension to a basis for this Hilbert space: for any subset A of $\{1, \ldots, n\}$ set

$$w_A = \prod_{\nu \in A} \epsilon_\nu , \quad \text{with } w_\emptyset = 1 .$$

It is plain upon inspection that the w_A are characters[3] of the group T^n and form an orthonormal basis of $L^2(\tau)$, the ***Walsh basis***.

Consider now the Banach space $L^2(\tau, \ell^\infty)$ of $\ell^\infty(k)$-valued functions f on T^n having

$$\| f \|_{L^2(\tau,\ell^\infty)} \stackrel{\text{def}}{=} \Big(\int \| f(t) \|^2_{\ell^\infty(k)} \, \tau(dt) \Big)^{1/2} < \infty .$$

[3] A map χ from a group into the circle $\{z \in \mathbb{C} : |z| = 1\}$ is a ***character*** if it is multiplicative: $\chi(st) = \chi(s)\chi(t)$, with $\chi(1) = 1$.

Its dual can be identified isometrically with the Banach space $L^2(\tau,\ell^1)$ of $\ell^1(k)$-valued functions f^* on T^n for which

$$\|f^*\|_{L^2(\tau,\ell^1)} \stackrel{\text{def}}{=} \Big(\int \|f^*(t)\|^2_{\ell^1}\,\tau(dt)\Big)^{1/2} < \infty\ ,$$

under the pairing $\langle f|f^*\rangle = \int \langle f(t)|f^*(t)\rangle\,\tau(dt)\ .$

Both spaces have finite dimension $k2^n$. $L^2(\tau,\ell^\infty)$ is the direct sum of the subspaces

$$E(\ell^\infty) \stackrel{\text{def}}{=} \Big\{\sum_{\nu=1}^n x_\nu\cdot\epsilon_\nu\ :\ x_\nu\in\ell^\infty(k)\Big\} \quad \text{and}$$

$$W(\ell^\infty) \stackrel{\text{def}}{=} \Big\{\sum x_A\cdot w_A\ :\ A\subset\{1,\ldots,n\}\ ,\ |A|\neq 1\ ,\ x_A\in\ell^\infty(k)\Big\}\ .$$

$|A|$ is, of course, the cardinality of the set $A\subset\{1,\ldots,n\}$, and $w_\emptyset=1$. It is convenient to denote the corresponding projections of $L^2(\tau,\ell^\infty)$ onto $E(\ell^\infty)$ and $W(\ell^\infty)$ by E and W, respectively. Here is a little information about the geometry of these subspaces, used below to estimate the right-hand side of inequality (4.1.12) on page 192:

Lemma 4.1.9 *Let $x_1,\ldots,x_n\in\ell^\infty(k)$. There is a function $f\in L^2(\tau,\ell^\infty(k))$ of the form*

$$f=\sum_{\nu=1}^n x_\nu\,\epsilon_\nu+\sum_{|A|\neq 1} x_A\,w_A$$

such that

$$\|f\|_{L^2(\tau,\ell^\infty)} \le K_1^{(\mathrm{A.8.5})}\cdot\Big(\sum_{\nu=1}^n\|x_\nu\|^2_{\ell^\infty}\Big)^{1/2}\ . \tag{4.1.19}$$

Proof. Set $f^\epsilon=\sum_\nu x_\nu\epsilon_\nu$, and let a denote the norm of the class of f^ϵ in the quotient $L^2(\tau,\ell^\infty)/W(\ell^\infty)$:

$$a=\inf\Big\{\,\|\,f^\epsilon+w\,\|_{L^2(\tau,\ell^\infty)}\ :\ w\in W(\ell^\infty)\Big\}\ .$$

Since $L^2\big(\tau,\ell^\infty(k)\big)$ is finite-dimensional, there is a function of the form $f=f^\epsilon+w$, $w\in W(\ell^\infty)$, with $\|\,f\,\|_{L^2(\tau,\ell^\infty)}=a$. This is the function promised by the statement.

To prove inequality (4.1.19), let B_a denote the open ball of radius a about zero in $L^2(\tau,\ell^\infty)$. Since the open convex set

$$\mathcal{C}\stackrel{\text{def}}{=} B_a-\big(f+W(\ell^\infty)\big)=\{\,g-(f+w)\ :\ g\in B_a\,,\ w\in W(\ell^\infty)\,\}$$

does not contain the origin, there is a linear functional f^* in the dual $L^2(\tau,\ell^1)$ of $L^2(\tau,\ell^\infty)$ that is negative on $\mathcal{C}$, and without loss of generality f^* can be chosen to have norm 1:

$$\int \|f^*(t)\|^2_{\ell^1}\,\tau(dt)=1\ . \tag{4.1.20}$$

Since $\langle g|f^*\rangle\le\langle f+w|f^*\rangle \qquad \forall g\in B_a(0)\,,\ \forall w\in W(\ell^\infty)\ ,$

it is evident that $\langle w|f^*\rangle = 0$ for all $w \in W(\ell^\infty)$, so that f^* is of the form

$$f^* = \sum_{\nu=1}^{n} x_\nu^* \, \epsilon_\nu \ , \qquad\qquad x_\nu^* \in \ell^1(k) \ .$$

Also, then $\langle g|f^*\rangle \le \langle f|f^*\rangle$ for all $g \in B_a(0)$, whence

$$a = \sup\{\langle g|f^*\rangle : g \in B_a(0)\} \le \langle f|f^*\rangle \le a \ .$$

Therefore

$$a = \langle f|f^*\rangle = \int \langle f(t)|f^*(t)\rangle \, \tau(dt) = \sum_{\nu=1}^{n} \langle x_\nu | x_\nu^*\rangle$$

$$\le \sum_{\nu=1}^{n} \|x_\nu\|_{\ell^\infty} \|x_\nu^*\|_{\ell^1} \le \Big(\sum_{\nu=1}^{n} \|x_\nu\|_{\ell^\infty}^2\Big)^{1/2} \cdot \Big(\sum_{\nu=1}^{n} \|x_\nu^*\|_{\ell^1}^2\Big)^{1/2} \ .$$

Now $$\Big(\sum_{\nu=1}^{n} \|x_\nu^*\|_{\ell^1}^2\Big)^{1/2} = \Big(\sum_{\nu=1}^{n} \Big(\sum_{\kappa=1}^{k} |x_\nu^{*\kappa}|\Big)^2\Big)^{1/2} \le \sum_{\kappa=1}^{k} \Big(\sum_{\nu=1}^{n} |x_\nu^{*\kappa}|^2\Big)^{1/2}$$

$$\le K_1^{(A.8.5)} \cdot \int \sum_{\kappa=1}^{k} \Big| \sum_{\nu} x_\nu^{*\kappa} \epsilon_\nu(t) \Big| \, \tau(dt)$$

$$= K_1 \cdot \int \Big\| \sum_{\nu=1}^{n} x_\nu^* \epsilon_\nu(t) \Big\|_{\ell^1(k)} \, \tau(dt)$$

$$\le K_1 \cdot \Big(\int \Big\| \sum_{\nu} x_\nu^* \epsilon_\nu(t) \Big\|_{\ell^1(k)}^2 \, \tau(dt) \Big)^{1/2}$$

by equation (4.1.20): $$= K_1 \cdot \|f^*\|_{L^2(\tau,\ell^1)} = K_1 \ ,$$

and we get the desired inequality

$$\|f\|_{L^2(\tau,\ell^\infty)} = a \le K_1^{(A.8.5)} \cdot \Big(\sum_{\nu=1}^{n} \|x_\nu\|_{\ell^\infty}^2\Big)^{1/2} \ . \qquad ■$$

We return to the continuous map $\mathcal{I} : \ell^\infty(k) \to L^p(\mu)$. We want to estimate the smallest constant $\eta_{p,2}(\mathcal{I})$ such that for any n and $x_1, \ldots, x_n \in \ell^\infty(k)$

$$\Big\| \Big(\sum_{\nu=1}^{n} |\mathcal{I} x_\nu|^2\Big)^{1/2} \Big\|_{L^p(\mu)} \le \eta_{p,2}(\mathcal{I}) \cdot \Big(\sum_{\nu=1}^{n} \| x_\nu \|_{\ell^\infty(k)}^2\Big)^{1/2} \ .$$

Lemma 4.1.10 *Let $x_1, \ldots, x_n \in \ell^\infty(k)$ be given, and let $f \in L^2(\tau, \ell^\infty)$ be any function with $Ef = \sum_{\nu=1}^{n} x_\nu \, \epsilon_\nu$, i.e.,*

$$f = \sum_{\nu=1}^{n} x_\nu \, \epsilon_\nu + \sum_{A \subset \{1,\ldots,n\}\,, |A| \ne 1} x_A \, w_A \ , \qquad\qquad x_A \in \ell^\infty(k) \ .$$

For any $\delta \in [0,1]$ *and* $p \in (0,2]$ *we have*

$$\Big\|\Big(\sum_{\nu=1}^{n} |\mathcal{I}x_\nu|^2\Big)^{1/2}\Big\|_{L^p(\mu)} \le 2^{0\vee(1-p)/p} K_p^{(\text{A.8.5})} \times \Big[\|\mathcal{I}\|_p/\sqrt{\delta} + \eta_{p,2}(\mathcal{I})\cdot\delta\Big]\cdot\|f\|_{L^2(\tau,\ell^\infty)} \ . \tag{4.1.21}$$

Proof. For $\theta \in [-1,1]$ set

$$\psi_\theta \stackrel{\text{def}}{=} \prod_{\nu=1}^{n}(1+\theta\epsilon_\nu) = \sum_{A\subset\{1,\ldots,n\}} \theta^{|A|} w_A \ .$$

Then $\psi_\theta \ge 0$, $\int|\psi_\theta(t)|\,\tau(dt) = \int\psi_\theta(t)\,\tau(dt) = 1$, and $\int w_A(t)\psi_\theta(t)\,\tau(dt) = \theta^{|A|}$. The function

$$\phi_\delta \stackrel{\text{def}}{=} \frac{1}{2\sqrt{\delta}}\Big(\psi_{\sqrt{\delta}} - \psi_{-\sqrt{\delta}}\Big)$$

is easily seen to have the following properties:

$$\gamma \stackrel{\text{def}}{=} \int |\phi_\delta(t)|\,\tau(dt) \le 1/\sqrt{\delta}\ ,$$

$$\int w_A(t)\phi_\delta(t)\,\tau(dt) = \begin{cases} 0 & \text{if } |A| \text{ is even,} \\ \sqrt{\delta}^{\,|A|-1} & \text{if } |A| \text{ is odd.}\end{cases} \tag{4.1.22}$$

For the proof proper of the lemma we analyze the convolution

$$f\star\phi_\delta(t) = \int f(st)\phi_\delta(s)\,\tau(ds)$$

of f with this function. For definiteness' sake write

$$f = \sum_{\nu=1}^{n} x_\nu\,\epsilon_\nu + \sum_{|A|\ne 1} x_A\,w_A = Ef + Wf\ .$$

As the w_A are characters,[3] including the $\epsilon_\nu = w_{\{\nu\}}$, (4.1.22) gives

$$w_A\star\phi_\delta(t) = \int w_A(st)\phi_\delta(s)\,\tau(ds) = w_A(t)\cdot\begin{cases} 0 & \text{if } |A| \text{ is even,} \\ \sqrt{\delta}^{\,|A|-1} & \text{if } |A| \text{ is odd;}\end{cases}$$

thus $$f\star\phi_\delta = \sum_{\nu=1}^{n} x_\nu\epsilon_\nu + \sum_{3\le|A|\ odd} x_A\,\sqrt{\delta}^{\,|A|-1} w_A = Ef + Wf\star\phi_\delta\ ,$$

whence $Ef = f\star\phi_\delta - Wf\star\phi_\delta$ and $\mathcal{I}Ef = \mathcal{I}f\star\phi_\delta - \mathcal{I}Wf\star\phi_\delta$.

Here $\mathcal{I}f$ denotes the function $t \mapsto \mathcal{I}f(t)$ from T to $L^p(\mu)$, etc. Plainly,

$$\Big\|\Big(\sum_{\nu=1}^{n}|\mathcal{I}x_\nu|^2\Big)^{1/2}\Big\|_{L^p(\mu)} = \Big\|\,\|\mathcal{I}Ef\|_{L^2(\tau)}\Big\|_{L^p(\mu)}\ .$$

Theorem A.8.26 permits the following estimate of the right-hand side:

$$\Big\|\|\mathcal{I}Ef\|_{L^2(\tau)}\Big\|_{L^p(\mu)} \le K_p \cdot \Big\|\|\mathcal{I}Ef\|_{L^p(\tau)}\Big\|_{L^p(\mu)}$$

$$\le 2^{0\vee(1-p)/p}K_p \cdot \Big[\Big\|\|\mathcal{I}f\star\phi_\delta\|_{L^p(\tau)}\Big\|_{L^p(\mu)} + \Big\|\|\mathcal{I}Wf\star\phi_\delta\|_{L^p(\tau)}\Big\|_{L^p(\mu)}\Big]$$

$$\le 2^{0\vee(1-p)/p}K_p \cdot \Big[\Big\|\|\mathcal{I}f\star\phi_\delta\|_{L^p(\mu)}\Big\|_{L^2(\tau)} + \Big\|\|\mathcal{I}Wf\star\phi_\delta\|_{L^2(\tau)}\Big\|_{L^p(\mu)}\Big]$$

$$\le 2^{0\vee(1-p)/p}K_p \cdot \Big[\|\mathcal{I}\|_p \cdot \|f\star\phi_\delta\|_{L^2(\tau,\ell^\infty)} + \Big\|\|\mathcal{I}Wf\star\phi_\delta\|_{L^2(\tau)}\Big\|_{L^p(\mu)}\Big]$$

$$= 2^{0\vee(1-p)/p}K_p \cdot [Q_1 + Q_2] \,. \tag{4.1.23}$$

The first term Q_1 can be bounded using Jensen's inequality (A.3.10) for the probability $|\phi_\delta|/\gamma \cdot \tau$, where $\gamma \stackrel{\text{def}}{=} \int |\phi_\delta(s)|\, \tau(ds) \le 1/\sqrt{\delta}$:

$$\int \|(f\star\phi_\delta)(t)\|^2_{\ell^\infty}\, \tau(dt) = \int \Big\|\int f(st)\phi_\delta(s)\, \tau(ds)\Big\|^2_{\ell^\infty} \tau(dt)$$

by A.3.28:
$$\le \int \Big(\int \|f(st)\|_{\ell^\infty} |\phi_\delta(s)|\, \tau(ds)\Big)^2 \tau(dt)$$

$$= \gamma^2 \int \Big(\int \|f(st)\|_{\ell^\infty} |\phi_\delta(s)|/\gamma\, \tau(ds)\Big)^2 \tau(dt)$$

$$\le \gamma^2 \int \int \|f(st)\|^2_{\ell^\infty} |\phi_\delta(s)|/\gamma\, \tau(ds)\, \tau(dt)$$

$$= \gamma^2 \int \|f(t)\|^2_{\ell^\infty}\, \tau(dt) \int |\phi_\delta(s)|/\gamma\, \tau(ds)$$

$$= \gamma^2 \int \|f(t)\|^2_{\ell^\infty}\, \tau(dt) \le \delta^{-1} \int \|f(t)\|^2_{\ell^\infty}\, \tau(dt) \,,$$

so that
$$\|f\star\phi_\delta\|_{L^2(\tau,\ell^\infty)} \le \frac{1}{\sqrt{\delta}} \cdot \|f\|_{L^2(\tau,\ell^\infty)} \,. \tag{4.1.24}$$

The function $Wf\star\phi_\delta$ in the second term Q_2 of inequality (4.1.23) has the form

$$(Wf\star\phi_\delta)(t) = \sum_{3\le|A|\ odd} x_A \sqrt{\delta}^{\,|A|-1} w_A(t) \,.$$

Thus
$$\|\mathcal{I}Wf\star\phi_\delta\|^2_{L^2(\tau)} = \int \Big(\sum_{3\le|A|\ odd} \mathcal{I}x_A \sqrt{\delta}^{\,|A|-1} w_A(t)\Big)^2 \tau(dt)$$

$$= \sum_{3\le|A|\ odd} (\mathcal{I}x_A)^2 \cdot \delta^{|A|-1}$$

$$\le \delta^2 \cdot \sum_{A\subset\{1,\ldots,n\}} (\mathcal{I}x_A)^2 = \delta^2 \cdot \|\mathcal{I}f\|^2_{L^2(\tau)} \,,$$

and therefore, from inequality (4.1.13),

$$\Big\| \|\mathcal{I}W f \star \phi_\delta\|_{L^2(\tau)} \Big\|_{L^p(\mu)} \le \delta \cdot \eta_{p,2}(\mathcal{I}) \cdot \|f\|_{L^2(\tau,\ell^\infty)} \ .$$

Putting this and (4.1.24) into (4.1.23) yields inequality (4.1.21). ▮

If for the function f of lemma 4.1.10 we choose the one provided by lemma 4.1.9, then inequality (4.1.21) turns into

$$\Big\| \Big(\sum_{\nu=1}^{n} |\mathcal{I}x_\nu|^2 \Big)^{1/2} \Big\|_{L^p(\mu)} \le 2^{0\vee(1-p)/p} K_p K_1$$
$$\times \Big[\frac{\|\mathcal{I}\|_p}{\sqrt{\delta}} + \eta_{p,2}(\mathcal{I}) \cdot \delta \Big] \cdot \Big(\sum_{\nu=1}^{n} \|x_\nu\|^2_{\ell^\infty} \Big)^{1/2} .$$

Since this inequality is true for all finite collections $\{x_1, \ldots, x_n\} \subset \mathcal{E}$, it implies

$$\eta_{p,2}(\mathcal{I}) \le 2^{0\vee(1-p)/p} K_p K_1 \Big[\frac{\|\mathcal{I}\|_p}{\sqrt{\delta}} + \eta_{p,2}(\mathcal{I}) \cdot \delta \Big] . \tag{$*$}$$

The function $\delta \mapsto \frac{\|\mathcal{I}\|_p}{\sqrt{\delta}} + \eta_{p,2}(\mathcal{I}) \cdot \delta$ takes its minimum at

$$\delta = \Big(\|\mathcal{I}\|_p / (2\eta_{p,2}) \Big)^{2/3} < 1 \ ,$$

where its value is $\big(2^{1/3} + 2^{-2/3}\big) \|\mathcal{I}\|_p^{2/3} \eta_{p,2}^{1/3}$. Therefore $(*)$ gives

$$\eta_{p,2}(\mathcal{I}) \le \big(2^{1/3} + 2^{-2/3}\big)\, 2^{0\vee(1-p)/p}\, K_p K_1 \cdot \|\mathcal{I}\|_p^{2/3} \eta_{p,2}^{1/3} \ ,$$

and so $\eta_{p,2}(\mathcal{I}) \le \Big(\big(2^{1/3} + 2^{-2/3}\big)\, 2^{0\vee(1-p)/p}\, K_p K_1 \Big)^{3/2} \cdot \|\mathcal{I}\|_p$

by (A.8.9): $\le \Big(2 \cdot 2^{0\vee(1-p)/p}\, 2^{1/p - 1/2}\, 2^{1/2} \Big)^{3/2} \cdot \|\mathcal{I}\|_p$

$$= \Big(2^{1\vee 1/p}\, 2^{1/p} \Big)^{3/2} \cdot \|\mathcal{I}\|_p = \Big(2\sqrt{2} \Big)^{1/p + 1\vee 1/p} \cdot \|\mathcal{I}\|_p \ :$$

Corollary 4.1.11 *A linear map $\mathcal{I} : \ell^\infty(k) \to L^p(\mu)$ is factorizable with*

$$\eta_{p,2}(\mathcal{I}) \le C_p \cdot \|\mathcal{I}\|_p \ , \tag{4.1.25}$$

where

$$C_p \le \Big(\big(2^{1/3} + 2^{-2/3}\big)\, 2^{0\vee(1-p)/p}\, K_p K_1 \Big)^{3/2}$$
$$\le \big(2\sqrt{2}\big)^{1/p + 1\vee 1/p} < 2^{3(2+p)/2p} \ .$$

Theorem 4.1.7, including the estimate (4.1.17) for C_p, is thus true when $\mathcal{E} = \ell^\infty(k)$.

Proof of Theorem 4.1.7. Given $\{x_1, \ldots, x_n\} \subset \mathcal{E}$, there is a finite subalgebra $\mathcal{A}'$ of $\mathcal{A}$, generated by finitely many atoms, say $\{A_1, \ldots, A_k\}$, such that every x_ν is a step function over $\mathcal{A}'$: $x_\nu = \sum_\kappa A_\kappa x_\nu^\kappa$. The linear map $\mathcal{I}' : \ell^\infty(k) \to L^p(\mu)$ that takes the κ^{th} standard basis vector of $\ell^\infty(k)$ to $\mathcal{I}A_\kappa$ has $\|\mathcal{I}'\|_p \le \|\mathcal{I}\|_p$ and takes $(x_\nu^\kappa)_{1\le\kappa\le k} \in \ell^\infty(k)$ to $\mathcal{I}x_\nu$. Since $\|(x_\nu^\kappa)_{1\le\kappa\le k}\|_{\ell^\infty(k)} = \|x_\nu\|_{\mathcal{E}}$, inequality (4.1.25) in corollary 4.1.11 gives

$$\Big\| \Big(\sum_{\nu=1}^{n} |\mathcal{I}x_\nu|^2 \Big)^{1/2} \Big\|_{L^p(\mu)} \le C_p^{(4.1.25)} \cdot \|\mathcal{I}\|_p \cdot \Big(\sum_{\nu=1}^{n} \|x_\nu\|_{\mathcal{E}}^2 \Big)^{1/2} ,$$

and another application of Rosenthal's criterion 4.1.4 yields the theorem. ∎

Proof of Theorem 4.1.2 for $0 < p \le 2 < q < \infty$. Theorem 4.1.7 does not extend to exponents $q > 2$ in general – it is due to the special nature of the stochastic integral, the "closeness of the arguments of $\mathcal{I}$ to its values" expressed for instance by exercise 2.1.14, that theorem 4.1.2 can be extended to $q > 2$. If $\boldsymbol{Z}$ is previsible, then "the values of $\mathcal{I}$ are *very close* to the arguments," and the factorization constant does not even depend on the length d of the vector $\boldsymbol{Z}$.

We start off by having what we know so far produce a probability $\mathbb{P}' = \mathbb{P}/g$ with $\|g\|_{L^{p/(2-p)}(\mathbb{P})} \le E_{p,2}$ for which $\boldsymbol{Z}$ is an L^2-integrator of size

$$\lceil \boldsymbol{Z} \rceil_{\mathcal{I}^2[\mathbb{P}']} \le D_{p,2}^{(4.1.5)} \cdot \lceil \boldsymbol{Z} \rceil_{\mathcal{I}^p[\mathbb{P}]} . \tag{4.1.26}$$

Then $\boldsymbol{Z}$ has a Doob–Meyer decomposition $\boldsymbol{Z} = \widehat{\boldsymbol{Z}} + \widetilde{\boldsymbol{Z}}$ with respect to $\mathbb{P}'$ (see theorem 4.3.1 on page 221) whose components have sizes

$$\lceil \widehat{\boldsymbol{Z}} \rceil_{\mathcal{I}^2[\mathbb{P}']} \le 2 \lceil \boldsymbol{Z} \rceil_{\mathcal{I}^2[\mathbb{P}']} \quad \text{and} \quad \lceil \widetilde{\boldsymbol{Z}} \rceil_{\mathcal{I}^2[\mathbb{P}']} \le 2 \lceil \boldsymbol{Z} \rceil_{\mathcal{I}^2[\mathbb{P}']} , \tag{4.1.27}$$

respectively. We shall estimate separately the factorization constants of the stochastic integrals driven by $\widehat{\boldsymbol{Z}}$ and by $\widetilde{\boldsymbol{Z}}$ and then apply exercise 4.1.6.

Our first claim is this: if $\mathcal{I} : \mathcal{E}^d \to L^p$ is the stochastic integral driven by a d-tuple $\boldsymbol{V}$ of finite variation processes, then, for $0 < p < q < \infty$,

$$\eta_{p,q}(\mathcal{I}) \le \Big\| \sum_{1\le\theta\le d} \lceil V^\theta \rceil_\infty \Big\|_{L^p} . \tag{4.1.28}$$

Since the right-hand side equals $\lceil \boldsymbol{V} \rceil_{\mathcal{I}^p}$ when $\boldsymbol{V}$ is previsible, this together with (4.1.27) and (4.1.26) will result in

$$\eta_{p,q}\Big(\int \cdot \, d\widehat{\boldsymbol{Z}} \Big) \le 2 \lceil \boldsymbol{Z} \rceil_{\mathcal{I}^2[\mathbb{P}']} \le 2\, D_{p,2}^{(4.1.5)} \cdot \lceil \boldsymbol{Z} \rceil_{\mathcal{I}^p[\mathbb{P}]} \tag{4.1.29}$$

for $0 < p < q < \infty$. To prove equation (4.1.28), let $\boldsymbol{X}^1, \ldots, \boldsymbol{X}^n$ be a finite collection of elementary integrands in $\mathcal{E}^d$. Then

$$\sum_\nu \Big| \int \boldsymbol{X}^\nu \, d\boldsymbol{V} \Big|^q \le \sum_\nu \Big| \|\boldsymbol{X}^\nu\|_{\mathcal{E}^d} \sum_{1\le\theta\le d} \lceil V^\theta \rceil_\infty \Big|^q .$$

Applying the L^p-mean to the q^{th} root gives

$$\Big\| \Big(\sum_\nu \Big| \int \boldsymbol{X}^\nu \, d\boldsymbol{V} \Big|^q \Big)^{1/q} \Big\|_{L^p} \le \Big\| \sum_{1 \le \theta \le d} ¦ V^\theta ¦_\infty \Big\|_{L^p} \cdot \Big(\sum_{\nu=1}^n \| \boldsymbol{X}_\nu \|^q_{\mathcal{E}^d} \Big)^{1/q} ,$$

which is equation (4.1.28).

Now to the martingale part. With $\boldsymbol{X}^\nu$ as above set

$$M^\nu \stackrel{\text{def}}{=} \boldsymbol{X}^\nu * \widetilde{\boldsymbol{Z}} , \qquad \nu = 1, \ldots, n ,$$

and for $\vec{m} = (m^1, \ldots, m^n)$ set

$$\Phi(\vec{m}) \stackrel{\text{def}}{=} Q^{2/q}(\vec{m}) \quad \text{, where} \quad Q(\vec{m}) \stackrel{\text{def}}{=} \sum_{\nu=1}^n \big| m^\nu \big|^q .$$

Then $\quad \Phi'_\mu(\vec{m}) = 2 Q^{\frac{2-q}{q}}(\vec{m}) \cdot \big| m^\mu \big|^{q-1} \operatorname{sgn}(m^\mu)$

$$\begin{aligned} \text{and} \quad \Phi''_{\mu\nu}(\vec{m}) &= 2(q-1) Q^{\frac{2-q}{q}}(\vec{m}) \cdot \big| m^\mu \big|^{q-2} \cdot \delta_{\mu\nu} \quad (\text{no sum over } \mu \text{ intended}) \\ &\quad + 2(2-q) Q^{\frac{2-2q}{q}}(\vec{m}) \cdot \big| m^\mu \big|^{q-1} \operatorname{sgn}(m^\mu) \big| m^\nu \big|^{q-1} \operatorname{sgn}(m^\nu) \qquad (*) \\ &\le 2(q-1) Q^{\frac{2-q}{q}}(\vec{m}) \cdot \big| m^\mu \big|^{q-2} \cdot \delta_{\mu\nu} , \end{aligned}$$

on the grounds that the $\mu\nu$-matrix in $(*)$ is negative semidefinite for $q > 2$. Our interest in Φ derives from the fact that criterion 4.1.4 asks us to estimate the expectation of $\Phi(\vec{M}_\infty)$. With $M^\mu_\lambda \stackrel{\text{def}}{=} (1-\lambda) M^\mu_{\cdot-} + \lambda M^\mu$ for short, Itô's formula results in the inequality

$$\begin{aligned} \Phi(\vec{M}_\infty) &\le \Phi(\vec{M}_0) + 2 \int_{0+}^\infty Q^{\frac{2-q}{q}}(\vec{M}_{\cdot-}) \cdot \sum_\mu \big| M^\mu_{\cdot-} \big|^{q-1} \operatorname{sgn}(M^\mu_{\cdot-}) \, dM^\mu \\ &\quad + 2(q-1) \int_0^1 (1-\lambda) \, d\lambda \int_{0+}^\infty Q^{\frac{2-q}{q}} \big(\vec{M}_\lambda \big) \cdot \sum_\mu \big| M^\mu_\lambda \big|^{q-2} \, d[M^\mu, M^\mu] , \\ &\le 2 \int_{0+}^\infty Q^{\frac{2-q}{q}}(\vec{M}_{\cdot-}) \cdot \sum_\mu \big| M^\mu_{\cdot-} \big|^{q-1} \operatorname{sgn}(M^\mu_{\cdot-}) \, dM^\mu \\ &\quad + 2(q-1) \int_0^1 (1-\lambda) \, d\lambda \int_0^\infty Q^{\frac{2-q}{q}} \big(\vec{M}_\lambda \big) \cdot \sum_\mu \big| M^\mu_\lambda \big|^{q-2} \, d[M^\mu, M^\mu] . \end{aligned}$$

Now $$[M^\mu, M^\mu] = \sum_{1 \le \eta, \theta \le d} X^\mu_\eta X^\mu_\theta * [\widetilde{Z}^\eta, \widetilde{Z}^\theta] , \tag{4.1.30}$$

whence[4] $$\mathbb{E}' \Big[\Phi(\vec{M}_\infty) \Big] \le 2(q-1) \int_0^1 (1-\lambda) \times$$

$$\times \, \mathbb{E}' \Big[\int_0^\infty Q^{\frac{2-q}{q}} \big(\vec{M}_\lambda \big) \sum_\mu \big| M^\mu_\lambda \big|^{q-2} X^\mu_\eta X^\mu_\theta \, d[\widetilde{Z}^\eta, \widetilde{Z}^\theta] \Big] \, d\lambda . \tag{4.1.31}$$

---

[4] Einstein's convention, adopted, implies summation over the same indices in opposite positions.

Consider first the case that $d = 1$, writing $\widetilde{Z}, X^\mu$ for the scalar processes $\widetilde{\boldsymbol{Z}}_1, \boldsymbol{X}_1^\mu$. Then $X_\eta^\mu X_\theta^\mu d[\widetilde{Z}^\eta, \widetilde{Z}^\theta] = (X^\mu)^2 \, d[\widetilde{Z}, \widetilde{Z}] \le \| X^\mu \|_{\mathcal{E}}^2 \, d[\widetilde{Z}, \widetilde{Z}]$, and using Hölder's inequality with conjugate exponents $q/2$ and $q/(q-2)$ in the sum over μ turns (4.1.31) into

$$\begin{aligned}
\mathbb{E}'\Big[\Phi(\vec{M}_\infty)\Big] &\le 2(q-1)\int_0^1 (1-\lambda) \\
&\qquad \mathbb{E}'\bigg[\int_0^\infty Q^{\frac{2-q}{q}}(\vec{M}_\lambda)\Big(\sum_\mu |M_\lambda^\mu|^q\Big)^{\frac{q-2}{q}} \\
&\qquad\qquad \times \Big(\sum_\mu \|X^\mu\|_{\mathcal{E}}^q\Big)^{2/q} d[\widetilde{Z}, \widetilde{Z}]\bigg]\, d\lambda \\
&= 2(q-1)\int_0^1 (1-\lambda)\, d\lambda \Big(\sum_\mu \|X^\mu\|_{\mathcal{E}}^q\Big)^{2/q} \mathbb{E}'\Big[[\widetilde{Z}, \widetilde{Z}]_\infty\Big] \\
&= (q-1)\Big\|\widetilde{Z}_\infty\Big\|_{L^2(\mathbb{P}')}^2 \Big(\sum_\mu \|X^\mu\|_{\mathcal{E}}^q\Big)^{2/q} \\
&= (q-1)\,\lceil\!\lceil \widetilde{Z} \rceil\!\rceil_{\mathcal{I}^2[\mathbb{P}']}^2 \Big(\sum_\mu \|X^\mu\|_{\mathcal{E}}^q\Big)^{2/q} .
\end{aligned} \tag{4.1.32}$$

Taking the square root results in

$$\eta_{2,q}\Big(\int \cdot\, d\widetilde{Z}\Big) \le \sqrt{q-1}\,\lceil\!\lceil \widetilde{Z} \rceil\!\rceil_{\mathcal{I}^2[\mathbb{P}']} \le 2\sqrt{q-1}\, D_{p,2}^{(4.1.5)} \cdot \lceil\!\lceil \boldsymbol{Z} \rceil\!\rceil_{\mathcal{I}^p[\mathbb{P}]} . \tag{4.1.33}$$

Now if $\boldsymbol{Z}$ and with it the $[\widetilde{Z}^\eta, \widetilde{Z}^\theta]$ are previsible, then the same inequality holds. To see this we pick an increasing previsible process V so that $d[\widetilde{Z}^\eta, \widetilde{Z}^\theta] \ll dV$, for instance the sum of the variations $\lceil\!\lceil[\widetilde{Z}^\eta, \widetilde{Z}^\theta]\rceil\!\rceil$, and let $G^{\eta\theta}$ be the previsible Radon–Nikodym derivative of $d[\widetilde{Z}^\eta, \widetilde{Z}^\theta]$ with respect to dV. According to lemma A.2.21 b), there is a Borel measurable function γ from the space $\mathcal{G}$ of $d \times d$-matrices to the unit box $\ell_1^\infty(d)$ such that

$$\sup\{x_\eta x_\theta G^{\eta\theta} : \boldsymbol{x} \in \ell_1^\infty\} = \gamma_\eta(G)\gamma_\theta(G)G^{\eta\theta} , \qquad G \in \mathcal{G} .$$

We compose this map γ with the $\mathcal{G}$-valued process $\big(G^{\eta\theta}\big)_{\eta,\theta=1}^d$ and obtain a previsible process $\boldsymbol{Y}$ with $|\boldsymbol{Y}| \le 1$. Let us write $M = \boldsymbol{Y} * \widetilde{\boldsymbol{Z}}$. Then

$$\mathbb{E}'[[M, M]_\infty] \le \lceil\!\lceil \widetilde{\boldsymbol{Z}} \rceil\!\rceil_{\mathcal{I}^2[\mathbb{P}']}^2$$

and in (4.1.30) $\qquad d[M^\mu, M^\mu] \le \|\boldsymbol{X}^\mu\|_{\mathcal{E}^d}^2 \cdot d[M, M]$.

We can continue as in inequality (4.1.32), replacing $[\widetilde{Z},\widetilde{Z}]$ by $[M,M]$, and arrive again at (4.1.33). Putting this inequality together with (4.1.29) into inequalities (4.1.26) and (4.1.27) gives, using exercise 4.1.6,

$$\eta_{p,q}\Big(\int \cdot\ d\boldsymbol{Z}\Big) \le 2^{1\vee 1/p}(\sqrt{q-1}+1)D_{p,2}\ \lceil\!\lceil \boldsymbol{Z} \rceil\!\rceil_{\mathcal{I}^p[\mathbb{P}]}$$

or
$$D_{p,q,d} \le 2^{1\vee 1/p}(1+\sqrt{q-1})D_{p,2} \le 3{\cdot}2^{1+4/p}{\cdot}(1+\sqrt{q-1}) \quad (4.1.34)$$

if $d=1$ or if $\boldsymbol{Z}$ is previsible. We leave it to the reader to show that in general

$$D_{p,q,d} \le \sqrt{d}\cdot d_{p,q}\cdot D_{p,2}\ . \quad (4.1.35)$$

Let $\mathbb{P}'' = \mathbb{P}/g'$ be the probability provided by exercise 4.1.5. It is not hard to see with Hölder's inequality that the estimates $\|g\|_{L^{p/(2-p)}(\mathbb{P})} < 2^{2/p}$ and $\|g'\|_{L^{2/(q-2)}(\mathbb{P}')} < 2^{q/2}$ lead to

$$E_{p,q} \le 4^{q/p} \ \text{ for } 0 < p \le 2 < q\,.$$

Proof of Theorem 4.1.2 (i) Only the case $2 < p < q < \infty$ has not yet been covered. It is not too interesting in the first place and can easily be reduced to the previous case by considering $\boldsymbol{Z}$ an L^2-integrator. We leave this to the reader.

Proof for $p = 0$

If $\boldsymbol{Z}$ is a single L^0-integrator, then proposition 4.1.1 together with a suitable analog of exercise 4.1.6 provides a proof of theorem 4.1.2 when $p=0$. This then can be extended rather simply to the case of finitely many integrators, except that the corresponding constants deteriorate with their number. This makes the argument inapplicable to random measures, which can be thought of as an infinity of infinitesimal integrators (page 173). So we go a different route.

Proof of Theorem 4.1.2 (ii) for $0 < q < 1$. Maurey [67] and Schwartz [68] have shown that the conclusion of theorem 4.1.2 (ii) holds for a continuous linear map $\mathcal{I} : E \to L^0(\mathbb{P})$ on any normed space E, provided the exponent q is strictly less than one.[5] In this subsection we extract the pertinent arguments from their immense work. Later we can then prove the general case $0 < q < \infty$ by applying theorem 4.1.2 (i) to the stochastic integral regarded as an $L^q(\mathbb{P}')$-integrator for a probability $\mathbb{P}' \approx \mathbb{P}$ produced here. For the arguments we need the information on symmetric stable laws and on the stable type of a map or space that is provided on pages 458–463.

The role of theorem 4.1.7 in the proof of theorem 4.1.2 is played for $p=0$ by the following result, reminiscent of proposition 4.1.1:

[5] Theorem 4.1.7 for $0 < p < 1$ is also first proved in their work.

Proposition 4.1.12 *Let E be a normed space, $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space, $\mathcal{I} : E \to L^0(\mathbb{P})$ a continuous linear map, and let $0 < q < 1$.*

(i) For every $\alpha \in (0,1)$ there is a smallest number $C_{[\alpha],q}[\|\mathcal{I}\|_{[\bullet]}] < \infty$, which depends only on the modulus of continuity $\|\mathcal{I}\|_{[\bullet]}$, α, and q, and which has the property that for every finite subset $\{x_1, \ldots, x_n\}$ of E

$$\Big\| \Big(\sum_{\nu=1}^{n} |\mathcal{I}(x_\nu)|^q \Big)^{1/q} \Big\|_{[\alpha]} \le C_{[\alpha],q}[\|\mathcal{I}\|_{[\bullet]}] \cdot \Big(\sum_{\nu=1}^{n} \|x_\nu\|_E^q \Big)^{1/q} . \tag{4.1.36}$$

(ii) For every $\alpha \in (0,1)$ there exist a positive measurable function $k_\alpha \le 1$ satisfying $\mathbb{E}[k_\alpha] \ge 1-\alpha$ and a constant ζ_α such that for all $x \in E$

$$\int |\mathcal{I}(x)|^q \, k_\alpha d\mathbb{P} \le \zeta_\alpha \cdot \|x\|_E^q .$$

(iii) There exists a probability $\mathbb{P}' = \mathbb{P}/g$ equivalent with $\mathbb{P}$ on $\mathcal{F}$ such that $\mathcal{I}$ is continuous as a map into $L^q(\mathbb{P}')$: for all $x \in E$

$$\|\mathcal{I}(x)\|_{L^q(\mathbb{P}')} \le D_q[\|\mathcal{I}\|_{[\bullet]}] \cdot \|x\|_E < \infty , \tag{4.1.37}$$

and such that $$\|g\|_{[\alpha;\mathbb{P}]} \le E_{[\alpha],q}[\|\mathcal{I}\|_{[\bullet]}] < \infty , \quad \forall\, \alpha \in (0,1) . \tag{4.1.38}$$

Again, $D_q[\|\mathcal{I}\|_{[\bullet]}]$ and $E_{[\alpha],q}[\|\mathcal{I}\|_{[\bullet]}]$ depend only on α, q, and $\|\mathcal{I}\|_{[\bullet]}$.

Proof. (i) Let $0 < p < q$ and let $(\gamma_\nu^{(q)})$ be a sequence of independent q-stable random variables, all defined on a probability space $(X, \mathcal{X}, dx)$ (see pages 458–463). In view of lemma A.8.33 (ii) we have, for every $\omega \in \Omega$,

$$\Big(\sum\nolimits_\nu |\mathcal{I}(x_\nu)(\omega)|^q \Big)^{1/q} \le B_{[\beta],q}^{(\mathrm{A.8.12})} \cdot \Big\| \sum\nolimits_\nu \mathcal{I}(x_\nu)(\omega) \gamma_\nu^{(q)} \Big\|_{[\beta;dx]} ,$$

and therefore

$$\begin{aligned} \Big\| \Big(\sum\nolimits_\nu |\mathcal{I}(x_\nu)|^q \Big)^{1/q} \Big\|_{[\alpha;\mathbb{P}]} &\le B_{[\beta],q} \cdot \Big\| \Big\| \sum\nolimits_\nu \mathcal{I}(x_\nu) \gamma_\nu^{(q)} \Big\|_{[\beta;dx]} \Big\|_{[\alpha;\mathbb{P}]} \\ \text{by A.8.16 with } 0<\delta<\alpha\beta: \quad &\le B_{[\beta],q} \cdot \Big\| \Big\| \mathcal{I}\Big(\sum\nolimits_\nu x_\nu \gamma_\nu^{(q)} \Big) \Big\|_{[\delta;\mathbb{P}]} \Big\|_{[\alpha\beta-\delta;dx]} \\ &\le B_{[\beta],q} \cdot \|\mathcal{I}\|_{[\delta]} \cdot \Big\| \Big\| \sum\nolimits_\nu x_\nu \gamma_\nu^{(q)} \Big\|_E \Big\|_{[\alpha\beta-\delta;dx]} \\ \text{by exercise A.8.15:} \quad &\le \frac{B_{[\beta],q} \cdot \|\mathcal{I}\|_{[\delta]}}{(\alpha\beta-\delta)^{1/p}} \cdot \Big\| \Big\| \sum\nolimits_\nu x_\nu \gamma_\nu^{(q)} \Big\|_E \Big\|_{L^p(dx)} \\ \text{by definition A.8.34:} \quad &\le \frac{B_{[\beta],q} \cdot \|\mathcal{I}\|_{[\delta]} \cdot T_{p,q}(E)}{(\alpha\beta-\delta)^{1/p}} \cdot \Big(\sum\nolimits_\nu \|x_\nu\|_E^q \Big)^{1/q} . \end{aligned}$$

Due to proposition A.8.39, the quantity $C_{[\alpha],q}[\|\mathcal{I}\|_{[\bullet]}]^{(4.1.36)}$ is finite. In fact,

$$C_{[\alpha],q}[\|\mathcal{I}\|_{[\bullet]}] \le \inf_{\substack{0<\beta<1 \\ 0<\delta<\alpha\beta \\ 0<p<q}} \frac{B_{[\beta],q}^{(\mathrm{A.8.12})} \cdot T_{p,q}(E) \cdot \|\mathcal{I}\|_{[\delta]}}{(\alpha\beta-\delta)^{1/p}}$$

$$\text{with } \delta = \alpha/4: \qquad \le \frac{B_{[1/2],q} T_{q/2,q}(E) \cdot \|\mathcal{I}\|_{[\alpha/4]}}{(\alpha/4)^{2/q}} \,. \tag{4.1.39}$$

Exercise 4.1.13 $C_{[\alpha],.8,\|\mathcal{I}\|_{[.]}} \le 5000\, \|\mathcal{I}\|_{[\alpha/8]}/\alpha^2$.

(ii) Following the proof of theorem 4.1.7 we would now like to apply Rosenthal's criterion 4.1.4 to produce k_α. It does not apply as stated, but there is an easy extension of its proof, due to Nikisin and used once before in proposition 4.1.1, that yields claim (ii). Namely, inequality (4.1.36) can be read as follows: for every random variable of the form

$$\phi = \phi_{x_1,\ldots,x_n} \stackrel{\text{def}}{=} \sum\nolimits_{\nu=1}^{n} |\mathcal{I}(x_\nu)|^q \,, \qquad n \in \mathbb{N}\,,\ x_\nu \in E \,,$$

on Ω there is the set

$$k_{x_1,\ldots,x_n} \stackrel{\text{def}}{=} \Big[|\phi_{x_1,\ldots,x_n}| \le C^q_{[\alpha],q}[\|\mathcal{I}\|_{[.]}] \cdot \Big(\sum\nolimits_{\nu=1}^{n} \|x_\nu\|_E^q \Big) \Big]$$

of measure $\mathbb{P}\big[k_{x_1,\ldots,x_n}\big] \ge 1-\alpha$ so that

$$\mathbb{E}[\phi_{x_1,\ldots,x_n} \cdot k_{x_1,\ldots,x_n}] \le C^q_{[\alpha],q}[\|\mathcal{I}\|_{[.]}] \cdot \sum\nolimits_{\nu=1}^{n} \|x_\nu\|_E^q \,.$$

Let K be the collection of positive random variables $k \le 1$ on Ω satisfying $\mathbb{E}[k] \ge 1-\alpha$. K is evidently convex and $\sigma(L^\infty, L^1)$-compact. The functions $k \mapsto \mathbb{E}[\phi_{x_1,\ldots,x_n} \cdot k]$ are lower semicontinuous as the suprema of integrals against chopped functions $\phi_{x_1,\ldots,x_n} \wedge j$, $j \in \mathbb{N}$. Therefore the functions

$$h_{x_1,\ldots,x_n} : \ k \mapsto C^q_{[\alpha],q}[\|\mathcal{I}\|_{[.]}] \cdot \sum\nolimits_{\nu=1}^{n} \|x_\nu\|_E^q - \mathbb{E}\big[\phi_{x_1,\ldots,x_n} \cdot k\big]$$

are upper semicontinuous on K, each one having a point of positivity $k_{x_1,\ldots,x_n} \in K$. Their collection clearly forms a convex cone $\mathcal{H}$. Ky–Fan's minimax theorem A.2.34 furnishes a common point $k_\alpha \in K$ of positivity, and this function evidently answers claim (ii), with $\zeta_\alpha = C^q_{[\alpha],q}[\|\mathcal{I}\|_{[.]}]$.

(iii) We construct now $\mathbb{P}'$ as on page 190: We may pick $\alpha \mapsto \zeta_\alpha \ge 1$ be decreasing on $(0,1)$ — for instance choosing α_1 so that $\|\mathcal{I}\|_{[\alpha_1]} \ge 1$

$$\text{we might set} \quad \zeta_\alpha = \zeta_\alpha^+ \stackrel{\text{def}}{=} \frac{B^q_{[1/2],q} \cdot T^q_{q/2,q}(E) \cdot \|\mathcal{I}\|^q_{[\alpha_1 \wedge \alpha/4]}}{(\alpha/4)^2} \qquad \text{(see (4.1.39))}.$$

$$\text{Then we set} \quad g' \stackrel{\text{def}}{=} \gamma' \sum\nolimits_{n=1}^{\infty} 2^{-n} \frac{k_{2^{-n}}}{\zeta_{2^{-n}}} \quad , \quad \text{where } \gamma' \in (\zeta_{1/2}, 4\zeta_{1/2})$$

is chosen so as to make $\mathbb{P}'$ a probability equivalent with $\mathbb{P}$. Then we proceed as after equation (4.1.4) on page 190 to produce (4.1.37) and (4.1.38):

$$\|\mathcal{I}(x)\|_{L^q(\mathbb{P}')} \le (4\zeta_{1/2})^{1/q} \cdot \|x\|_E \quad \text{and} \quad \|g\|_{[\alpha;\mathbb{P}]} \le \frac{8\zeta_{\alpha/4}}{\alpha\zeta_{1/2}} \,. \qquad ■$$

Applying this with $E = \mathcal{E}$ and $\mathcal{I} = \int \cdot\, d\boldsymbol{Z}$ finishes the proof of theorem 4.1.2 for $p = 0 < q < 1$. The choice $\zeta = \zeta^+$ results in the estimates

$$D_{q,d}^{(4.1.8)}[\boldsymbol{z}_{\bullet}] \le D_q^{(4.1.37)}[\boldsymbol{z}_{\bullet}] \le 4^{4/q} \cdot B_{[1/2],q} T_{q/2,q}(E) \cdot \boldsymbol{z}_{\alpha_1 \wedge 1/8} ,$$

where $\boldsymbol{z}_{\bullet} \stackrel{\text{def}}{=} \|\mathcal{I}\|_{[\bullet]} = \lceil\!\lceil \boldsymbol{Z} \rceil\!\rceil_{[\bullet]} : (0,1) \to (0,\infty)$ with $z_{\alpha_1} \ge 1$, and

$$E_{[\alpha],q}^{(4.1.9)}[\boldsymbol{z}_{\bullet}] \le E_{[\alpha],q}^{(4.1.38)}[\boldsymbol{z}_{\bullet}] \le \frac{32\boldsymbol{z}^q_{\alpha_1 \wedge \alpha/16}}{\alpha^3 \cdot \boldsymbol{z}^q_{\alpha_1 \wedge 1/8}} \le \frac{32\boldsymbol{z}^q_{\alpha_1 \wedge \alpha/16}}{\alpha^3} . \tag{4.1.40}$$

Proof of Theorem 4.1.2 (ii) for $1 \le q < \infty$. Let $0 < p < 1$. We know now that there is a probability $\mathbb{P}' = \mathbb{P}/g$ with respect to which $\boldsymbol{Z}$ is a global L^p-integrator. Its size and that of g are controlled by the inequalities

$$\lceil\!\lceil \boldsymbol{Z} \rceil\!\rceil_{\mathcal{I}^p[\mathbb{P}']} \le D_{p,d}^{(4.1.8)}[\lceil\!\lceil \boldsymbol{Z} \rceil\!\rceil_{[\bullet]}] \quad \text{and} \quad \|g\|_{[\alpha;\mathbb{P}]} \le E_{[\alpha],p}^{(4.1.9)}[\lceil\!\lceil \boldsymbol{Z} \rceil\!\rceil_{[\bullet]}] .$$

By part (i) of theorem 4.1.2 there exists a probability $\mathbb{P}'' = \mathbb{P}'/g'$ with respect to which $\boldsymbol{Z}$ is an L^q-integrator of size

$$\lceil\!\lceil \boldsymbol{Z} \rceil\!\rceil_{\mathcal{I}^q[\mathbb{P}'']} \le D_{p,d}^{(4.1.8)}[\lceil\!\lceil \boldsymbol{Z} \rceil\!\rceil_{[\bullet]}] \cdot D_{p,q,d}^{(4.1.5)} .$$

This implies $\quad D_{q,d}^{(4.1.8)}[\lceil\!\lceil \boldsymbol{Z} \rceil\!\rceil_{[\bullet]}] \le D_{p,q,d}^{(4.1.5)} \cdot D_{p,d}^{(4.1.8)}[\lceil\!\lceil \boldsymbol{Z} \rceil\!\rceil_{[\bullet]}]$;

similarly, $\quad E_{[\alpha],q}^{(4.1.9)}[\lceil\!\lceil \boldsymbol{Z} \rceil\!\rceil_{[\bullet]}] \le \Big(E_{[\alpha/2],p}^{(4.1.9)}[\lceil\!\lceil \boldsymbol{Z} \rceil\!\rceil_{[\bullet]}]\Big)^{q/p} \cdot E_{p,q}^{(4.1.6)} \cdot \Big(\frac{2}{\alpha}\Big)^{(q-p)/p}$.

The parameter $p \in (0,1)$ is arbitrary but the same in the previous two inequalities. We leave the last inequality above as an exercise (a sketch of the proof can be found in the Answers).

The following corollary makes a good exercise to test our understanding of the flow of arguments above; it is also needed in chapter 5.

Corollary 4.1.14 (Factorization for Random Measures) *(i) Let ζ be a spatially bounded global $L^p(\mathbb{P})$-random measure, where $0 < p < 2$. There exists a probability $\mathbb{P}'$ equivalent with $\mathbb{P}$ on $\mathcal{F}_\infty$ with respect to which ζ is a spatially bounded global L^2-random measure; furthermore, $d\mathbb{P}'/d\mathbb{P}$ is bounded and there exist universal constants D_p and E_p depending only on p such that*

$$\lceil\!\lceil \zeta \rceil\!\rceil_{\mathcal{I}^2[\mathbb{P}']} \le D_p \cdot \lceil\!\lceil \zeta \rceil\!\rceil_{\mathcal{I}^p[\mathbb{P}]} , \qquad D_p = D_{p,2,d}^{(4.1.5)} ,$$

and such that the Radon–Nikodym derivative $g \stackrel{\text{def}}{=} d\mathbb{P}/d\mathbb{P}'$ is bounded away from zero and satisfies

$$\|g\|_{L^{p/(2-p)}(\mathbb{P})} \le E_p , \qquad E_p = E_{p,2}^{(4.1.6)} ,$$

which has the consequence that for any $r > 0$ and $f \in \mathcal{F}_\infty$

$$\|f\|_{L^r(\mathbb{P})} \le E_p^{p/2r} \cdot \|f\|_{L^{2r/p}[\mathbb{P}']} .$$

(ii) Let ζ be a spatially bounded global $L^0(\mathbb{P})$-random measure with modulus of continuity[6] $\lceil\!\lceil \zeta \rceil\!\rceil_{[.]}$. *There exists a probability $\mathbb{P}' = \mathbb{P}/g$ equivalent with $\mathbb{P}$ on $\mathcal{F}_\infty$ with respect to which ζ is a global L^2-integrator; furthermore, there exist universal constants $D[\lceil\!\lceil \zeta \rceil\!\rceil_{[.]}]$ and $E = E_{[\alpha]}[\lceil\!\lceil \zeta \rceil\!\rceil_{[.]}]$, depending only on $\alpha \in (0,1)$ and the modulus of continuity $\lceil\!\lceil \zeta \rceil\!\rceil_{[.]}$,*

such that $$\lceil\!\lceil \zeta \rceil\!\rceil_{\mathcal{I}^q[\mathbb{P}']} \le D[\lceil\!\lceil \zeta \rceil\!\rceil_{[.]}] , \qquad D = D^{(4.1.8)}_{2,d} ,$$

and $$\|g\|_{[\alpha]} \le E_{[\alpha]}[\lceil\!\lceil \zeta \rceil\!\rceil_{[.]}] \quad \forall \alpha \in (0,1) , \; E = E^{(4.1.9)}_{[\alpha],2}[\lceil\!\lceil \zeta \rceil\!\rceil_{[.]}] ,$$

– *this implies* $$\|f\|_{[\alpha+\beta;\mathbb{P}]} \le \Big(E_{[\alpha]}[\lceil\!\lceil \zeta \rceil\!\rceil_{[.]}]/\beta\Big)^{1/r} \|f\|_{L^r(\mathbb{P}')}$$

for any $f \in \mathcal{F}_\infty$, $r > 0$ and $\alpha, \beta \in (0,1)$.

(iii) When ζ is previsible, the exponent 2 can be replaced by any $q < \infty$.

4.2 Martingale Inequalities

Exercise 3.8.12 shows that the square bracket of an integrator of finite variation is just the sum of the squares of the jumps, a quantity of modest interest. For a martingale integrator M, though, the picture is entirely different: the size of the square function controls the size of the martingale, even of its integrator norm. In fact, in the range $1 \le p < \infty$ the quantities $\lceil\!\lceil M \rceil\!\rceil_{\mathcal{I}^p}$, $\| M^\star_\infty \|_{L^p}$, and $\| S_\infty[M] \|_{L^p}$ are all equivalent in the sense that there are universal constants C_p such that

$$\lceil\!\lceil M \rceil\!\rceil_{\mathcal{I}^p} \le C_p \cdot \|M^\star_\infty\|_{L^p} , \; \|M^\star_\infty\|_{L^p} \le C_p \cdot \|S_\infty[M]\|_{L^p} ,$$

and $$\|S_\infty[M]\|_{L^p} \le C_p \cdot \lceil\!\lceil M \rceil\!\rceil_{\mathcal{I}^p} \tag{4.2.1}$$

for all martingales M. These and related inequalities are proved in this section.

Fefferman's Inequality

The K^q-seminorms are auxiliary seminorms on L^q-integrators, defined for $2 \le q \le \infty$. They appear in Fefferman's famed inequality (4.2.2), and they simplify the proof of inequality (4.2.1) and other inequalities of interest. Towards their definition let us introduce, for every L^0-integrator Z, the class $\mathcal{K}[Z]$ of all $g \in L^2[\mathcal{F}_\infty]$ having the property that

$$\mathbb{E}\Big[[Z,Z]_\infty - [Z,Z]_{T-} \;\Big|\; \mathcal{F}_T \Big] \le \mathbb{E}\big[g^2 \;\big|\; \mathcal{F}_T \big]$$

at all stopping times T. $\mathcal{K}[Z]$ is used to define the seminorm $\| Z \|_{\mathcal{K}^q}$ by

$$\| Z \|_{\mathcal{K}^q} = \inf \big\{ \| g \|_{L^q} : \; g \in \mathcal{K}[Z] \big\} , \qquad 2 \le q \le \infty.$$

---

[6] $\lceil\!\lceil \zeta \rceil\!\rceil_{[\alpha]} \stackrel{\text{def}}{=} \sup\big\{ \| \int \check{X} \, d\zeta \|_{[\alpha;\mathbb{P}]} : \; \check{X} \in \check{\mathcal{E}} , \; |\check{X}| \le 1 \big\}$ for $0 < \alpha < 1$; see page 56.

As usual, this number is ∞ if $\mathcal{K}[Z]$ is void. When $q=\infty$ it is customary to write $\|Z\|_{\mathcal{K}^\infty} = \|Z\|_{BMO}$ and to say Z has ***bounded mean oscillation*** if this number is finite. We collect now a few properties of the seminorm $\|\ \|_{\mathcal{K}^q}$.

Exercise 4.2.1 $\|Z\|_{\mathcal{K}^q} \le \|S_\infty[Z]\|_{L^q}$. $d[Z,Z] \le d[Z',Z']$ implies $\|Z\|_{\mathcal{K}^q} \le \|Z'\|_{\mathcal{K}^q}$.

Lemma 4.2.2 *Let Z be an L^0-integrator and $I \ge 0$ an adapted increasing right-continuous process. Then, for $1 \le p < 2$ with conjugate $p' \overset{\text{def}}{=} p/(p-1)$*

$$\mathbb{E}\Big[\int_0^\infty I\,d[Z,Z]\Big] \le \inf\Big\{\mathbb{E}[I_\infty\cdot g^2] : g \in \mathcal{K}[Z]\Big\}$$
$$\le \Big(\mathbb{E}[I_\infty^{p/(2-p)}]\Big)^{(2-p)/p} \cdot \|Z\|^2_{\mathcal{K}^{p'}}\ .$$

Proof. With the usual understanding that $[Z,Z]_{0-} = 0 = I_{0-}$, integration by parts gives

$$\int_0^\infty I\,d[Z,Z] = \int_0^\infty \big([Z,Z]_\infty - [Z,Z]_{\leftarrow}\big)\,dI$$
$$= \int_0^\infty \big([Z,Z]_\infty - [Z,Z]_{T^\lambda-}\big)\cdot[T^\lambda<\infty]\,d\lambda\ ,$$

where $T^\lambda = \inf\{t : I_t \ge \lambda\}$ are the stopping times appearing in the change-of-variable theorem 2.4.7. Since $[T^\lambda<\infty] \in \mathcal{F}_{T^\lambda}$,

$$\mathbb{E}\Big[\int_0^\infty I\,d[Z,Z]\Big] = \mathbb{E}\Big[\int_0^\infty \mathbb{E}\big[\,[Z,Z]_\infty - [Z,Z]_{T^\lambda-}\ \big|\ \mathcal{F}_{T^\lambda}\big]\cdot[T^\lambda<\infty]\,d\lambda\Big]$$
$$\le \inf\Big\{\mathbb{E}\Big[\int_0^\infty g^2\cdot[T^\lambda<\infty]\,d\lambda\Big] : g\in\mathcal{K}[Z]\Big\}$$
$$\le \inf\Big\{\mathbb{E}\Big[g^2\cdot\int_0^\infty [I_\infty>\lambda]\,d\lambda\Big] : g\in\mathcal{K}[Z]\Big\}$$
$$= \inf\big\{\mathbb{E}\big[I_\infty\cdot g^2\big] : g\in\mathcal{K}[Z]\big\}$$
$$\le \Big(\mathbb{E}\big[I_\infty^{p/(2-p)}\big]\Big)^{(2-p)/p}\cdot\|Z\|^2_{\mathcal{K}^{p'}}\ .$$

The last inequality comes from an application of Hölder's inequality with conjugate exponents $p/(2-p)$ and $p'/2$. ∎

On an L^2-bounded martingale M the norm $\|M\|_{\mathcal{K}^q}$ can be rewritten in a useful way:

Lemma 4.2.3 *For any stopping time T*

$$\mathbb{E}\big[\,[M,M]_\infty - [M,M]_{T-}\ \big|\ \mathcal{F}_T\,\big] = \mathbb{E}\big[\,(M_\infty - M_{T-})^2\ \big|\ \mathcal{F}_T\,\big]\ ,$$

and consequently $\mathcal{K}[M]$ equals the collection

$$\Big\{g\in\mathcal{F}_\infty : \mathbb{E}\big[(M_\infty-M_{T-})^2\ \big|\ \mathcal{F}_T\big] \le \mathbb{E}\big[g^2\ \big|\ \mathcal{F}_T\big] \quad \forall\ \textit{stopping times } T\Big\}\ .$$

Proof. Let $U \geq T$ be a stopping time such that $(M^U - M^T)_{.-}$ is bounded. By proposition 3.8.19

$$(M^U - M^T)^2 = 2(M^U - M^T)_{.-}{*}(M^U - M^T) + [M,M]^U - [M,M]^T,$$

and so

$$[M,M]_U - [M,M]_{T-} = (M_U - M_T)^2 + (\Delta M_T)^2 - 2\int_{T+}^{U} (M - M^T)_{.-}\, d(M - M^T)\;.$$

The integral is the value at U of a martingale that vanishes at T, so its conditional expectation on $\mathcal{F}_T$ vanishes (theorem 2.5.22). Consequently,

$$\begin{aligned}
&\mathbb{E}\big[\,[M,M]_U - [M,M]_{T-}\,\big|\,\mathcal{F}_T\,\big] = \mathbb{E}\big[\,(M_U - M_T)^2 + (\Delta M_T)^2\,\big|\,\mathcal{F}_T\,\big]\\
=&\mathbb{E}\big[\,(M_U - M_{T-} - \Delta M_T)^2 + (\Delta M_T)^2\,\big|\,\mathcal{F}_T\,\big]\\
=&\mathbb{E}\big[\,(M_U - M_{T-})^2 - 2(M_U - M_{T-})\Delta M_T + 2(\Delta M_T)^2\,\big|\,\mathcal{F}_T\,\big]\\
=&\mathbb{E}\big[\,(M_U - M_{T-})^2\,\big|\,\mathcal{F}_T\,\big]\;.
\end{aligned}$$

Now U can be chosen arbitrarily large. As $U \to \infty$, $[M,M]_U \to [M,M]_\infty$ and $M_U^2 \to M_\infty^2$ in L^1-mean, whence the claim. ∎

Since $|M_\infty - M_{T-}| \leq 2M_\infty^\star$, the following consequence is immediate:

Corollary 4.2.4 *For any martingale M, $2M_\infty^\star \in \mathcal{K}^q[M]$ and consequently*

$$\|\,M\,\|_{\mathcal{K}^q} \leq 2\,\|\,M_\infty^\star\,\|_{L^q}\;, \qquad\qquad 2 \leq q < \infty\;.$$

Lemma 4.2.5 *Let I, D be positive bounded right-continuous adapted processes, with I increasing, D decreasing, and such that $I{\cdot}D$ is still increasing. Then for any bounded martingale N with $|N_\infty| \leq D_\infty$ and any $q \in [2,\infty)$*

$$2I_\infty{\cdot}D_\infty \in \mathcal{K}[I_{.-}{*}N] \quad \textit{and thus} \quad \|\,I_{.-}{*}N\,\|_{\mathcal{K}^q} \leq 2\,\|\,I_\infty \cdot D_\infty\,\|_{L^q}\;.$$

Proof. Let T be a stopping time. The stochastic integral in the quantity

$$Q \stackrel{\text{def}}{=} \mathbb{E}\big[\,\big((I_{.-}{*}N)_\infty - (I_{.-}{*}N)_{T-}\big)^2\,\big|\,\mathcal{F}_T\,\big]$$

that must be estimated is the limit of

$$I_{T-}{\cdot}\Delta N_T + \sum_{k=0}^{\infty} I_{S_k}{\cdot}(N_{S_{k+1}} - N_{S_k})$$

as the partition $\mathcal{S} = \{T = S_0 \leq S_1 \leq S_2 \leq \ldots\}$ runs through a sequence $(\mathcal{S}^n)$ whose mesh tends to zero. (See theorem 3.7.23 and proposition 3.8.21.)

If we square this and take the expectation, the usual cancellation occurs, and

$$\begin{aligned} Q &\le \lim_{\mathcal{S}} \mathbb{E}\Big[(I_{T-}\cdot\Delta N_T)^2 + \sum_{0\le k} I^2_{S_k}\cdot(N^2_{S_{k+1}} - N^2_{S_k}) \;\Big|\; \mathcal{F}_T\Big] \\ &\le \lim_{\mathcal{S}} \mathbb{E}\Big[(I_{T-}\cdot\Delta N_T)^2 + \sum_{0\le k} I^2_{S_{k+1}}\cdot N^2_{S_{k+1}} - \sum_{0\le k} I^2_{S_k}\cdot N^2_{S_k} \;\Big|\; \mathcal{F}_T\Big] \\ &= \mathbb{E}\Big[I^2_{T-}\cdot(N_T - N_{T-})^2 + I^2_\infty\cdot N^2_\infty - I^2_T\cdot N^2_T \;\Big|\; \mathcal{F}_T\Big] \\ &\le \mathbb{E}\Big[I^2_{T-}\cdot\big(N^2_T - 2N_T N_{T-} + N^2_{T-}\big) + I^2_\infty\cdot N^2_\infty - I^2_{T-}\cdot N^2_T \;\Big|\; \mathcal{F}_T\Big] \\ &\le \mathbb{E}\Big[I^2_{T-}\cdot\big(2|N_T||N_{T-}| + |N^2_{T-}|\big) + I^2_\infty\cdot N^2_\infty \;\Big|\; \mathcal{F}_T\Big] \end{aligned}$$

results. Now $|N_T| \le D_T \le D_{T-}$ and $|N_{T-}| \le D_{T-}$, so we continue

$$Q \le \mathbb{E}\big[3I^2_{T-}\cdot D^2_{T-} + I^2_\infty\cdot D^2_\infty \;\big|\; \mathcal{F}_T\big] \le 4\cdot\mathbb{E}\big[I^2_\infty\cdot D^2_\infty \;\big|\; \mathcal{F}_T\big] .$$

This says, in view of lemma 4.2.3, that $2I_\infty\cdot D_\infty \in \mathcal{K}[I_{.-}{*}N]$. ▮

Exercise 4.2.6 The conclusion persists for unbounded I and D.

Theorem 4.2.7 (Fefferman's Inequality) *For any two L^0-integrators Y, Z and $1 \le p \le 2$*

$$\mathbb{E}\big[\,¦[Y,Z]¦_\infty\big] \le \sqrt{2/p}\cdot\|\,S_\infty[Y]\,\|_{L^p}\cdot\|\,Z\,\|_{\mathcal{K}^{p'}} . \tag{4.2.2}$$

Proof. Let us abbreviate $S = S[Y]$. The mean value theorem produces

$$S^p_t - S^p_s = \big(S^2_t\big)^{p/2} - \big(S^2_s\big)^{p/2} = (p/2)\,\sigma^{p/2\,-1}\cdot\big(S^2_t - S^2_s\big) ,$$

where σ is a point between S^2_s and S^2_t. Since $p \le 2$, we have

$$p/2 - 1 \le 0$$

and
$$(S^2_t)^{p/2\,-1} \le \sigma^{p/2\,-1} ;$$

thus
$$(p/2)\cdot S^{p-2}_t\cdot\big(S^2_t - S^2_s\big) \le S^p_t - S^p_s ,$$

and by the same token
$$(p/2)\cdot S^{p-2}_0\cdot S^2_0 \le S^p_0 .$$

We read this as a statement about the measures $d(S^p)$ and $d(S^2)$:

$$S^{p-2}\cdot d(S^2) \le (2/p)\,d(S^p)$$

(exercise 4.2.9). In conjunction with the theorem 3.8.9 of Kunita–Watanabe, this yields the estimate

$$\begin{aligned} ¦[Y,Z]¦_\infty &= \int_0^\infty S^{p/2\,-1}\cdot S^{1-\,p/2}\,d¦[Y,Z]¦ \\ &\le \Big(\int_0^\infty S^{p-2}\cdot d(S^2)\Big)^{1/2}\cdot\Big(\int_0^\infty S^{2-p}\,d[Z,Z]\Big)^{1/2} \\ &\le \sqrt{2/p}\cdot\Big(\int_0^\infty d(S^p)\Big)^{1/2}\cdot\Big(\int_0^\infty S^{2-p}\,d[Z,Z]\Big)^{1/2} . \end{aligned}$$

Upon taking the expectation and applying the Cauchy–Schwarz inequality,

$$\mathbb{E}[\,\vert [Y,Z] \vert_\infty] \le \sqrt{2/p} \cdot \big(\mathbb{E}[S^p_\infty]\big)^{1/2} \cdot \Big(\mathbb{E}\Big[\int_0^\infty S^{2-p}\, d[Z,Z]\Big]\Big)^{1/2}$$

follows. From lemma 4.2.2 with $I = S^{2-p}$,

$$\begin{aligned}\mathbb{E}[\,\vert [Y,Z]\vert_\infty] &\le \sqrt{2/p} \cdot (\mathbb{E}[S^p_\infty])^{1/2} \cdot \Big(\big(\mathbb{E}[S^p_\infty]\big)^{(2-p)/p} \cdot \|Z\|^2_{\mathcal{K}^{p'}}\Big)^{1/2} \\ &= \sqrt{2/p} \cdot \|S_\infty\|_{L^p} \cdot \|Z\|_{\mathcal{K}^{p'}} \ .\end{aligned}$$

From corollary 4.2.4 and Doob's maximal theorem 2.5.19, the following consequence is immediate:

Corollary 4.2.8 *Let Z be an L^0-integrator and M a martingale. Then*

$$\begin{aligned}\mathbb{E}\big[\,\vert [Z,M]\vert_\infty\big] &\le 2\sqrt{2/p} \cdot \|S_\infty[Z]\|_{L^p} \cdot \|M^\star_\infty\|_{L^{p'}} \\ &\le 2\sqrt{2p} \cdot \|S_\infty[Z]\|_{L^p} \cdot \|M_\infty\|_{L^{p'}} \ .\end{aligned} \qquad 1 \le p \le 2$$

Exercise 4.2.9 Let $y, z, f : [0,\infty) \to [0,\infty)$ be right-continuous with left limits, y and z increasing. If $f_0 \cdot y_0 \le z_0$ and $f_t \cdot (y_t - y_s) \le z_t - z_s \quad \forall\, s < t$, then

$$f \cdot dy \le dz \ .$$

Exercise 4.2.10 Let M, N be two locally square integrable martingales. Then

$$\mathbb{E}[\ \vert [M,N] \vert_\infty\ ] \le \sqrt{2} \cdot \mathbb{E}[S_\infty[M]] \cdot \|N\|_{BMO} \ .$$

This was Fefferman's original result, enabling him to show that the martingales N with $\|N\|_{BMO} < \infty$ form the dual of the subspace of martingales in $\mathcal{I}^1$.

Exercise 4.2.11 For any local martingale M and $1 \le p \le 2$,

$$\vert M \vert_{\mathcal{I}^p} \le C_p \cdot \|S_\infty[M]\|_{L^p} \tag{4.2.3}$$

with $C_2 = 1$ and $C_p \le 2\sqrt{2p}$.

The Burkholder–Davis–Gundy Inequalities

Theorem 4.2.12 *Let $1 \le p < \infty$ and M a local martingale. Then*

$$\|M^\star_\infty\|_{L^p} \le C_p \cdot \|S_\infty[M]\|_{L^p} \tag{4.2.4}$$

and

$$\|S_\infty[M]\|_{L^p} \le C_p \cdot \|M^\star_\infty\|_{L^p} \ . \tag{4.2.5}$$

The arguments below provide the following bounds for the constants C_p:

$$C_p^{(4.2.4)} \le \begin{cases} \sqrt{10p}\,, & 1 \le p < 2, \\ 2\,, & p = 2, \\ p\,\sqrt{e/2}\,, & 2 < p < \infty \end{cases} ; \ C_p^{(4.2.5)} \le \begin{cases} 6/\sqrt{p}\,, & 1 \le p \le 2, \\ 1\,, & p = 2, \\ \sqrt{2p}\,, & 2 \le p < \infty. \end{cases} \tag{4.2.6}$$

Proof of (4.2.4) for $2 \le p < \infty$. Let $K > 0$ and set $T = \inf\{t : |M_t| > K\}$. Itô's formula gives

$$
\begin{aligned}
|M_T|^p &= |M_0|^p + p \int_{0+}^T |M_{.-}|^{p-1} \operatorname{sgn} M_{.-} \, dM \\
&\quad + p(p-1) \int_0^1 (1-\lambda) \int_{0+}^T \big|(1-\lambda) M_{.-} + \lambda M\big|^{p-2} \, d[M,M] \, d\lambda \\
&\le p \int_{0+}^T |M_{.-}|^{p-1} \operatorname{sgn} M_{.-} \, dM + \frac{p(p-1)}{2} \int_0^T |M|^{\star(p-2)} \, d[M,M] \, .
\end{aligned}
$$

If $\| S_\infty[M] \|_{L^p} = \infty$, there is nothing to prove. And in the opposite case, $M_T^\star \le K + S_T[M]$ belongs to L^p, and M^T is a global L^p-integrator (theorem 2.5.30). The stochastic integral on the left in the last line above has a bounded integrand and is the value at T of a martingale vanishing at zero (exercise 3.7.9), and thus has expectation zero. Applying Doob's maximal theorem 2.5.19 and Hölder's inequality with conjugate exponents $p/(p-2)$ and $p/2$, we get

$$
\begin{aligned}
\mathbb{E}\big[|M_T|^{\star p}\big] &\le \frac{p^p}{(p-1)^p} \cdot \mathbb{E}\big[|M_T|^p\big] \\
&\le \frac{p^p \cdot p(p-1)}{(p-1)^p \cdot 2} \cdot \mathbb{E}\Big[\int_0^T |M|^{\star(p-2)} \, d[M,M]\Big] \\
&\le \frac{p^2 \cdot p^{p-1}}{2 \cdot (p-1)^{p-1}} \cdot \mathbb{E}\Big[M_T^{\star(p-2)} \cdot \big(S_T[M]\big)^2\Big] \qquad (4.2.7) \\
&\le \frac{p^2}{2}\Big(1 + \frac{1}{p-1}\Big)^{p-1} \cdot \Big(\mathbb{E}[M_T^{\star p}]\Big)^{(p-2)/p} \Big(\mathbb{E}\big[(S_T[M])^p\big]\Big)^{2/p} < \infty \, .
\end{aligned}
$$

Division by $\mathbb{E}\big[M_T^{\star p}\big]^{1-\,2/p}$ and taking the square root results in

$$
\| M_T^\star \|_{L^p} \le p \cdot \sqrt{\Big(1 + \frac{1}{p-1}\Big)^{p-1} \Big/ 2} \cdot \| S_T[M] \|_{L^p} \le p\sqrt{e/2} \cdot \| S_T[M] \|_{L^p} \, .
$$

Now we let K and with it T increase without bound. ∎

Exercise 4.2.13 For $2 \le q < \infty$, $\| M \|_{\mathcal{K}^q}/2 \le \big\| M_\infty^\star \big\|_{L^q} \le \sqrt{e/2}\, q \cdot \| M \|_{\mathcal{K}^q}$.

Proof of (4.2.4) for $1 \le p \le 2$. Let $\mathcal{T}$ be the collection of stopping times reducing M to a martingale. Doob's maximal theorem 2.5.19 and exercise 4.2.11 produce

$$
\| M_\infty^\star \|_{L^p} \le p' \sup_{t<\infty, T \in \mathcal{T}} \Big\| M_t^T \Big\|_{L^p} \le p' C_p^{(4.2.3)} \cdot \| S_\infty[M] \|_{L^p} \, .
$$

This applies only for $p > 1$, though, and the estimate $p' \, 2\sqrt{2p}$ of the constant has a pole at $p = 1$. So we must argue differently for the general case. We use the maximal theorem for integrators 2.3.6: an application of exercise 4.2.11 gives

$$\|M_\infty^\star\|_{L^p} \le C_p^{\star(2.3.5)} C_p^{(4.2.3)} \cdot \|S_\infty[M]\|_{L^p} \le 6.7 \cdot 2^{1/p}\sqrt{p} \cdot \|S_\infty[M]\|_{L^p} .$$

The constant is a factor of 4 larger than the $\sqrt{10p}$ of the statement. We borrow the latter value from Garsia [36]. ▮

Proof of (4.2.5) for $1 \le p \le 2$. By homogeneity we may assume that $\|M_\infty^\star\|_{L^p}=1$. Then, using Hölder's inequality with conjugate exponents $2/p$ and $2/(2-p)$,

$$\begin{aligned}\mathbb{E}[\,(S_\infty[M])^p] &= \mathbb{E}\Big[\Big(M_\infty^{\star(p-2)} \cdot [M,M]_\infty\Big)^{p/2} \cdot M_\infty^{\star p(2-p)/2}\Big] \\ &\le \Big(\mathbb{E}\big[M_\infty^{\star(p-2)} \cdot [M,M]_\infty\big]\Big)^{p/2} ,\end{aligned}$$

i.e., $$\|S_\infty[M]\|_{L^p}^2 \le \mathbb{E}\big[M_\infty^{\star(p-2)} \cdot [M,M]_\infty\big] .$$

With $$\begin{aligned}[M,M]_\infty &= M_\infty^2 - 2\int_0^\infty M_{.-}\,dM \\ &\le M_\infty^{\star 2} - 2\int_0^\infty M_{.-}\,dM ,\end{aligned}$$

this turns into $$\|S_\infty[M]\|_{L^p}^2 \le 1 + 2 \cdot \Big|\mathbb{E}\Big[M_\infty^{\star(p-2)} \cdot \int_0^\infty M_{.-}\,dM\Big]\Big| .$$

Now, if $M_\infty^{\star(p-2)}$ is bounded let N be the martingale that has $N_\infty = M_\infty^{\star(p-2)}$. We employ lemma 4.2.5 with $I = M^\star$ and $D = M^{*(p-2)}$. The previous inequality can be continued as

$$\begin{aligned}\|S_\infty[M]\|_{L^p}^2 &\le 1 + 2 \cdot \Big|\mathbb{E}\big[N_\infty \cdot M_{.-}{*}M_\infty\big]\Big| \\ &= 1 + 2 \cdot \Big|\mathbb{E}\big[[M, M_{.-}{*}N]_\infty\big]\Big| \\ \text{by theorem 4.2.7:} \quad &\le 1 + 2\sqrt{2/p} \cdot \|S_\infty[M]\|_{L^p} \cdot \|M_{.-}{*}N\|_{\mathcal{K}^{p'}} \\ \text{by exercise 4.2.1:} \quad &\le 1 + 2\sqrt{2/p} \cdot \|S_\infty[M]\|_{L^p} \cdot \big\|M_{.-}^\star{*}N\big\|_{\mathcal{K}^{p'}} \\ \text{by lemma 4.2.5:} \quad &\le 1 + 4\sqrt{2/p} \cdot \|S_\infty[M]\|_{L^p} \cdot \Big\|M_\infty^{\star(p-1)}\Big\|_{L^{p'}} \\ &= 1 + 4\sqrt{2/p} \cdot \|S_\infty[M]\|_{L^p} .\end{aligned}$$

Completing the square we get

$$\|S_\infty[M]\|_{L^p} \le \sqrt{1+8/p} + \sqrt{8/p} < 6/\sqrt{p} .$$

If $M_\infty^{\star(p-2)}$ is not bounded, then taking the supremum over bounded martingales N with $N_\infty \le M_\infty^{\star(p-2)}$ achieves the same thing. ▮

Proof of (4.2.5) for $2 \le p < \infty$. Set $S = S[M]$. An easy argument as in the proof of theorem 4.2.7 gives

$$d(S^p) \le \frac{p}{2}\cdot S^{p-2}\, d(S^2) = \frac{p}{2}\cdot S^{p-2}\, d[M,M]\;,$$

and thus
$$\mathbb{E}[S^p_\infty] \le \frac{p}{2}\cdot \mathbb{E}\Big[\int_0^\infty S^{p-2}\, d[M,M]\Big]\;.$$

Lemma 4.2.2 and corollary 4.4.4 allow us to continue with

$$\mathbb{E}[S^p_\infty] \le 2p\cdot \mathbb{E}\big[S^{p-2}_\infty \cdot M^{\star 2}_\infty\big] \le 2p\cdot \big(\mathbb{E}[S^p_\infty]\big)^{(p-2)/p}\cdot \big(\mathbb{E}[M^{\star p}_\infty]\big)^{2/p}\;.$$

We divide by $\mathbb{E}\big[S^p_\infty\big]^{1-2/p}$, take the square root, and arrive at

$$\|\, S_\infty[M]\,\|_{L^p} \le \sqrt{2p}\cdot \|\, M^\star_\infty\,\|_{L^p}\;.$$

∎

Here is an interesting little application of the Burkholder–Davis–Gundy inequalities:

Exercise 4.2.14 (A Strong Law of Large Numbers) In a generalization of exercise 2.5.17 on page 75 prove the following: let $F_1, F_2, \ldots$ be a sequence of random variables that have bounded q^{th} moments for some fixed $q > 1$: $\|\, F_\nu\,\|_{L^q} \le \sigma_q$, all having the same expectation p. Assume that the conditional expectation of F_{n+1} given $F_1, F_2, \ldots, F_n$ equals p as well, for $n = 1, 2, 3, \ldots$. [To paraphrase: knowledge of previous executions of the experiment may influence the law of its current replica only to the extent that the expectation does not change and the q^{th} moments do not increase overly much.] Then

$$\lim_{n\to\infty} \frac{1}{n}\sum_{\nu=1}^n F_\nu = p \quad \text{almost surely.}$$

The Hardy Mean

The following observation will throw some light on the merit of inequality (4.2.4). Proposition 3.8.19 gives $[X{*}M, X{*}M] = X^2{*}[M,M]$ for elementary integrands X. Inequality (4.2.4) applied to the local martingale $X{*}M$ therefore yields

$$\Big\|\int X\, dM\Big\|_{L^p} \le C_p\cdot\Big(\int\Big(\int_0^\infty X^2\, d[M,M]\Big)^{p/2} d\mathbb{P}\Big)^{1/p} \tag{4.2.8}$$

for $1 \le p < \infty$. The corresponding assignment

$$F \;\mapsto\; \|F\|^{\mathcal{H}*}_{M\text{–}p} \stackrel{\text{def}}{=} \Big(\int^*\Big(\int^* F_t^2(\omega)\, d[M,M]_t(\omega)\Big)^{p/2} \mathbb{P}(d\omega)\Big)^{1/p} \tag{4.2.9}$$

is a ***pathwise mean*** in the sense that it computes first for every single path $t \mapsto F_t(\omega)$ separately a quantity, $\big(\int^* F_t^2(\omega)\, d[M,M]_t(\omega)\big)^{1/2}$ in this case, and then applies a p-mean to the resulting random variable. It is called the ***Hardy mean***. It controls the integral in the sense that

$$\left\| \int X \, dM \right\|_{L^p} \le C_p^{(4.2.4)} \cdot \| X \|_{M\text{–}p}^{\mathcal{H}*}$$

for elementary integrands X, and it can therefore be used to extend the elementary integral just as well as Daniell's mean can. It offers "pathwise" or "ω–by–ω" control of the integrand, and such is of paramount importance for the solution of stochastic differential equations – for more on this see sections 4.5 and 5.2.

The Hardy mean is the favorite mean of most authors who treat stochastic integration *for martingale integrators* and exponents $p \ge 1$.

How do the Hardy mean and Daniell's mean compare? The minimality of Daniell's mean on previsible processes (exercises 3.6.16 and 3.5.7(ii)) gives

$$\| F \|_{M\text{–}p}^{*} \le C_p^{(4.2.4)} \cdot \| F \|_{M\text{–}p}^{\mathcal{H}*} \tag{4.2.10}$$

for $1 \le p < \infty$ and *all previsible* F. In fact, if M is continuous, so that $S[M]$ agrees with the previsible square function $s[M]$, then inequality (4.2.10) extends to all $p > 0$ (exercise 4.3.20). On the other hand, proposition 3.8.19 and equation (3.7.5) produce for all elementary integrands X

$$\left(\int \left(\int_0^\infty X^2 \, d[M,M] \right)^{p/2} d\mathbb{P} \right)^{1/p} \le K_p^{(3.8.6)} \cdot ¦ X{*}M ¦_{\mathcal{I}^p} = K_p \cdot \| X \|_{M\text{–}p}^{*} ,$$

which due to proposition 3.6.1 results in the converse of inequality (4.2.10):

$$\| F \|_{M\text{–}p}^{\mathcal{H}*} \le K_p \cdot \| F \|_{M\text{–}p}^{*}$$

for $0 < p < \infty$ and for all functions F on the ambient space. In view of the integrability criterion 3.4.10, both $\| \ \|_{M\text{–}p}^{*}$ and $\| \ \|_{M\text{–}p}^{\mathcal{H}*}$ have the same *previsible* integrable processes:

$$\mathcal{P} \cap \mathfrak{L}^1[[\![\ ]\!]_{M\text{–}p}^{*}] = \mathcal{P} \cap \mathfrak{L}^1[\| \ \|_{M\text{–}p}^{\mathcal{H}*}] \ \text{ and } \ [\![\ ]\!]_{M\text{–}p}^{*} \approx \| \ \|_{M\text{–}p}^{\mathcal{H}*}$$

on this space for $1 \le p < \infty$, and even for $0 < p < \infty$ if M happens to be continuous. Here is an instance where $\| \ \|_{M\text{–}p}^{\mathcal{H}*}$ is nevertheless preferable: Suppose that M is *continuous*; then so is $[M, M]$, and $\| \ \|_{M\text{–}p}^{\mathcal{H}*}$ annihilates the graph of any random time – Daniell's mean $\| \ \|_{M\text{–}p}^{*}$ may well fail to do so. Now a well-measurable process differs from a predictable one only on the graphs of countably many stopping times (exercise A.5.18). Thus a well-measurable process is $\| \ \|_{M\text{–}p}^{\mathcal{H}*}$-measurable, and all well-measurable processes with finite mean are integrable if the mean $\| \ \|_{M\text{–}p}^{\mathcal{H}*}$ is employed. For another instance see the proof of theorem 4.2.15. ∎

Martingale Representation on Wiener Space

Consider an L^p-integrator $\boldsymbol{Z}$. The definite integral $\boldsymbol{X} \mapsto \int \boldsymbol{X}\, d\boldsymbol{Z}$ is a map from $\mathfrak{L}^1[\boldsymbol{Z}\text{–}p]$ to L^p. One might reasonably ask what its kernel and range are. Not much can be said when $\boldsymbol{Z}$ is arbitrary; but if it is a Wiener process and $p > 1$, then there is a complete answer (see also theorem 4.6.10 on page 261):

Theorem 4.2.15 *Assume that* $\boldsymbol{W} = (W^1, \ldots, W^d)$ *is a standard d-dimensional Wiener process on its natural filtration* $\mathcal{F}_{\boldsymbol{.}}[\boldsymbol{W}]$, *and let* $1 < p < \infty$. *Then for every* $f \in L^p(\mathcal{F}_\infty[\boldsymbol{W}])$ *there is a unique* $\boldsymbol{W}$*–p-integrable vector* $\boldsymbol{X} = (X^1, \ldots, X^d)$ *of previsible processes so that*

$$f = \mathbb{E}[f] + \int_0^\infty \boldsymbol{X}\, d\boldsymbol{W} \;.$$

Put slightly differently, the martingale $M^f_{\boldsymbol{.}} \stackrel{\text{def}}{=} \mathbb{E}[f|\mathcal{F}_{\boldsymbol{.}}[\boldsymbol{W}]]$ *has the representation*

$$M^f_t = \mathbb{E}[f] + \int_0^t \boldsymbol{X}\, d\boldsymbol{W} \;.$$

Proof (Si–Jian Lin). Denote by $\boldsymbol{H}$ the discrete space $\{1, \ldots, d\}$ and by $\check{\boldsymbol{B}}$ the set $\boldsymbol{H} \times \boldsymbol{B}$ equipped with its elementary integrands $\check{\mathcal{E}} \stackrel{\text{def}}{=} C(\boldsymbol{H}) \otimes \mathcal{E}$. As on page 109, a d-vector of processes on $\boldsymbol{B}$ is identified with a function on $\check{\boldsymbol{B}}$. According to theorem 2.5.19 and exercise 4.2.18, the stochastic integral

$$\boldsymbol{X} \mapsto \int \boldsymbol{X}\, d\boldsymbol{W} = \sum_{\eta=1}^{d} \int_0^\infty X^\eta\, dW^\eta$$

is up to a constant an isometry of $\mathcal{P} \cap \mathfrak{L}^1[\|\ \|^*_{\boldsymbol{W}\text{–}p}]$ onto a subspace of $L^p(\mathcal{F}_\infty[\boldsymbol{W}])$. Namely, for $1 < p < \infty$ and with $M \stackrel{\text{def}}{=} \boldsymbol{X}{*}\boldsymbol{W}$

$$\Big\| \int \boldsymbol{X}\, d\boldsymbol{W} \Big\|_p = \|M_\infty\|_p$$

by theorems 2.5.19 and 4.2.12: $\quad \sim \|M^\star_\infty\|_p \sim \|S_\infty[M]\|_p = \|\boldsymbol{X}\|^{\mathcal{H}*}_{\boldsymbol{W}\text{–}p}$

by definition (4.2.9): $\quad \sim \|\boldsymbol{X}\|^*_{\boldsymbol{W}\text{–}p}\;.$

The image of $\mathfrak{L}^1[\|\ \|^*_{\boldsymbol{W}\text{–}p}]$ under the stochastic integral is thus a complete and therefore closed subspace $\mathcal{S} \subset \big\{f \in L^p(\mathcal{F}_\infty[\boldsymbol{W}]) : \mathbb{E}[f] = 0\big\}$. Since bounded pointwise convergence implies mean convergence in L^p, the subspace $\mathcal{S}_b(\mathbb{C})$ of bounded functions in the complexification of $\mathbb{R} \oplus \mathcal{S}$ forms a bounded monotone class. According to exercise A.3.5, it suffices to show that $\mathcal{S}_b(\mathbb{C})$ contains a complex multiplicative class $\mathcal{M}$ that generates $\mathcal{F}_\infty[\boldsymbol{W}]$. $\mathcal{S}_b(\mathbb{C})$ will then contain all bounded $\mathcal{F}_\infty[\boldsymbol{W}]$-measurable random variables and its closure all of $L^p_{\mathbb{C}}$. We take for $\mathcal{M}$ the multiples of random variables of the form

$$\exp\big(i\boldsymbol{\phi}{*}\boldsymbol{W}_\infty\big) = e^{i\int_0^\infty \sum_\eta \phi^\eta(s)\, dW^\eta_s}\;,$$

the ϕ^η being real bounded Borel functions that vanishing past some instant each. $\mathcal{M}$ is clearly closed under multiplication and complex conjugation and contains the constants. To see that it is contained in $\mathcal{S}_b(\mathbb{C})$, consider the Doléans–Dade exponential

$$\mathscr{E}_t = 1 + \int_0^t i\mathscr{E}_s \sum_\eta \phi^\eta(s)\, dW_s^\eta = 1 + \big(\mathscr{E}{*}i(\boldsymbol{\phi}{*}\boldsymbol{W})\big)_t$$

by (3.9.4): $$= \exp\Big(i\boldsymbol{\phi}{*}\boldsymbol{W}_t + 1/2\int_0^t |\boldsymbol{\phi}(s)|^2\, ds\Big)$$

of $i\boldsymbol{\phi}{*}\boldsymbol{W}$. Clearly $\mathscr{E}_\infty = \exp\big(i\boldsymbol{\phi}{*}\boldsymbol{W}_\infty + c\big)$ belongs to $\mathcal{S}_b(\mathbb{C})$, and so does the scalar multiple $\exp\big(i\boldsymbol{\phi}{*}\boldsymbol{W}_\infty\big)$. To see that the σ-algebra $\mathcal{F}$ generated by $\mathcal{M}$ contains $\mathcal{F}_\infty[\boldsymbol{W}]$, differentiate $\exp\big(i\tau\int\boldsymbol{\phi}\,d\boldsymbol{W}\big)$ at $\tau = 0$ to conclude that $\int\boldsymbol{\phi}\,d\boldsymbol{W} \in \mathcal{F}$. Then take $\phi^\eta = [0,t]$ for $\eta = 1,\ldots,d$ to see that $\boldsymbol{W}_t \in \mathcal{F}$ for all t. We leave to the reader the following straightforward generalization from finite auxiliary space to continuous auxiliary space: ∎

Corollary 4.2.16 (Generalization to Wiener Random Measure) *Let β be a Wiener random measure with intensity rate ν on the auxiliary space $\boldsymbol{H}$, as in definition 3.10.5. The filtration $\mathcal{F}_{\bullet}$ is the one generated by β (ibidem).*

(i) For $0 < p < \infty$, THE *Daniell mean and the* ***Hardy mean***

$$\check{F} \;\mapsto\; \|F\|^{\mathcal{H}*}_{\beta\text{–}p} \stackrel{\text{def}}{=} \Big(\int^* \Big(\int^* \check{F}_s^2(\eta;\omega)\,\nu(d\eta)ds\Big)^{p/2} \mathbb{P}(d\omega)\Big)^{1/p}$$

agree on the previsibles $\check{F} \in \check{\mathcal{P}} \stackrel{\text{def}}{=} \mathcal{B}^\bullet(\boldsymbol{H}) \otimes \mathcal{P}$, up to a multiplicative constant.

(ii) For every $f \in L^p(\mathcal{F}_\infty)$, $1 < p < \infty$, there is a $\beta{-}p$-integrable predictable random function $\check{X}$, unique up to indistinguishability, so that

$$f = \mathbb{E}[f] + \int_{\check{\boldsymbol{B}}} \check{X}(\eta,s)\,\beta(d\eta,ds)\ .$$

Additional Exercises

Exercise 4.2.17 $\|Z\|_{\mathcal{K}^q} \le \|S_\infty[Z]\|_{L^q} \le \sqrt{q/2}\cdot\|Z\|_{\mathcal{K}^q}$ for $2 \le q \le \infty$.

Exercise 4.2.18 Let $1 \le p < \infty$ and M a local martingale. Then

$$\big\|M^\star_\infty\big\|_{L^p} \le C_p \cdot \lceil\!\lceil M \rceil\!\rceil_{\mathcal{I}^p}\ , \tag{4.2.11}$$

$$\lceil\!\lceil M \rceil\!\rceil_{\mathcal{I}^p} \le C_p \cdot \|S_\infty[M]\|_{L^p}\ , \tag{4.2.12}$$

and $$\big\|(X{*}M)^\star_T\big\|_{L^p} \le C_p^{(4.2.4)} \cdot \Big\|\Big(\int_0^T X^2\, d[M,M]\Big)^{1/2}\Big\|_{L^p}$$

for any previsible X and stopping time T, with

$$C_p^{(4.2.11)} \le \begin{cases} C_p^{(4.2.4)}\cdot K_p^{(3.8.6)} \le 2^{1/p}\sqrt{5p} \le 5 & \text{for } 1 \le p \le 1.3, \\ p' = p/(p{-}1) \le 5 & \text{for } 1.3 \le p \le 2, \\ p' = p/(p{-}1) \le 2 & \text{for } 2 \le p < \infty, \end{cases}$$

and $$C_p^{(4.2.12)} \le C_p^{(4.2.3)} \wedge C_p^{(4.2.4)} \le \begin{cases} 2\sqrt{2p} & \text{for } 1 \le p < 2, \\ 1 & \text{for } p = 2, \\ \sqrt{e/2} \cdot p & \text{for } 2 < p < \infty. \end{cases}$$

Inequality (4.2.6) permits an estimate of the constant $A_p^{(2.5.6)}$:

$$A_p^{(2.5.6)} \le C_p^{(4.2.12)} \cdot C_p^{(4.2.5)} \cdot p' \le \begin{cases} 17p' & \text{for } 1 < p < 2, \\ 1 & \text{for } p = 2, \\ \sqrt{e}p^{3/2}p' & \text{for } 2 < p < \infty. \end{cases}$$

Exercise 4.2.19 Let $p, q, r > 1$ with $1/r = 1/q + 1/p$ and M an L^p-bounded martingale. If X is previsible and its maximal function is measurable and finite in L^q-mean, then X is M–r-integrable.

Exercise 4.2.20 A standard Wiener process W is an L^p-integrator for all $p < \infty$, of size $\lceil W^t \rceil_{\mathcal{I}^p} \le p\sqrt{et/2}$ for $p > 2$ and $\lceil W^t \rceil_{\mathcal{I}^p} \le \sqrt{t}$ for $0 < p \le 2$.

Exercise 4.2.21 Let $T^{c+} = \inf\{t : |W_t| > c\}$ and $T^c = \inf\{t : |W_t| \ge c\}$, where W is a standard Wiener process and $c \ge 0$. Then $\mathbb{E}[T^{c+}] = \mathbb{E}[T^c] = c^2$.

Exercise 4.2.22 (Martingale Representation in General) For $1 \le p < \infty$ let $\mathcal{H}_0^p$ denote the Banach space of $\mathbb{P}$-martingales M on $\mathcal{F}_{.}$ that have $M_0 = 0$ and that are global L^p-integrators. The ***Hardy space*** $\mathcal{H}_0^p$ carries the integrator norm $M \mapsto \lceil M \rceil_{\mathcal{I}^p} \sim \|S_\infty[M]\|_p$ (see inequality (4.2.1)). A closed linear subspace $\mathcal{S}$ of $\mathcal{H}_0^p$ is called ***stable*** if it is closed under stopping ($M \in \mathcal{S} \implies M^T \in \mathcal{S} \quad \forall T \in \mathfrak{T}$). The ***stable span*** $\mathcal{A}^{\|}$ of a set $\mathcal{A} \subset \mathcal{H}_0^p$ is defined as the smallest closed stable subspace containing $\mathcal{A}$. It contains with every finite collection $\boldsymbol{M} = \{M^1, \ldots, M^n\} \subset \mathcal{A}$, considered as a random measure having auxiliary space $\{1, \ldots, n\}$, and for every $\boldsymbol{X} = (X_i) \in \mathfrak{L}^1[\boldsymbol{M}\text{–}p]$, the indefinite integral $\boldsymbol{X}{*}\boldsymbol{M} = \sum_i X_i{*}M^i$; in fact, $\mathcal{A}^{\|}$ is the closure of the collection of all such indefinite integrals.

If $\mathcal{A}$ is finite, say $\mathcal{A} = \{M^1, \ldots, M^n\}$, and a) $[M^i, M^j] = 0$ for $i \ne j$ or b) $\boldsymbol{M}$ is previsible or b') the $[M^i, M^j]$ are previsible or c) $p = 2$ or d) $n = 1$, then the set $\{\boldsymbol{X}{*}\boldsymbol{M} : \boldsymbol{X} \in \mathfrak{L}^1[\boldsymbol{M}\text{–}p]\}$ of indefinite integrals is closed in $\mathcal{H}_0^p$ and therefore equals $\mathcal{A}^{\|}$; in other words, then every martingale in $\mathcal{A}^{\|}$ has a representation as an indefinite integral against the M^i.

Exercise 4.2.23 (Characterization of $\mathcal{A}^{\|}$) The dual $\mathcal{H}_0^{p*}$ of $\mathcal{H}_0^p$ equals $\mathcal{H}_0^{p'}$ when the conjugate exponent p' is finite and equals BMO_0 when $p = 1$ and then $p' = \infty$; the pairing is $(M, M') \mapsto \langle M|M' \rangle \stackrel{\text{def}}{=} \mathbb{E}[M_\infty \cdot M'_\infty]$ in both cases ($M \in \mathcal{H}_0^p, M' \in \mathcal{H}_0^{p*}$). A martingale M' in $\mathcal{H}_0^{p*}$ is called ***strongly perpendicular*** to $M \in \mathcal{H}_0^p$, denoted $M \perp\!\!\!\perp M'$, if $[M, M']$ is a (then automatically uniformly integrable) martingale. M' is strongly perpendicular to all $M \in \mathcal{A} \subset \mathcal{H}_0^p$ if and only if it is perpendicular to every martingale in $\mathcal{A}^{\|}$, that is to say, if and only if $\mathbb{E}[M'_\infty \cdot M_\infty] = 0 \quad \forall\, M \in \mathcal{A}^{\|}$. The collection of all such martingales $M' \in \mathcal{H}_0^{p*}$ is denoted by $\mathcal{A}^{\perp\!\!\!\perp}$. It is a stable subspace of $\mathcal{H}_0^{p*}$, and $(\mathcal{A}^{\perp\!\!\!\perp})^{\perp\!\!\!\perp} = \mathcal{A}^{\|}$.

Exercise 4.2.24 (Continuation: Martingale Measures) Let $G' \stackrel{\text{def}}{=} 1 + M'$, with $\mathcal{A}^{\perp\!\!\!\perp} \ni M' > -1$. Then $\mathbb{P}' \stackrel{\text{def}}{=} G'_\infty \mathbb{P}$ is a probability, equivalent with $\mathbb{P}$ and equal to $\mathbb{P}$ on $\mathcal{F}_0$, for which every element of $\mathcal{A}^{\|}$ is a martingale. For this reason such $\mathbb{P}'$ is called a ***martingale measure*** for $\mathcal{A}$. The set $\mathfrak{M}[\mathcal{A}]$ of martingale measures for $\mathcal{A}$ is evidently convex and contains $\mathbb{P}$. $\mathcal{A}^{\perp\!\!\!\perp}$ contains no bounded martingale other than zero if and only if $\mathbb{P}$ is an extremal point of $\mathfrak{M}[\mathcal{A}]$. Assume now $\boldsymbol{M} = \{M^1, \ldots, M^n\} \subset \mathcal{H}_0^p$ has bounded jumps, and $M^i \perp\!\!\!\perp M^j$ for $i \ne j$. Then every martingale $M \in \mathcal{H}_0^p$ has a representation $M = \boldsymbol{X}{*}\boldsymbol{M}$ with $\boldsymbol{X} \in \mathfrak{L}^1[\boldsymbol{M}\text{–}p]$ if and only if $\mathbb{P}$ is an extremal point of $\mathfrak{M}[\boldsymbol{M}]$.

4.3 The Doob–Meyer Decomposition

Throughout the remainder of the chapter the probability $\mathbb{P}$ *is fixed, and the filtration* $(\mathcal{F}_\bullet, \mathbb{P})$ *satisfies the natural conditions. As usual, mention of* $\mathbb{P}$ *is suppressed in the notation.*

In this section we address the question of finding a canonical decomposition for an L^p-integrator Z. The classes in which the constituents of Z are sought are the finite variation processes and the local martingales. The next result is about as good as one might expect. Its estimates hold only in the range $1 \le p < \infty$.

Theorem 4.3.1 *An adapted process* Z *is a local* L^1*-integrator if and only if it is the sum of a right-continuous previsible process* $\widehat{Z}$ *of finite variation and a local martingale* $\widetilde{Z}$ *that vanishes at time zero. The decomposition*

$$Z = \widehat{Z} + \widetilde{Z}$$

is unique up to indistinguishability and is termed the ***Doob–Meyer decomposition*** *of* Z. *If* Z *has continuous paths, then so do* $\widehat{Z}$ *and* $\widetilde{Z}$. *For* $1 \le p < \infty$ *there are universal constants* $\widehat{C}_p$ *and* $\widetilde{C}_p$ *such that*

$$\lceil \widehat{Z} \rceil_{\mathcal{I}^p} \le \widehat{C}_p \cdot \lceil Z \rceil_{\mathcal{I}^p} \quad \text{and} \quad \lceil \widetilde{Z} \rceil_{\mathcal{I}^p} \le \widetilde{C}_p \cdot \lceil Z \rceil_{\mathcal{I}^p} . \tag{4.3.1}$$

The size of the martingale part $\widetilde{Z}$ *is actually controlled by the square function of* Z *alone:*

$$\lceil \widetilde{Z} \rceil_{\mathcal{I}^p} \le \widetilde{C}'_p \cdot \| S_\infty[Z] \|_{L^p} . \tag{4.3.2}$$

The previsible finite variation part $\widehat{Z}$ *is also called the* ***compensator*** *or* ***dual previsible projection*** *of* Z, *and the local martingale part* $\widetilde{Z}$ *is called its* ***compensatrix*** *or "****Z compensated.****"*

The proof below (see page 227 ff.) furnishes the estimates

$$\widetilde{C}'^{(4.3.2)}_p \le \begin{cases} 2\sqrt{2p} < 4 & \text{for } 1 \le p < 2, \\ 1 & \text{for } p = 2, \\ pC^{(4.2.5)}_{p'} \le 6p/\sqrt{p'} & \text{for } 2 < p < \infty, \end{cases}$$

$$\widetilde{C}^{(4.3.1)}_p \le \begin{cases} 4.1 & \text{for } 1 \le p < 2, \\ 1 & \text{for } p = 2, \\ 6p & \text{for } 2 < p < \infty, \end{cases} \tag{4.3.3}$$

$$\widehat{C}^{(4.3.1)}_p \le \begin{cases} 1 & \text{for } p = 1, \\ 5.1 & \text{for } 1 < p < 2, \\ 2 & \text{for } p = 2, \\ 6.5\,p & \text{for } 2 < p < \infty. \end{cases}$$

In the range $0 \le p < 1$, a weaker statement is true: an L^p-integrator is the sum of a local martingale and a process of finite variation; but the decomposition is neither canonical nor unique, and the sizes of the summands cannot in general be estimated. These matters are taken up below (section 4.4).

Doléans–Dade Measures and Processes

The main idea in the construction of the Doob–Meyer decomposition 4.3.1 of a local L^1-integrator Z is to analyze its ***Doléans-Dade measure*** μ_Z. This is defined on all bounded previsible and locally Z–1-integrable processes X by

$$\mu_Z(X) = \mathbb{E}\Big[\int X\,dZ\Big]$$

and is evidently a σ-finite σ-additive measure on the previsibles $\mathcal{P}$ that vanishes on evanescent processes. Suppose it were known that every measure μ on $\mathcal{P}$ with these properties has a ***predictable representation*** in the form

$$\mu(X) = \mathbb{E}\Big[\int X\,dV^\mu\Big]\,, \qquad X \in \mathcal{P}_b\,,$$

where V^μ is a right-continuous predictable process of finite variation – such V^μ is known as a ***Doléans–Dade process for*** $\boldsymbol{\mu}$. Then we would simply set $\widehat{Z} \stackrel{\text{def}}{=} V^{\mu_Z}$ and $\widetilde{Z} \stackrel{\text{def}}{=} Z - \widehat{Z}$. Inasmuch as

$$\mathbb{E}\Big[\int X\,d\widetilde{Z}\Big] = \mathbb{E}\Big[\int X\,dZ\Big] - \mathbb{E}\Big[\int X\,dV^{\mu_Z}\Big] = 0$$

on (many) previsibles $X \in \mathcal{P}_b$, the difference $\widetilde{Z}$ would be a (local) martingale and $Z = \widehat{Z} + \widetilde{Z}$ would be a Doob–Meyer decomposition of Z: the battle plan is laid out.[7]

It is convenient to investigate first the case when μ is totally finite:

Proposition 4.3.2 *Let μ be a σ-additive measure of bounded variation on the σ-algebra $\mathcal{P}$ of predictable sets and assume that μ vanishes on evanescent sets in $\mathcal{P}$. There exists a right-continuous predictable process V^μ of integrable total variation $\lvert V^\mu\rvert_\infty$, unique up to indistinguishability, such that for all bounded previsible processes X*

$$\mu(X) = \mathbb{E}\Big[\int_0^\infty X\,dV^\mu\Big]\,. \tag{4.3.4}$$

Proof. Let us start with a little argument showing that if such a Doléans–Dade process V^μ exists, then it is unique. To this end fix t and $g \in L^\infty(\mathcal{F}_t)$, and let M^g be the bounded right-continuous martingale whose value at any instant s is $M^g_s = \mathbb{E}[g|\mathcal{F}_s]$ (example 2.5.2). Let $M^g_{.-}$ be the left-continuous

[7] There are other ways to establish theorem 4.3.1. This particular construction, via the correspondence $Z \to \mu_Z$ and $\mu \to V^\mu$, is however used several times in section 4.5.

version of M^g and in (4.3.4) set $X = M_0^g \cdot [\![0]\!] + M_{.-}^g \cdot (\!(0,t]\!]$. Then from corollary 3.8.23

$$\mu(X) = \mathbb{E}[M_0^g V_0^\mu] + \mathbb{E}\Big[\int_{0+}^t M_{.-}^g \, dV^\mu\Big] = \mathbb{E}[g V_t^\mu] \, .$$

In other words, V_t^μ is a Radon–Nikodym derivative of the measure

$$\mu^t : g \mapsto \mu\big(M_0^g \cdot [\![0]\!] + M_{.-}^g \cdot (\!(0,t]\!]\big) \, , \qquad\qquad g \in L^\infty(\mathcal{F}_t) \, ,$$

with respect to $\mathbb{P}$, both μ^t and $\mathbb{P}$ being regarded as measures on $\mathcal{F}_t$. This determines V_t^μ up to a modification. Since V^μ is also right-continuous, it is unique up to indistinguishability (exercise 1.3.28).

For the existence we reduce first of all the situation to the case that μ is positive, by splitting μ into its positive and negative parts. We want to show that then there exists an increasing right-continuous predictable process I with $\mathbb{E}[I_\infty] < \infty$ that satisfies (4.3.4) for all $X \in \mathcal{P}_b$. To do that we stand the uniqueness argument above on its head and *define* the random variable $I_t \in L^1_+(\mathcal{F}_t, \mathbb{P})$ as the Radon–Nikodym derivative of the measure μ^t on $\mathcal{F}_t$ with respect to $\mathbb{P}$. Such a derivative does exist: μ^t is clearly additive. And if (g_n) is a sequence in $L^\infty(\mathcal{F}_t)$ that decreases pointwise $\mathbb{P}$-a.s. to zero, then $\big(M^{g_n}\big)$ decreases pointwise, and thanks to Doob's maximal lemma 2.5.18, $\inf_n\big(M_{.-}^{g_n}\big)$ is zero except on an evanescent set. Consequently,

$$\lim_{n\to\infty} \mu^t(g_n) = \lim_{n\to\infty} \mu\big(M_0^{g_n} \cdot [\![0]\!] + M_{.-}^{g_n} \cdot (\!(0,t]\!]\big) = 0 \, .$$

This shows at the same time that μ^t is σ-additive and that it is absolutely continuous with respect to the restriction of $\mathbb{P}$ to $\mathcal{F}_t$. The Radon–Nikodym theorem A.3.22 provides a derivative $I_t = d\mu^t/d\mathbb{P} \in L^1_+(\mathcal{F}_t, \mathbb{P})$. In other words, I_t is defined by the equation

$$\mu\big(M_0^g \cdot [\![0]\!] + M_{.-}^g \cdot (\!(0,t]\!]\big) = \mathbb{E}[M_t^g \cdot I_t] \, , \qquad\qquad g \in L^\infty(\mathcal{F}_\infty) \, .$$

Taking differences in this equation results in

$$\begin{aligned} \mu\big(M_{.-}^g \cdot (\!(s,t]\!]\big) &= \mathbb{E}\big[M_t^g I_t - M_s^g I_s\big] = \mathbb{E}\big[g \cdot (I_t - I_s)\big] \qquad (4.3.5) \\ &= \mathbb{E}\Big[\int g \cdot (\!(s,t]\!] \, dI\Big] \end{aligned}$$

for $0 \le s < t \le \infty$. Taking $g = [I_s > I_t]$ we see that I is increasing. Namely, the left-hand side of equation (4.3.5) is then positive and the right-hand side negative, so that both must vanish. This says that $I_t \ge I_s$ a.s. Taking $t_n \downarrow s$ and $g = [\inf_n I_{t_n} > I_s]$ we see similarly that I is right-continuous in L^1-mean. I is thus a global L^1-integrator, and we may and shall replace it by its right-continuous modification (theorem 2.3.4). Another look at (4.3.5) reveals that μ equals the Doléans–Dade measure of I, at least on processes of the form $g\cdot(\!(s,t]\!]$, $g\in\mathcal{F}_s$. These processes generate the predictables, and so $\mu=\mu^I$ on all of $\mathcal{P}$. In particular,

$$\mathbb{E}[M_tI_t-M_0I_0]=\mu\big(M_{.-}\cdot(\!(0,t]\!]\big)=\mathbb{E}\Big[\int_{0+}^t M_{.-}\,dI\Big]$$

for bounded martingales M. Taking differences turns this into

$$\mathbb{E}\Big[\int g\cdot(\!(t,\infty)\!)\,dI\Big]=\mathbb{E}\Big[\int M^g_{.-}\cdot(\!(t,\infty)\!)\,dI\Big]$$

for all bounded random variables g with attached right-continuous martingales $M^g_t=\mathbb{E}[g|\mathcal{F}_t]$. Now $M^g_{.-}\cdot(\!(t,\infty)\!)$ is the predictable projection of $(\!(t,\infty)\!)\cdot g$ (corollary A.5.15 on page 439), so the equality above can be read as

$$\mathbb{E}\Big[\int X\,dI\Big]=\mathbb{E}\Big[\int X^{\mathcal{P},\mathbb{P}}\,dI\Big]\,, \tag{$*$}$$

at least for X of the form $(\!(t,\infty)\!)\cdot g$. Now such X generate the measurable σ-algebra on $\boldsymbol{B}$, and the bounded monotone class theorem implies that $(*)$ holds for all bounded measurable processes X (ibidem).

On the way to proving that I is predictable another observation is useful: *at a predictable time S the jump ΔI_S is measurable on $\mathcal{F}_{S-}$*:

$$\Delta I_S\in\mathcal{F}_{S-}\ . \tag{$**$}$$

To see this, let f be a bounded $\mathcal{F}_S$-measurable function and set $g\stackrel{\text{def}}{=}f-\mathbb{E}[f|\mathcal{F}_{S-}]$ and $M^g_t\stackrel{\text{def}}{=}\mathbb{E}[g|\mathcal{F}_t]$. Then M^g is a bounded martingale that vanishes at any time strictly prior to S and is constant after S. Thus $M^g\cdot[\![0,S]\!]=M^g\cdot[\![S]\!]$ has predictable projection $M^g_{.-}[\![S]\!]=0$ and

$$\mathbb{E}\big[f\cdot\big(\Delta I_S-\mathbb{E}[\Delta I_S|\mathcal{F}_{S-}]\big)\big]=\mathbb{E}[g\Delta I_S]=\mathbb{E}\Big[\int M^g[\![0,S]\!]\,dI\Big]=0\ .$$

This is true for all $f\in\mathcal{F}_S$, so $\Delta I_S=\mathbb{E}[\Delta I_S|\mathcal{F}_{S-}]$.

Now let $a\ge 0$ and let P be a previsible subset of $[\Delta I>a]$, chosen so that $\mathbb{E}[\int P\,dI]$ is maximal. We want to show that $N\stackrel{\text{def}}{=}[\Delta I>a]\setminus P$ is evanescent. Suppose it were not. Then

$$0<\mathbb{E}\Big[\int N\,dI\Big]=\mathbb{E}\Big[\int N^{\mathcal{P},\mathbb{P}}\,dI\Big]\,,$$

so that $N^{\mathcal{P},\mathbb{P}}$ could not be evanescent. According to the predictable section theorem A.5.14, there would exist a predictable stopping time S with $[\![S]\!]\subset\big[N^{\mathcal{P},\mathbb{P}}>0\big]$ and $\mathbb{P}[S<\infty]>0$. Then

$$0<\mathbb{E}\big[N^{\mathcal{P},\mathbb{P}}_S[S<\infty]\big]=\mathbb{E}\big[N_S[S<\infty]\big]\ .$$

Now either $N_S = 0$ or $\Delta I_S > a$. The predictable[8] reduction $S' \stackrel{\text{def}}{=} S_{[\Delta I_S > a]}$ still would have $\mathbb{E}\big[N_{S'}[S' < \infty]\big] > 0$, and consequently

$$\mathbb{E}\Big[\int N \cap [\![S']\!]\, dI\Big] > 0 \,.$$

Then $P_0 \stackrel{\text{def}}{=} [\![S']\!] \setminus P$ would be a previsible non-evanescent subset of N with $\mathbb{E}[\int P_0\, dI] > 0$, in contradiction to the maximality of P.

That is to say, $[\Delta I > a] = P$ is previsible, for all $a \geq 0$: ΔI is previsible. Then so is $I = I_- + \Delta I$; and since this process is right-continuous, it is even predictable. ∎

Exercise 4.3.3 A right-continuous increasing process $I \in \mathfrak{D}$ is previsible if and only if its jumps occur only at predictable stopping times and if, in addition, the jump ΔI_T at a stopping time T is measurable on the strict past $\mathcal{F}_{T-}$ of T.

Exercise 4.3.4 Let $V = {}^c V + {}^j V$ be the decomposition of the càdlàg predictable finite variation process V into continuous and jump parts (see exercise 2.4.6). Then the sparse set $[\Delta V \neq 0] = [\Delta {}^j V \neq 0]$ is previsible and is, in fact, the disjoint union of the graphs of countably many predictable stopping times [use theorem A.5.14].

Exercise 4.3.5 A supermartingale $Z \geq 0$ right-continuous in probability has a Doob–Meyer decomposition $Z = \widehat{Z} + \widetilde{Z}$ with $\lceil \widehat{Z} \rceil_{\mathcal{I}^1} < \infty$ and with uniformly integrable martingale part $\widetilde{Z}$ iff $\{Z_T : T \in \mathfrak{T}[\mathcal{F}_.]\,,\ T < \infty\}$ is uniformly integrable.

Proof of Theorem 4.3.1: Necessity, Uniqueness, and Existence

Since a local martingale is a local L^1-integrator (corollary 2.5.29) and a predictable process of finite variation has locally bounded variation (exercise 3.5.4 and corollary 3.5.16) and is therefore a local L^p-integrator for every $p > 0$ (proposition 2.4.1), a process having a Doob–Meyer decomposition is necessarily a local L^1-integrator.

Next the uniqueness. Suppose that $Z = \widehat{Z} + \widetilde{Z} = \widehat{Z}' + \widetilde{Z}'$ are two Doob–Meyer decompositions of Z. Then $M \stackrel{\text{def}}{=} \widetilde{Z} - \widetilde{Z}' = \widehat{Z}' - \widehat{Z}$ is a predictable local martingale of finite variation that vanishes at zero. We know from exercise 3.8.24 (i) that M is evanescent.

Let us make here an observation to be used in the existence proof. Suppose that Z stops at the time T: $Z = Z^T$. Then $Z = \widehat{Z} + \widetilde{Z}$ and $Z = \big(\widehat{Z}\big)^T + \big(\widetilde{Z}\big)^T$ are both Doob–Meyer decompositions of Z, so they coincide. That is to say, if Z has a Doob–Meyer decomposition at all, then its predictable finite variation and martingale parts also stop at time T. Doing a little algebra one deduces from this that if Z vanishes strictly before time S, i.e., on $[\![0, S)\!)$, and is constant after time T, i.e., on $[\![T, \infty)\!)$, then the parts of its Doob–Meyer decomposition, should it have any, show the same behavior.

Now to the existence. Let (T_n) be a sequence of stopping times that reduce Z to global L^1-integrators and increase to infinity. If we can produce Doob–Meyer decompositions

$$Z^{T_{n+1}} - Z^{T_n} = V^n + M^n$$

[8] See $(**)$ and lemma 3.5.15 (iv).

for the global L^1-integrators on the left, then $Z = \sum_n V^n + \sum_n M^n$ will be a Doob–Meyer decomposition for Z – note that at every point $\varpi \in \boldsymbol{B}$ this is a finite sum. In other words, *we may assume that Z is a global L^1-integrator.*

Consider then its Doléans–Dade measure μ:

$$\mu(X) = \mathbb{E}\Big[\int X\, dZ\Big] , \qquad X \in \mathcal{P}_b ,$$

and let $\widehat{Z}$ be the predictable process V^μ of finite variation provided by proposition 4.3.2. From

$$\mathbb{E}\Big[\int X\, d(Z - \widehat{Z})\Big] = 0$$

it follows that $\widetilde{Z} \stackrel{\text{def}}{=} Z - \widehat{Z}$ is a martingale. $Z = \widehat{Z} + \widetilde{Z}$ is the sought-after Doob–Meyer decomposition. ∎

Exercise 4.3.6 Let $T > 0$ be a predictable stopping time and Z a global L^1-integrator with Doob–Meyer decomposition $Z = \widehat{Z} + \widetilde{Z}$. Then the jump $\Delta\widehat{Z}_T$ equals $\mathbb{E}\left[\Delta Z_T \middle| \mathcal{F}_{T-}\right]$. The predictable finite variation and martingale parts of any continuous local L^1-integrator are again continuous. For any local L^1-integrator Z

$$\Big\| S_\infty[\widehat{Z}] \Big\|_{L^q} \le \sqrt{q/2} \cdot \| S_\infty[Z] \|_{L^q} , \qquad 2 \le q < \infty .$$

Exercise 4.3.7 If I and J are increasing processes with $\mu_I \le \mu_J$ – which we also write $dI \le dJ$, see page 406 – then $d\widehat{I} \le d\widehat{J}$ and $\widehat{I} \le \widehat{J}$.

Exercise 4.3.8 A local L^1-integrator Z with bounded jumps is locally an L^q-integrator for all $q \in (0, \infty)$ (see corollary 4.4.3 on page 234 for much more).

Exercise 4.3.9 Let Z be a local L^1-integrator. There are arbitrarily large stopping times U such that Z agrees on the right-open interval $[\![0, U)\!)$ with a process that is a global L^q-integrator for all $q \in (0, \infty)$.

Exercise 4.3.10 Let X be a bounded previsible process. The Doob–Meyer decomposition of $X{*}Z$ is $X{*}Z = X{*}\widehat{Z} + X{*}\widetilde{Z}$.

Exercise 4.3.11 Let μ, V^μ be as in proposition 4.3.2 and let D be a bounded previsible process. The Doléans–Dade process of the measure $D \cdot \mu$ is $D{*}V^\mu$.

Exercise 4.3.12 Let V, V' be previsible positive increasing processes with associated Doléans–Dade measures $\mu_V, \mu_{V'}$ on $\mathcal{P}$. The following are equivalent: (i) for almost every $\omega \in \Omega$ the measure $dV_t(\omega)$ is absolutely continuous with respect to $dV'_t(\omega)$ on $\mathcal{B}^\bullet(\mathbb{R}_+)$; (ii) μ_V is absolutely continuous with respect to $\mu_{V'}$; and (iii) there exists a previsible process G such that $\mu_V = G{\cdot}\mu_{V'}$. In this case $dV_t(\omega) = G_t(\omega){\cdot}dV'_t(\omega)$ on $\mathbb{R}_+$, for almost all $\omega \in \Omega$.

Exercise 4.3.13 Let V be an adapted right-continuous process of integrable total variation $|V|$ and with $V_0 = 0$, and let $\mu = \mu_V$ be its Doléans–Dade measure. We know from proposition 2.4.1 that $|V|_{\mathcal{I}^p} \le \| |V|_\infty \|_{L^p}$, $0 < p < \infty$. If V is previsible, then the variation process $|V|$ is indistinguishable from the Doléans–Dade process of $|\mu|$, and equality obtains in this inequality: for $0 < p < \infty$

$$\| Y \|_{V-p} = \Big\| \int Y\, d|V| \Big\|_{L^p} \quad \text{and} \quad |V|_{\mathcal{I}^p} = \Big\| |V|_\infty \Big\|_{L^p} , \qquad Y \in \mathcal{E}_+ .$$

Exercise 4.3.14 (Fundamental Theorem of Local Martingales [76]) A local martingale M is the sum of a finite variation process and a locally square integrable local martingale (for more see corollary 4.4.3 and proposition 4.4.1).

Exercise 4.3.15 (Emery) With the natural conditions in force, let $0 < p < \infty$, $0 < q < \infty$, and $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. There are universal constants $C_{p,q}$ such that for every global L^q-integrator Z and previsible integrand X

$$\lceil X{*}Z \rceil_{\mathcal{I}^r} \le C_{p,q} \cdot \| X^\star_\infty \|_{L^p} \cdot \lceil Z \rceil_{\mathcal{I}^q} .$$

Proof of Theorem 4.3.1: The Inequalities

Let $Z = \widehat{Z} + \widetilde{Z}$ be the Doob–Meyer decomposition of Z. We may assume that Z is a global L^p-integrator, else there is nothing to prove. If $p = 1$, then

$$\lceil \widehat{Z} \rceil_{\mathcal{I}^1} = \sup\Big\{ \mathbb{E}\Big[\int X \, dZ \Big] : \ X \in \mathcal{E}_1 \Big\} \le \lceil Z \rceil_{\mathcal{I}^1} ,$$

so inequality (4.3.1) holds with $\widehat{C}_1 = 1$.

For $p \ne 1$ we go after the martingale term instead. Since $\widetilde{Z}$ vanishes at time zero, it suffices to estimate the size of $(X{*}\widetilde{Z})_\infty$ for $X \in \mathcal{E}_1$ with $X_0 = 0$. Let then M be a martingale with $\| M_\infty \|_{p'} \le 1$. Let T be a stopping time such that $(X{*}\widetilde{Z}_{.-})^T$ and $M^T_{.-}$ are bounded and $[\widehat{Z}, M]^T$ is a martingale (exercise 3.8.24 (ii)). Then the first two terms on the right of

$$(X{*}\widetilde{Z}^T)M^T = X{*}\widetilde{Z}_{.-}{*}M^T + (XM_{.-}){*}\widetilde{Z}^T + X{*}[\widetilde{Z}, M]^T$$

are martingales and vanish at zero. Further, $X{*}[\widetilde{Z}, M]^T$ and $X{*}[Z, M]^T$ differ by the martingale $X{*}[\widehat{Z}, M]^T$. Therefore

$$\mathbb{E}\big[(X{*}\widetilde{Z})_T M_T\big] = \mathbb{E}\Big[\int_{0+}^T X \, d[Z, M]\Big] \le \mathbb{E}\Big[\lceil [Z, M] \rceil_\infty \Big] .$$

Now $X{*}\widetilde{Z}$ is constant after some instant. Taking the supremum over T and $X \in \mathcal{E}_1$ thus gives

$$\lceil \widetilde{Z} \rceil_{\mathcal{I}^p} \le \sup\Big\{ \mathbb{E}\Big[\lceil [Z, M] \rceil_\infty \Big] : \ \| M_\infty \|_{L^{p'}} \le 1 \Big\} . \qquad (*)$$

If $1 < p < 2$, we continue this inequality using corollary 4.2.8:

$$\lceil \widetilde{Z} \rceil_{\mathcal{I}^p} \le 2\sqrt{2p} \, \| S_\infty[Z] \|_{L^p} \le 2\sqrt{2p} \, K_p^{(3.8.6)} \lceil Z \rceil_{\mathcal{I}^p} \le 4.1 \cdot \lceil Z \rceil_{\mathcal{I}^p} .$$

If $2 \le p$, we continue at $(*)$ instead with an application of exercise 3.8.10:

$$\lceil \widetilde{Z} \rceil_{\mathcal{I}^p} \le \|S_\infty[Z]\|_{L^p} \cdot \sup\big\{ \|S_\infty[M]\|_{L^{p'}} : \ \|M_\infty\|_{L^{p'}} \le 1 \big\} \qquad (**)$$

$$\le \|S_\infty[Z]\|_{L^p} \cdot C_{p'}^{(4.2.5)} \cdot \sup\big\{ \|M^\star_\infty\|_{L^{p'}} : \ \|M_\infty\|_{L^{p'}} \le 1 \big\}$$

by 2.5.19: $$\le \|S_\infty[Z]\|_{L^p} \cdot C_{p'}^{(4.2.5)} \cdot p \le \|S_\infty[Z]\|_{L^p} \cdot (6/\sqrt{p'}) \cdot p$$

by 3.8.4: $$\le (6p/\sqrt{p'}) \cdot \lceil Z \rceil_{\mathcal{I}^p} \le 6p \lceil Z \rceil_{\mathcal{I}^p} .$$

For $p = 2$ use $\|S_\infty[M]\|_{L^{p'}} \le 1$ at (**) instead. This proves the stated inequalities for $\widetilde{Z}$; the ones for $\widehat{Z}$ follow by subtraction. Theorem 4.3.1 is proved in its entirety. ▮

Remark 4.3.16 The main ingredient in the proof of proposition 4.3.2 and thus of theorem 4.3.1 was the fact that an increasing process I that satisfies

$$\mathbb{E}\big[M_t I_t\big] = \mathbb{E}\Big[\int_0^t M_{\cdot-}\, dI\Big] \tag{$*$}$$

for all bounded martingales M is previsible. It was Paul–André Meyer who called increasing processes with $(*)$ ***natural*** and then proceeded to show that they are previsible [71], [72]. At first sight there is actually something *unnatural* about all this [73, page 111]. Namely, while the interest in previsible processes as *integrands* is perfectly natural in view of our experience with Borel functions, of which they are the stochastic analogs, it may not be altogether obvious what good there is in having *integrators* previsible. In answer let us remark first that the previsibility of $\widehat{Z}$ enters essentially into the proof of the estimates (4.3.1)–(4.3.2). Furthermore, it will lead to *previsible pathwise control* of integrators, which permits a controlled analysis of stochastic differential equations driven by integrators with jumps (section 4.5).

The Previsible Square Function

If Z is an L^p-integrator with $p \ge 2$, then $[Z,Z]$ is an L^1-integrator and therefore has a Doob–Meyer decomposition

$$[Z,Z] = \widehat{[Z,Z]} + \widetilde{[Z,Z]} \, .$$

Its previsible finite variation part $\widehat{[Z,Z]}$ is called the ***previsible*** or ***oblique bracket*** or ***angle bracket*** and is denoted by $\langle Z,Z\rangle$. Note that $\langle Z,Z\rangle_0 = [Z,Z]_0 = Z_0^2$. The square root $s[Z] \stackrel{\text{def}}{=} \sqrt{\langle Z,Z\rangle}$ is called the ***previsible square function*** of Z. The processes $\langle Z,Z\rangle$ and $s[Z]$ evidently can be defined unequivocally also in case Z is merely a local L^2-integrator. If Z is continuous, then clearly $\langle Z,Z\rangle = [Z,Z]$ and $s[Z] = S[Z]$.

Let Y,Z be local L^2-integrators. According to the inequality of Kunita–Watanabe (theorem 3.8.9), $[Y,Z]$ is a local L^1-integrator and has a Doob–Meyer decomposition

$$[Y,Z] = \widehat{[Y,Z]} + \widetilde{[Y,Z]}$$

with

$$\widehat{[Y,Z]}_0 = [Y,Z]_0 = Y_0{\cdot}Z_0 \, .$$

Its previsible finite variation part $\widehat{[Y,Z]}$ is called the ***previsible*** or ***oblique bracket*** or ***angle bracket*** and is denoted by $\langle Y,Z\rangle$. Clearly if either of Y,Z is continuous, then $\langle Y,Z\rangle = [Y,Z]$.

Exercise 4.3.17 The previsible bracket has the same general properties as $[\cdot,\cdot]$:

(i) The theorem of Kunita–Watanabe holds for it: for any two local L^2-integrators Y, Z there exists a set $\Omega_0 \in \mathcal{F}_\infty$ of full measure $\mathbb{P}[\Omega_0] = 1$ such that for all $\omega \in \Omega_0$ and any two $\mathcal{B}^\bullet(\mathbb{R}) \otimes \mathcal{F}_\infty$-measurable processes U, V

$$\int_0^\infty |UV|\, d\,\vert\langle Y,Z\rangle\vert \le \Big(\int_0^\infty U^2\, d\langle Y,Y\rangle\Big)^{1/2} \cdot \Big(\int_0^\infty V^2\, d\langle Z,Z\rangle\Big)^{1/2} .$$

(ii) $s[Y+Z] \le s[Y] + s[Z]$, except possibly on an evanescent set.

(iii) For any stopping time T and $p,q,r > 0$ with $1/r = 1/p + 1/q$

$$\Big\| \,\vert\langle Y,Z\rangle\vert_T \Big\|_{L^r} \le \| s_T[Z] \|_{L^p} \cdot \| s_T[Y] \|_{L^q}$$

(iv) Let Z^1, Z^2 be local L^2-integrators and X^1, X^2 processes integrable for both. Then

$$\langle X^1 {*} Z^1, X^2 {*} Z^2\rangle = (X^1 \cdot X^2) {*} \langle Z^1, Z^2\rangle .$$

Exercise 4.3.18 With respect to the previsible bracket the martingales M with $M_0 = 0$ and the previsible finite variation processes V are perpendicular: $\langle M, V\rangle = 0$. If Z is a local L^2-integrator with Doob–Meyer decomposition $Z = \widehat{Z} + \widetilde{Z}$, then

$$\langle Z,Z\rangle = \langle \widehat{Z},\widehat{Z}\rangle + \langle \widetilde{Z},\widetilde{Z}\rangle .$$

For $0 < p \le 2$ the previsible square function $s[M]$ can be used as a control for the integrator size of a local martingale M much as the square function $S[M]$ controls it in the range $1 \le p < \infty$ (theorem 4.2.12). Namely,

Proposition 4.3.19 *For a locally L^2-integrable martingale M and $0 < p \le 2$*

$$\|M^\star_\infty\|_{L^p} \le C_p \cdot \|s_\infty[M]\|_{L^p} , \tag{4.3.6}$$

with universal constants

$$C_2^{(4.3.6)} \le 2$$

and

$$C_p^{(4.3.6)} \le 4\sqrt{2/p} , \qquad p \ne 2 .$$

Exercise 4.3.20 For a *continuous* local martingale M the Burkholder–Davis–Gundy inequality (4.2.4) extends to all $p \in (0,\infty)$ and implies

$$\vert M^t\vert_{\mathcal{I}^p} \le C_p \cdot \Big\| s_\infty[M^t] \Big\|_{L^p} \tag{4.3.7}$$

for all t, with

$$C_p^{(4.3.7)} \le \begin{cases} C_p^{(4.3.6)} & \text{for } 0 < p \le 2 \\ C_p^{(4.2.4)} & \text{for } 1 \le p < \infty. \end{cases}$$

Proof of Proposition 4.3.19. First the case $p = 2$: thanks to Doob's maximal theorem 2.5.19 and exercise 3.8.11

$$\mathbb{E}\Big[M_T^{\star 2}\Big] \le 4\,\mathbb{E}\Big[M_T^2\Big] = 4\,\mathbb{E}\Big[[M,M]_T\Big] = 4\,\mathbb{E}\Big[\langle M,M\rangle_T\Big]$$

for arbitrarily large stopping times T. Upon letting $T \to \infty$ we get $\mathbb{E}\big[M_\infty^{\star 2}\big] \le 4\,\mathbb{E}\big[\langle M,M\rangle_\infty\big]$.

Now the case $p < 2$. By reduction to arbitrarily large stopping times we may assume that M is a global L^2-integrator. Let $s = s[M]$. Literally as in the proof of Fefferman's inequality 4.2.7 one shows that $s^{p-2} \cdot d(s^2) \le (2/p)\, d(s^p)$ and so

$$\int_0^t s^{p-2}\, d\langle M, M\rangle \le (2/p) \cdot s_t^p \;. \tag{$*$}$$

Next let $\epsilon > 0$ and define $\overline{s} = s[M] + \epsilon$ and $\overline{M} = \overline{s}^{(p-2)/2} {*} M$. From the first part of the proof

$$\mathbb{E}\Big[\overline{M}_t^{\star 2}\Big] \le 4 \cdot \mathbb{E}\Big[\overline{M}_t^2\Big] = 4 \cdot \mathbb{E}\Big[\langle \overline{M}, \overline{M}\rangle_t\Big] = 4 \cdot \mathbb{E}\Big[\int_0^t \overline{s}^{p-2}\, d\langle M, M\rangle\Big]$$

by $(*)$: $$\le 4 \cdot \mathbb{E}\Big[\int_0^t s^{p-2}\, d\langle M, M\rangle\Big] \le (8/p) \cdot \mathbb{E}\big[s_t^p\big] \;. \tag{$**$}$$

Next observe that for $t \ge 0$

$$\begin{aligned} M_t &= \int_0^t \overline{s}^{(2-p)/2}\, d\overline{M} = \overline{s}_t^{(2-p)/2} \cdot \overline{M}_t - \int_{0+}^t \overline{M}_{\cdot -}\, d\overline{s}^{(2-p)/2} \\ &\le 2 \cdot \overline{s}_t^{(2-p)/2} \cdot \overline{M}_t^\star \;. \end{aligned}$$

The same inequality holds for $-M$, and since the process on the previous line increases with t,

$$M_t^\star \le 2 \cdot \overline{s}_t^{(2-p)/2} \cdot \overline{M}_t^\star \;.$$

From this, using Hölder's inequality with conjugate exponents $2/(2-p)$ and $2/p$ and inequality $(**)$,

$$\begin{aligned} \mathbb{E}\big[M_t^{\star p}\big] &\le 2^p \cdot \mathbb{E}\Big[\overline{s}_t^{p(2-p)/2} \cdot \overline{M}_t^{\star p}\Big] \le 2^p \cdot \Big(\mathbb{E}\big[\overline{s}_t^p\big]\Big)^{(2-p)/2} \cdot \Big(\mathbb{E}\big[\overline{M}_t^{\star 2}\big]\Big)^{p/2} \\ &\le 2^p (8/p)^{p/2} \cdot \Big(\mathbb{E}\big[\overline{s}_t^p\big]\Big)^{(2-p)/2} \cdot \Big(\mathbb{E}\big[s_t^p\big]\Big)^{p/2} \xrightarrow[\epsilon \to 0]{} \Big(4\sqrt{2/p}\Big)^p \cdot \mathbb{E}\Big[s_t^p\Big] \;. \end{aligned}$$

We take the p^{th} root and get $\|\, M_t^\star \,\|_{L^p} \le 4\sqrt{2/p} \cdot \|\, s_t[M] \,\|_{L^p}$. ∎

Exercise 4.3.21 Suppose that Z is a global L^1-integrator with Doob–Meyer decomposition $Z = \widehat{Z} + \widetilde{Z}$. Here is an a priori L^p-mean estimate of the compensator $\widehat{Z}$ for $1 \le p < \infty$: let $\mathcal{P}_b$ denote the bounded previsible processes and set

$$\|Z\|_p^\wedge \stackrel{\text{def}}{=} \sup\Big\{\mathbb{E}\Big[\int X\, dZ\Big] : \; X \in \mathcal{P}_b \;,\; \big\|X_\infty^\star\big\|_{L^{p'}} \le 1\Big\} \;.$$

Then $$\|Z\|_p^\wedge \le \Big\| \,\lceil\!\lceil \widehat{Z} \rceil\!\rceil_\infty \Big\|_{L^p} = \lceil\!\lceil \widehat{Z} \rceil\!\rceil_{\mathcal{I}^p} \le p \cdot \|Z\|_p^\wedge \;.$$

Exercise 4.3.22 Let I be a positive increasing process with Doob–Meyer decomposition $I = \widehat{I} + \widetilde{I}$. In this case there is a better estimate of $\widehat{C}_p^{(4.3.1)}$ and $\widetilde{C}_p^{(4.3.1)}$ than inequality (4.3.3) provides. Namely, for $1 \le p < \infty$,

$$\lceil\!\lceil \widehat{I} \rceil\!\rceil_{\mathcal{I}^p} = \Big\| \widehat{I}_\infty \Big\|_{L^p} \le p \cdot \lceil\!\lceil I \rceil\!\rceil_{\mathcal{I}^p} \quad \text{and} \quad \lceil\!\lceil \widetilde{I} \rceil\!\rceil_{\mathcal{I}^p} \le (p+1) \cdot \lceil\!\lceil I \rceil\!\rceil_{\mathcal{I}^p} \;.$$

Exercise 4.3.23 Suppose that Z is a continuous L^p-integrator for some $p \geq 2$. Then $S[\widetilde{Z}] = S[Z]$ and inequality (4.2.4) can be used to improve the estimate (4.3.3) minutely to

$$\widetilde{C}_p \leq p\sqrt{e/2}\ .$$

The Doob–Meyer Decomposition of a Random Measure

Let ζ be a random measure with auxiliary space $\boldsymbol{H}$ and elementary integrands $\check{\mathcal{E}}$ (see section 3.10). There is a straightforward generalization of theorem 4.3.1 to ζ.

Theorem 4.3.24 *Suppose ζ is a local L^1-random measure. There exist a unique previsible **strict** random measure $\widehat{\zeta}$ and a unique local martingale random measure $\widetilde{\zeta}$ that vanishes at zero, both local L^1-random measures, so that*

$$\zeta = \widehat{\zeta} + \widetilde{\zeta}\ .$$

In fact, there exist an increasing predictable process V and Radon measures $\nu_\varpi = \nu_{s,\omega}$ on $\boldsymbol{H}$, one for every $\varpi = (s,\omega) \in \boldsymbol{B}$ and usually written $\nu_s = \nu_s^\zeta$, so that $\widehat{\zeta}$ has the disintegration

$$\int_{\check{\boldsymbol{B}}} \check{H}_s(\eta)\,\widehat{\zeta}(d\eta, ds) = \int_0^\infty \int_{\boldsymbol{H}} \check{H}_s(\eta)\,\nu_s(d\eta)\ dV_s\ , \tag{4.3.8}$$

*which is valid for every $\check{H} \in \check{\mathcal{P}}$. We call $\widehat{\zeta}$ the **intensity** or **intensity measure** or **compensator** of ζ, and ν_s its **intensity rate**. $\widetilde{\zeta}$ is the **compensated random measure**. For $1 \leq p < \infty$ and all $h \in \mathcal{E}_+[\boldsymbol{H}]$ and $t \geq 0$ we have the estimates (see definition 3.10.1)*

$$\left\lceil\!\left\lceil \widehat{\zeta}^{t,h} \right\rceil\!\right\rceil_{\mathcal{I}^p} \leq \widehat{C}_p^{(4.3.1)} \left\lceil\!\left\lceil \zeta^{t,h} \right\rceil\!\right\rceil_{\mathcal{I}^p} \quad \textit{and} \quad \left\lceil\!\left\lceil \widetilde{\zeta}^{t,h} \right\rceil\!\right\rceil_{\mathcal{I}^p} \leq \widetilde{C}_p^{(4.3.1)} \left\lceil\!\left\lceil \zeta^{t,h} \right\rceil\!\right\rceil_{\mathcal{I}^p}\ .$$

Proof. Regard the measure $\theta : \check{H} \mapsto \mathbb{E}[\int \check{H}\ d\zeta]$, $\check{H} \in \check{\mathcal{P}}$, as a σ-finite scalar measure on the product $\check{\boldsymbol{B}} \stackrel{\text{def}}{=} \boldsymbol{H} \times \boldsymbol{B}$ equipped with $C_{00}(\boldsymbol{H}) \otimes \mathcal{E}$. According to corollary A.3.42 on page 418 there is a disintegration $\theta = \int_{\boldsymbol{B}} \nu_\varpi\ \mu(d\varpi)$, where μ is a positive σ-additive measure on $\mathcal{E}$ and, for every $\varpi \in \boldsymbol{B}$, ν_ϖ a Radon measure on $\boldsymbol{H}$, so that

$$\int_{\boldsymbol{H}\times\boldsymbol{B}} \check{H}(\eta,\varpi)\ \theta(d\eta, d\varpi) = \int_{\boldsymbol{B}} \int_{\boldsymbol{H}} \check{H}(\eta,\varpi)\nu_\varpi(d\eta)\ \mu(d\varpi)$$

for all θ-integrable functions $\check{H} \in \check{\mathcal{P}}$. Since μ clearly annihilates evanescent sets, it has a Doléans–Dade process V^μ. We simply define $\widehat{\zeta}$ by

$$\int \check{H}_s(\eta,\omega)\ \widehat{\zeta}(d\eta, ds; \omega) = \int\int \check{H}_s(\eta,\omega)\ \nu_{(s,\omega)}(d\eta)\ dV_s^\mu(\omega)\ , \qquad \omega \in \Omega\ ,$$

which clearly has previsible indefinite integrals $\check{H}*\widehat{\zeta}^T$ for arbitrarily large stopping times T and $\check{H} \in \check{\mathcal{E}}$, making it locally a ***previsible strict random measure***. Then we set $\widetilde{\zeta} \stackrel{\text{def}}{=} \zeta - \widehat{\zeta}$. Clearly $\check{H}*\widetilde{\zeta}^T$ is a martingale for arbitrarily large T and $\check{H} \in \check{\mathcal{E}}$, making it a ***local martingale random measure***. . ∎

If ζ is the jump measure $\jmath_{\boldsymbol{Z}}$ of an integrator $\boldsymbol{Z}$, then $\widehat{\zeta} = \widehat{\jmath_{\boldsymbol{Z}}}$ is called the ***jump intensity*** of $\boldsymbol{Z}$ and $\nu_s = \nu_s^{\boldsymbol{Z}}$ the ***jump intensity rate***. In this case both $\widehat{\zeta} = \widehat{\jmath_{\boldsymbol{Z}}}$ and $\widetilde{\zeta} = \widetilde{\jmath_{\boldsymbol{Z}}}$ are strict random measures. We say that $\boldsymbol{Z}$ has ***continuous jump intensity*** $\widehat{\jmath_{\boldsymbol{Z}}}$ if $\widehat{Y}_{\boldsymbol{.}} \stackrel{\text{def}}{=} \int_{[\![0,\boldsymbol{.}]\!]} |\boldsymbol{y}^2| \wedge 1 \; \widehat{\jmath_{\boldsymbol{Z}}}(d\boldsymbol{y}, ds)$ has continuous paths.

Proposition 4.3.25 *The following are equivalent: (i) $\boldsymbol{Z}$ has continuous jump intensity; (ii) the jumps of $\boldsymbol{Z}$, if any, occur only at totally inaccessible stopping times; (iii) $H*\widehat{\jmath_{\boldsymbol{Z}}}$ has continuous paths for every previsible Hunt function H.*

Definition 4.3.26 *A process with these properties, in other words, a process that has negligible jumps at any predictable stopping time is called* ***quasi-left-continuous****. A random measure ζ is quasi-left-continuous if and only if all of its indefinite integrals $\check{X}*\zeta$ are, $\check{X} \in \check{\mathcal{E}}$.*

Proof. $(i) \implies (ii)$ Let S be a predictable stopping time. If $\Delta \boldsymbol{Z}_S$ is non-negligible, then clearly neither is the jump $\Delta Y_S = |\Delta \boldsymbol{Z}_S|^2 \wedge 1$ of the increasing process $Y_t \stackrel{\text{def}}{=} \int_{[\![0,t]\!]} |\boldsymbol{y}|^2 \wedge 1 \; \jmath_{\boldsymbol{Z}}(d\boldsymbol{y}, ds)$. Since $\Delta Y_S \ge 0$, then $\Delta \widehat{Y}_S = \mathbb{E}\big[\Delta Y_S | \mathcal{F}_{S-}\big]$ is not negligible either (see exercise 4.3.6) and $\boldsymbol{Z}$ does not have continuous jump intensity. The other implications are even simpler to see. ∎

Exercise 4.3.27 If $\boldsymbol{Z}$ is a vector of L^1-integrators, then $\big(\widehat{\boldsymbol{Z}}, {}^c[Z^\eta, Z^\theta], \widehat{\jmath_{\boldsymbol{Z}}}\big)$ is called the ***characteristic triple*** of $\boldsymbol{Z}$. The expectation of any random variable of the form $\Phi(\boldsymbol{Z}_t)$, $\Phi \in C_b^2$, can be expressed in terms of $\boldsymbol{Z}_0$, $\boldsymbol{Z}_{\boldsymbol{.}-}$ and the characteristic triple.

4.4 Semimartingales

A process Z is called a ***semimartingale*** if it can be written as the sum of a process V of finite variation and a local martingale M. A semimartingale is clearly an L^0-integrator (proposition 2.4.1, corollary 2.5.29, and proposition 2.1.9). It is shown in proposition 4.4.1 below that the converse is also true: an L^0-integrator is a semimartingale. Stochastic integration in some generality was first developed for semimartingales $Z = V + M$. It was an amalgam of integration with respect to a finite variation process V, known forever, and of integration with respect to a square integrable martingale, known since Courrège [16] and Kunita–Watanabe [60] generalized Itô's procedure. A succinct account can be found in [75]. Here is a rough description: the dZ-integral of a process F is defined as $\int F\, dV + \int F\, dM$, the first summand being understood as a pathwise Lebesgue–Stieltjes integral, and the second as the extension of the elementary M-integral under the Hardy mean of definition (4.2.9). A problem with this approach is that the decomposition $Z = V + M$ is not unique, so that the results of any calculation have to be proven independent of it. There is a very simple example which shows that the class of processes F that can be so integrated depends on the decomposition (example 4.4.4 on page 234).

Integrators Are Semimartingales

Proposition 4.4.1 *An L^0-integrator Z is a semimartingale; in fact, there is a decomposition $Z = V + M$ with $|\Delta M| \le 1$.*

Proof. Recall that Z^n is Z stopped at n. ${}^nZ \stackrel{\text{def}}{=} Z^{n+1} - Z^n$ is a global $L^0(\mathbb{P})$-integrator that vanishes on $[\![0, n]\!]$, $n = 0, 1, \ldots$. According to proposition 4.1.1 or theorem 4.1.2, there is a probability ${}^n\mathbb{P}$ equivalent with $\mathbb{P}$ on $\mathcal{F}_\infty$ such that ${}^nZ$ is a global $L^1({}^n\mathbb{P})$-integrator, which then has a Doob–Meyer decomposition ${}^nZ = \widehat{{}^nZ} + \widetilde{{}^nZ}$ with respect to ${}^n\mathbb{P}$. Due to lemma 3.9.11, $\widetilde{{}^nZ}$ is the sum of a finite variation process and a local $\mathbb{P}$-martingale. Clearly then so is ${}^nZ$, say ${}^nZ = {}^nV + {}^nM$. Both ${}^nV$ and ${}^nM$ vanish on $[\![0, n]\!]$ and are constant after time $n+1$. The (ultimately constant) sum $Z = \sum {}^nV + \sum {}^nM$ exhibits Z as a $\mathbb{P}$-semimartingale.

We prove the second claim "locally" and leave its "globalization" as an exercise. Let then an instant $t > 0$ and an $\epsilon > 0$ be given. There exists a stopping time T_1 with $\mathbb{P}[T_1 < t] < \epsilon/3$ such that Z^{T_1} is the sum of a finite variation process $V^{(1)}$ and a martingale $M^{(1)}$. Now corollary 2.5.29 provides a stopping time T_2 with $\mathbb{P}[T_2 < t] < \epsilon/3$ and such that the stopped martingale $M^{(1)T_2}$ is the sum of a process $V^{(2)}$ of finite variation and a global L^2-integrator $Z^{(2)}$. $Z^{(2)}$ has a Doob–Meyer decomposition $Z^{(2)} = \widehat{Z}^{(2)} + \widetilde{Z}^{(2)}$ whose constituents are global L^2-integrators. The following little lemma 4.4.2 furnishes a stopping time T_3 with $\mathbb{P}[T_3 < t] < \epsilon/3$ and such that $\widetilde{Z}^{(2)T_3} = V^{(3)} + M$, where V^3 is a process of finite variation and M a martingale whose jumps are uniformly bounded by 1. Then $T = T_1 \wedge T_2 \wedge T_3$ has $\mathbb{P}[T < t] < \epsilon$, and

$$Z^T = V + M \quad \text{, where } V = \Big(V^{(3)} + \widehat{Z}^{(2)T} + V^{(2)T} + V^{(1)T}\Big)$$

is a process of finite variation: Z^T meets the description of the statement. ∎

Lemma 4.4.2 *Any L^2-bounded martingale M can be written as a sum $M = V + M'$, where V is a right-continuous process with integrable total variation $\lvert V \rvert_\infty$ and M' a locally square integrable globally $\mathcal{I}^1$-bounded martingale whose jumps are uniformly bounded by 1.*

Proof. Define the finite variation process V' by

$$V'_t = \sum\big\{\Delta M_s : \ s \le t, \ |\Delta M_s| \ge 1/2\big\} , \qquad t \le \infty .$$

This sum converges a.s. absolutely, since by theorem 3.8.4

$$\begin{aligned} \lvert V' \rvert_\infty &= \sum\big\{|\Delta M_s| : \ |\Delta M_s| > 1/2\big\} \\ &\le 2 \cdot \textstyle\sum_{s<\infty} \big(\Delta M_s\big)^2 \le 2 \cdot [M, M]_\infty \end{aligned}$$

is integrable. V' is thus a global L^1-integrator, and so is $Z = M - V'$, a process whose jumps are uniformly bounded by $1/2$. Z has a Doob–Meyer decomposition $Z = \widehat{Z} + \widetilde{Z}$. By exercise 4.3.6 the jump of $\widehat{Z}$ is uniformly bounded by $1/2$, and therefore by subtraction $|\Delta\widetilde{Z}| \le 1$. The desired decomposition is $M = \big(\widehat{Z} + V'\big) + \widetilde{Z}$: $M' = \widetilde{Z}$ is reduced to a uniformly bounded martingale by the stopping times $\inf\{t : |M'|_t \ge K\}$, which can be made arbitrarily large by the choice of K (lemma 2.5.18). V' has integrable total variation as remarked above, and clearly so does $\widehat{Z}$ (inequality (4.3.1) and exercise 4.3.13). ∎

Corollary 4.4.3 *Let $p > 0$. An L^0-integrator Z is a local L^p-integrator if and only if $|\Delta Z|^\star_T \in L^p$ at arbitrarily large stopping times T or, equivalently, if and only if its square function $S[Z]$ is a local L^p-integrator. In particular, an L^0-integrator with bounded jumps is a local L^p-integrator for all $p < \infty$.*

Proof. Note first that $|\Delta Z|^\star_t$ is in fact measurable on $\mathcal{F}_t$ (corollary A.5.13). Next write $Z = V + M$ with $|\Delta M| < 1$. By the choice of K we can make the time $T \stackrel{\text{def}}{=} \inf\{t : ¦V¦_t \vee M^\star_t > K\} \wedge K$ arbitrarily large. Clearly $M^\star_T < K + 1$, so M^T is an L^p-integrator for all $p < \infty$ (theorem 2.5.30). Since $\Delta ¦V¦ \le 1 + |\Delta Z|$, we have $¦V¦_T \le K + 1 + |\Delta Z|^\star_K \in L^p$, so that V^T is an L^p-integrator as well (proposition 2.4.1). ∎

Example 4.4.4 (S. J. Lin) Let N be a Poisson process that jumps at the times $T_1, T_2, \ldots$ by 1. It is an increasing process that at time T_n has the value n, so it is a local L^q-integrator for all $q < \infty$ and has a Doob–Meyer decomposition $N = \widehat{N} + \widetilde{N}$; in fact $\widehat{N}_t = t$. Considered as a semimartingale, there are two representations of the form $N = V + M$: $N = N + 0$ and $N = \widehat{N} + \widetilde{N}$.

Now let $H_t = (\!(0, T_1]\!]_t / t$. This predictable process is pathwise Lebesgue–Stieltjes–integrable against N, with integral $1/T_1$. So the disciple choosing the decomposition $N = N + 0$ has no problem with the definition of the integral $\int H\, dN$. A person viewing N as the semimartingale $\widehat{N} + \widetilde{N}$ – which is a very *natural* thing to do[9] – and attempting to integrate H with $d\widehat{N}_t$ and with $d\widetilde{N}_t$ and then to add the results will fail, however, since $\int H_t(\omega)\, d\widehat{N}_t(\omega) = \int H_t(\omega)\, dt = \infty$ for all $\omega \in \Omega$. In other words, the class of processes integrable for a semimartingale Z depends in general on its representation $Z = V + M$ if such an ad hoc integration scheme is used.

We leave to the reader the following mitigating fact: if there exists some representation $Z = V + M$ such that the previsible process F is pathwise dV-integrable and is dM-integrable in the sense of the Hardy mean of definition (4.2.9), then F is Z–0-integrable in the sense of chapter 3, and the integrals coincide.

Various Decompositions of an Integrator

While there is nothing unique about the finite variation and martingale parts in the decomposition $Z = V + M$ of an L^0-integrator, there are in fact some canonical parts and decompositions, all related to the location and size of its

[9] See remark 4.3.16 on page 228.

jumps. Consider first the increasing L^0-integrator $Y \stackrel{\text{def}}{=} h_0{*}\jmath_Z$, where h_0 is the prototypical sure Hunt function $y \mapsto |y|^2 \wedge 1$ (page 180). Clearly Y and Z jump at exactly the same times (by different amounts, of course). According to corollary 4.4.3, Y is a local L^2-integrator and therefore has a Doob–Meyer decomposition $Y = \widehat{Y} + \widetilde{Y}$, whose only use at present is to produce the sparse previsible set $P \stackrel{\text{def}}{=} [\Delta\widehat{Y} \neq 0]$ (see exercise 4.3.4). Let us set

$$ {}^pZ \stackrel{\text{def}}{=} P{*}Z \quad \text{and} \quad {}^qZ \stackrel{\text{def}}{=} Z - {}^pZ = (1-P){*}Z \ . \tag{4.4.1} $$

By exercise 3.10.12 (iv) we have $\jmath_{{}^qZ} = (1-P)\cdot\jmath_Z$, and this random measure has continuous previsible part $\widehat{\jmath_{{}^qZ}} = (1-P)\cdot\widehat{\jmath_Z}$. In other words, ${}^qZ$ has continuous jump intensity. Thanks to proposition 4.3.25, $\Delta{}^qZ_S = 0$ at all predictable stopping times S:

Proposition 4.4.5 *Every L^0-integrator Z has a unique decomposition*

$$ Z = {}^pZ + {}^qZ $$

with the following properties: there exists a previsible set P, a union of the graphs of countably many predictable stopping times, such that ${}^pZ = P{*}{}^pZ$[10]*; and ${}^qZ$ jumps at totally inaccessible stopping times only, which is to say that ${}^qZ$ is quasi-left-continuous. For $0 < p < \infty$, the maps $Z \mapsto {}^pZ$ and $Z \mapsto {}^qZ$ are linear contractive projections on $\mathcal{I}^p$.*

Proof. If $Z = {}^pZ + {}^qZ = Z = {}^pZ' + {}^qZ'$, then ${}^pZ - {}^pZ' = {}^qZ' - {}^qZ$ is supported by a sparse previsible set yet jumps at no predictable stopping time, so must vanish. This proves the uniqueness. The linearity and contractivity follow from this and the construction (4.4.1) of ${}^pZ$ and ${}^qZ$, which is therefore canonical. ∎

Exercise 4.4.6 Every random measure ζ has a unique decomposition $\zeta = {}^p\zeta + {}^q\zeta$ with the following properties: there exists a previsible set P, a union of the graphs of countably many predictable stopping times, such that ${}^p\zeta = (\boldsymbol{H}\times P)\cdot\zeta$[10,11]; and ${}^q\zeta$ jumps at totally inaccessible stopping times only, in the sense that $\check{H}{*}{}^q\zeta$ does for all $\check{H} \in \check{\mathcal{P}}$. For $0 < p < \infty$, the maps $\zeta \mapsto {}^p\zeta$ and $\zeta \mapsto {}^q\zeta$ are linear contractive projections on the space of L^p-random measures.

Proposition 4.4.7 (The Continuous Martingale Part of an Integrator) *An L^0-integrator Z has a canonical decomposition*

$$ Z = {}^{\tilde{c}}Z + {}^rZ \ , $$

where ${}^{\tilde{c}}Z$ is a continuous local martingale with ${}^{\tilde{c}}Z_0 = 0$ and with continuous bracket $[{}^{\tilde{c}}Z, {}^{\tilde{c}}Z] = {}^c[Z,Z]$ and where the remainder ${}^rZ$ has continuous bracket ${}^c[{}^rZ, {}^rZ] = 0$. There are universal constants C_p such that at all instants t

$$ \lceil\!\lceil {}^{\tilde{c}}Z^t \rceil\!\rceil_{\mathcal{I}^p} \le C_p \lceil\!\lceil Z^t \rceil\!\rceil_{\mathcal{I}^p} \quad \text{and} \quad \lceil\!\lceil {}^rZ^t \rceil\!\rceil_{\mathcal{I}^p} \le C_p \lceil\!\lceil Z^t \rceil\!\rceil_{\mathcal{I}^p} \ , \ 0 < p < \infty \ . \tag{4.4.2} $$

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[10] We might paraphrase this by saying "d^pZ is supported by a ***sparse previsible set***."
[11] This is to mean of course that $\check{H}{*}{}^p\zeta = (\check{H}\cdot(\boldsymbol{H}\times P)){*}\zeta$ for all $\check{H} \in \check{\mathcal{P}}$.

Exercise 4.4.8 Z and ${}^qZ$ have the same continuous martingale part. Taking the continuous martingale part is stable under stopping: ${}^{\tilde c}(Z^T) = ({}^{\tilde c}Z)^T$ at all $T \in \mathfrak{T}$. Consequently (4.4.2) persists if t is replaced by a stopping time T.

Exercise 4.4.9 Every random measure ζ has a canonical decomposition as $\zeta = {}^{\tilde c}\zeta + {}^r\zeta$, given by $\check H * {}^{\tilde c}\zeta = {}^{\tilde c}(\check H * \zeta)$ and $\check H * {}^r\zeta = {}^r(\check H * \zeta)$. For $0 < p < \infty$, the maps $\zeta \mapsto {}^{\tilde c}\zeta$ and $\zeta \mapsto {}^r\zeta$ are linear projections on the space of L^p-random measures with $\lceil\!\lceil {}^{\tilde c}\zeta^{h,t} \rceil\!\rceil_{\mathcal{I}^p} \le C^{p(4.4.2)} \lceil\!\lceil \zeta^{h,t} \rceil\!\rceil_{\mathcal{I}^p}$ and $\lceil\!\lceil {}^r\zeta^{h,t} \rceil\!\rceil_{\mathcal{I}^p} \le C^{p(4.4.2)} \lceil\!\lceil \zeta^{h,t} \rceil\!\rceil_{\mathcal{I}^p}$, $h \in \mathcal{E}_+[\boldsymbol{H}]$.

Proof of 4.4.7. First the uniqueness. If also $Z = {}^{\tilde c}Z' + {}^rZ'$, then ${}^{\tilde c}Z - {}^{\tilde c}Z'$ is a continuous martingale whose continuous bracket vanishes, since it is that of ${}^rZ' - {}^rZ$; thus ${}^{\tilde c}Z - {}^{\tilde c}Z'$ must be constant, in fact, since ${}^{\tilde c}Z_0 = {}^{\tilde c}Z'_0$, it must vanish.

Next the inequalities:

$$\lceil\!\lceil {}^{\tilde c}Z^t \rceil\!\rceil_{\mathcal{I}^p} \le C_p^{(4.3.7)} \Big\| S_t[{}^{\tilde c}Z] \Big\|_p = C_p \|\sigma_t[Z]\|_p \le C_p \|S_t[Z]\|_p$$

$$\le C_p K_p^{(3.8.6)} \lceil\!\lceil Z^t \rceil\!\rceil_{\mathcal{I}^p} .$$

Now to the existence. There is a sequence (T^i) of bounded stopping times with disjoint graphs so that every jump of Z occurs at one of them (exercise 1.3.21). Every T^i can be decomposed as the infimum of an accessible stopping time T^i_A and a totally inaccessible stopping time T^i_I (see page 122). For every i let $(S^{i,j})$ be a sequence of predictable stopping times so that $[\![T^i_A]\!] \subset \bigcup_j [\![S^{i,j}]\!]$, and let Δ denote the union of the graphs of the $S^{i,j}$. This is a previsible set, and $\Delta * Z$ is an integrator whose jumps occur only on Δ (proposition 3.8.21) and whose continuous square function vanishes. The jumps of $Z' \overset{\text{def}}{=} Z - \Delta * Z = (1 - \Delta) * Z$ occur at the totally inaccessible times T^i_I.

Assume now for the moment that Z is an L^2-integrator; then clearly so is Z'. Fix an i and set $J^i \overset{\text{def}}{=} \Delta Z'_{T^i_I} \cdot [\![T^i, \infty)\!)$. This is an L^2-integrator of total variation $|\Delta Z'_{T^i_I}| \in L^2$ and has a Doob–Meyer decomposition $J^i = \widehat{J}^i + \widetilde{J}^i$. The previsible part $\widehat{J}^i$ is continuous: if $[\Delta \widehat{J}^i > 0]$ were non-evanescent, it would contain the non-evanescent graph of a previsible stopping time (theorem A.5.14), at which the jump of Z' could not vanish (exercise 4.3.6), which is impossible due to the total inaccessibility of the jump times of Z'. Therefore $\widetilde{J}^i$ has exactly the same single jump as J^i, namely $\Delta Z'_{T^i_I}$ at T^i_I. Now at all instants t

$$\Big\| \Big| \sum_{I<i\le J} \widetilde{J}^i \Big|^\star_t \Big\|_{L^2} \le 2 \Big\| S_t\Big[\sum_{I<i\le J} \widetilde{J}^i \Big] \Big\|_{L^2} = 2 \Big(\mathbb{E}\Big[\sum_{I<i\le J} |\Delta Z'_{T^i_I}|^2 \Big] \Big)^{1/2}$$

$$\le 2 \Big(\mathbb{E}\Big[\sum_{I<i\le J} |\Delta Z_{T^i}|^2 \Big] \Big)^{1/2} \xrightarrow[I<J\,;\,I,J\to\infty]{} 0 .$$

The sum $M \overset{\text{def}}{=} \sum_i \widetilde{J}^i$ therefore converges uniformly almost surely and in L^2 and defines a martingale M that has exactly the same jumps as Z'. Then

$$Z'' \overset{\text{def}}{=} Z' - M = (1 - \Delta) * Z - M$$

is a continuous L^2-integrator whose square function is ${}^c[Z,Z]$. Its martingale part evidently meets the description of ${}^{\tilde c}Z$.

Now if Z is merely an $L^0(\mathbb{P})$-integrator, we fix a t and find a probability $\mathbb{P}' \approx \mathbb{P}$ on $\mathcal{F}_t$ with respect to which Z^t is an L^2-integrator. We then write Z^t as ${}^{\tilde c}Z'^t + {}^rZ'^t$, where ${}^{\tilde c}Z'^t$ is the canonical $\mathbb{P}'$-martingale part of Z^t. Thanks to the Girsanov–Meyer lemma 3.9.11, whose notations we employ here again,

$$ {}^{\tilde c}Z'^t = \Big({}^{\tilde c}Z_0'^t - G_{\bullet-}{*}[{}^{\tilde c}Z'^t, G']\Big) + \Big(G_{\bullet-}{*}({}^{\tilde c}Z'^tG') - ({}^{\tilde c}Z'^tG)_{\bullet-}{*}G'\Big) $$

is the sum of two continuous processes, of which the first has finite variation and the second is a local $\mathbb{P}$-martingale, which we call ${}^{\tilde c}Z^t$. Clearly ${}^{\tilde c}Z^t$ is the canonical local $\mathbb{P}$-martingale part of the stopped process Z^t. We can do this for arbitrarily large times t. From the uniqueness, established first, we see that the sequence $\big({}^{\tilde c}Z^n\big)$ is ultimately constant, except possibly on an evanescent set. Clearly ${}^{\tilde c}Z \stackrel{\text{def}}{=} \lim_{n\to\infty} {}^{\tilde c}Z^n$ is the desired canonical continuous martingale part of Z; and ${}^rZ$ is defined as the difference $Z - {}^{\tilde c}Z$. ∎

In view of exercise 4.4.8 we now have a canonical decomposition

$$ \boldsymbol{Z} = {}^p\boldsymbol{Z} + {}^{\tilde c}\boldsymbol{Z} + {}^r\boldsymbol{Z} $$

of an L^0-integrator $\boldsymbol{Z}$, with the linear projections

$$ \boldsymbol{Z} \mapsto {}^p\boldsymbol{Z}\ ,\ \ \boldsymbol{Z} \mapsto {}^{\tilde c}\boldsymbol{Z}\ \ \text{, and}\ \ \boldsymbol{Z} \mapsto {}^r\boldsymbol{Z} $$

continuous from $\mathcal{I}^p$ to $\mathcal{I}^p$ for all $p \ge 0$. $\boldsymbol{Z} \mapsto {}^p\boldsymbol{Z}$ is actually contractive, the other two have $\lceil\!\lceil {}^{\tilde c}\boldsymbol{Z} \rceil\!\rceil_{\mathcal{I}^p} \le C^{p(4.4.2)} \lceil\!\lceil \boldsymbol{Z} \rceil\!\rceil_{\mathcal{I}^p}$ and $\lceil\!\lceil {}^r\boldsymbol{Z} \rceil\!\rceil_{\mathcal{I}^p} \le C^{p(4.4.2)} \lceil\!\lceil \boldsymbol{Z} \rceil\!\rceil_{\mathcal{I}^p}$ for $p > 0$ (see exercises 4.4.8 and 4.4.9).

Exercise 4.4.10 ${}^r\boldsymbol{Z}$ can be decomposed further. Set

$$ {}^{\tilde s}\boldsymbol{Z}_t \stackrel{\text{def}}{=} \int_{[\![0,t]\!]} \boldsymbol{y}\cdot[|\boldsymbol{y}|\le 1]\, d\widetilde{\jmath_{{}^r\boldsymbol{Z}}} = \int_{[\![0,t]\!]} \boldsymbol{y}\cdot[|\boldsymbol{y}|\le 1]\, d\widetilde{\jmath_{\boldsymbol{Z}}}\ , $$

$$ {}^l\boldsymbol{Z}_t \stackrel{\text{def}}{=} \int_{[\![0,t]\!]} \boldsymbol{y}\cdot[|\boldsymbol{y}| > 1]\, d\jmath_{{}^r\boldsymbol{Z}} = \int_{[\![0,t]\!]} \boldsymbol{y}\cdot[|\boldsymbol{y}| > 1]\, d\widetilde{\jmath_{\boldsymbol{Z}}}\ , $$

and $$ {}^{\hat v}\boldsymbol{Z} \stackrel{\text{def}}{=} {}^r\boldsymbol{Z} - {}^{\tilde s}\boldsymbol{Z} - {}^l\boldsymbol{Z}\ . $$

Then ${}^{\tilde s}\boldsymbol{Z}$ is a martingale with zero continuous part but jumps bounded by 1, the ***small jump martingale part***; ${}^l\boldsymbol{Z}$ is a finite variation process without a continuous part and constant between its jumps, which are of size at least 1 and occur at discrete times, the ***large jump part***; and ${}^{\hat v}\boldsymbol{Z}$ is a continuous finite variation process. The projections ${}^r\boldsymbol{Z} \mapsto {}^{\tilde s}\boldsymbol{Z}$, ${}^r\boldsymbol{Z} \mapsto {}^l\boldsymbol{Z}$, and ${}^r\boldsymbol{Z} \mapsto {}^{\hat v}\boldsymbol{Z}$ are not even linear, much less continuous in $\mathcal{I}^p$. From this obtain a decomposition

$$ \boldsymbol{Z} = ({}^{lp}\boldsymbol{Z} + {}^{\tilde sp}\boldsymbol{Z}) + ({}^{\tilde c}\boldsymbol{Z}) + ({}^{\hat v}\boldsymbol{Z} + {}^{\tilde s}\boldsymbol{Z} + {}^l\boldsymbol{Z}) \tag{4.4.3} $$

and describe its ingredients.

4.5 Previsible Control of Integrators

A general L^p-integrator's integral is controlled by Daniell's mean; if the integrator happens to be a martingale M and $p \geq 1$, then its integral can be controlled pathwise by the finite variation process $[M,M]$ (see definition (4.2.9) on page 216). For the solution of stochastic differential equations it is desirable to have such pathwise control of the integral not only for general integrators instead of merely for martingales but also *by a previsible* increasing process instead of $[M,M]$, and *for a whole vector of integrators simultaneously.* Here is what we can accomplish in this regard – see also theorem 4.5.25 concerning random measures:

Theorem 4.5.1 *Let $\boldsymbol{Z}$ be a d-tuple of local L^q-integrators, where $2 \leq q < \infty$. Fix a (small) $\alpha \in (0,1)$. There exists a strictly increasing previsible process $\Lambda{=}\Lambda^{\langle q\rangle}[\boldsymbol{Z}]$ such that for every $p{\in}[2,q]$, every stopping time T, and every d-tuple $\boldsymbol{X} = (X_1, \ldots, X_d)$ of previsible processes*

$$\big\| |\boldsymbol{X}{*}\boldsymbol{Z}|_T^\star \big\|_{L^p} \leq C_p^\diamond \cdot \max_{\rho=1^\diamond, p^\diamond} \left\| \Big(\int_0^T |\boldsymbol{X}|_s^\rho \, d\Lambda_s \Big)^{1/\rho} \right\|_{L^p} , \tag{4.5.1}$$

with $|\boldsymbol{X}| \overset{\text{def}}{=} |\boldsymbol{X}|_\infty = \max_{1\leq\eta\leq d} |X_\eta|$ and universal constant $C_p^\diamond \leq 9.5p$.

Here $\quad 1^\diamond = 1^\diamond[\boldsymbol{Z}] \overset{\text{def}}{=} \begin{cases} 2 & \text{if } \boldsymbol{Z} \text{ is a martingale} \\ 1 & \text{otherwise,} \end{cases}$

and $\quad p^\diamond = p^\diamond[\boldsymbol{Z}] \overset{\text{def}}{=} \begin{cases} 1 & \text{if } \boldsymbol{Z} \text{ is continuous and has finite variation} \\ 2 & \text{if } \boldsymbol{Z} \text{ is continuous and } \widetilde{\boldsymbol{Z}} \neq 0 \\ p & \text{if } \boldsymbol{Z} \text{ has jumps.} \end{cases}$

Furthermore, the previsible controller Λ can be estimated by

$$\mathbb{E}\Big[\Lambda_T^{\langle q\rangle}[\boldsymbol{Z}]\Big] \leq \alpha \mathbb{E}[T] + 3\Big(¦\boldsymbol{Z}^T¦_{\mathcal{I}^q} \vee ¦\boldsymbol{Z}^T¦_{\mathcal{I}^q}^q \Big) \tag{4.5.2}$$

and $\qquad \Lambda_T^{\langle q\rangle}[\boldsymbol{Z}] \geq \alpha T \quad$ *at all stopping times T.*

Remark 4.5.2 The controller Λ constructed in the proof below satisfies the four inequalities

$$\alpha \cdot dt \leq d\Lambda_t \; ; \tag{4.5.3}$$

and

$$|X_\eta|_s \, d¦\widehat{Z}^\eta¦_s \leq |\boldsymbol{X}|_s \, d\Lambda_s \; ,$$

$$d\big|X_\eta X_\theta{*}\langle Z^\eta, Z^\theta\rangle\big|_s^\star \leq |\boldsymbol{X}|_s^2 \, d\Lambda_s \; ,$$

$$\int_{\mathbb{R}_*^d} |\langle \boldsymbol{X}|\boldsymbol{y}\rangle|^q \, \widehat{\jmath_{\boldsymbol{Z}}}(d\boldsymbol{y}, ds) \leq |\boldsymbol{X}|_s^q \, d\Lambda_s \; ,$$

for all previsible $\boldsymbol{X}$, and is in fact the smallest increasing process doing so. This makes it somewhat canonical (when α is fixed) and justifies naming it ***THE previsible controller of $\boldsymbol{Z}$***.

The imposition of inequality (4.5.3) above is somewhat artificial. Its purpose is to ensure that Λ is strictly increasing and that the predictable (see exercise 3.5.19) stopping times

$$T^\lambda \stackrel{\text{def}}{=} \inf\{t : \Lambda_t \geq \lambda\} \quad \text{and} \quad T^{\lambda+} \stackrel{\text{def}}{=} \inf\{t : \Lambda_t > \lambda\} \tag{4.5.4}$$

agree and are bounded (by λ/α); in most naturally occurring situations one of the drivers Z^η is time, and then this inequality is automatically satisfied. The collection $T^\bullet = \{T^\lambda : \lambda > 0\}$ will henceforth simply be called ***THE time transformation for Z*** (or for Λ). The process $\Lambda^{\langle q\rangle}[\boldsymbol{Z}]$ or the parameter λ, its value, is occasionally referred to as the ***intrinsic time***. It and the time transformation $T^\bullet$ are the main tools in the existence, uniqueness, and stability proofs for stochastic differential equations driven by $\boldsymbol{Z}$ in chapter 5.

The proof of theorem 4.5.1 will show that if $\boldsymbol{Z}$ is quasi-left-continuous, then Λ is continuous. This happens in particular when $\boldsymbol{Z}$ is a Lévy process (see section 4.6 below) or is the solution of a differential equation driven by a Lévy process (exercise 5.2.17 and page 349). If Λ is continuous, then the time transformation $\lambda \mapsto T^\lambda$ is evidently strictly increasing without bound.

Exercise 4.5.3 Suppose that inequality (4.5.1) holds whenever $\boldsymbol{X} \in \mathcal{P}^d$ is bounded and T reduces $\boldsymbol{Z}$ to a global L^p-integrator. Then it holds in the generality stated.

Controlling a Single Integrator

The remainder of the section up to page 249 is devoted to the proof of theorem 4.5.1. We start with the case $d = 1$: in this subsection $\boldsymbol{Z}$ is a single local L^q-integrator Z for some $q \in [2, \infty)$.

The main tools are the ***higher order brackets*** $Z^{[\rho]}$ defined for all t

by[12]

$$Z_t^{[1]} \stackrel{\text{def}}{=} Z_t \ , \quad Z_t^{[2]} \stackrel{\text{def}}{=} [Z,Z]_t = {}^c[Z,Z]_t + \int_{[\![0,t]\!]} y^2 \, \jmath_Z(dy,ds) \ ,$$

$$Z_t^{[\rho]} \stackrel{\text{def}}{=} \sum_{0\leq s\leq t} (\Delta Z_s)^\rho = \int_{[\![0,t]\!]} y^\rho \, \jmath_Z(dy,ds) \ , \qquad \text{for } \rho = 3,4,\ldots,$$

and

$$\mathopen{\vert\!\vert} Z^{[\rho]} \mathclose{\vert\!\vert}_t \stackrel{\text{def}}{=} \sum_{0\leq s\leq t} \left|\Delta Z_s\right|^\rho = \int_{[\![0,t]\!]} |y|^\rho \, \jmath_Z(dy,ds) \ ,$$

defined for any real $\rho > 2$ and satisfying

$$\mathopen{\vert\!\vert} Z^{[\rho]} \mathclose{\vert\!\vert}_t^{1/\rho} \leq \Big(\sum_{0\leq s\leq t} |\Delta Z_s|^2\Big)^{1/2} \leq S_t[Z] \ . \tag{4.5.5}$$

For integer ρ, $\mathopen{\vert\!\vert} Z^{[\rho]} \mathclose{\vert\!\vert}$ is the variation process of $Z^{[\rho]}$. Observe now that equation (3.10.7) can be rewritten in terms of the $Z^{[\rho]}$ as follows:

---

[12] $[\![0,t]\!]$ is the product $\mathbb{R}_*^d \times [\![0,t]\!]$ of ***auxiliary space*** $\mathbb{R}_*^d$ with the stochastic interval $[\![0,t]\!]$.

Lemma 4.5.4 *For an n-times continuously differentiable function Φ on $\mathbb{R}$ and any stopping time T*

$$\Phi(Z_T) = \Phi(Z_0) + \sum_{\nu=1}^{n-1} \frac{1}{\nu!} \int_{0+}^{T} \Phi^{(\nu)}(Z_{.-})\, dZ^{[\nu]}$$

$$+ \int_0^1 \frac{(1-\lambda)^{n-1}}{(n-1)!} \int_{0+}^{T} \Phi^{(n)}\big(Z_{.-} + \lambda \Delta Z\big)\, dZ^{[n]}\, d\lambda\,.$$

Let us apply lemma 4.5.4 to the function $\Phi(z) = |z|^p$, with $1 < p < \infty$. If n is a natural number strictly less than p, then Φ is n-times continuously differentiable. With $\epsilon = p - n$ we find, using item A.2.43 on page 388,

$$|Z_t|^p = |Z_0|^p + \sum_{\nu=1}^{n-1} \binom{p}{\nu} \int_{0+}^{t} |Z|_{.-}^{p-\nu} \cdot (\operatorname{sgn} Z_{.-})^\nu\, dZ^{[\nu]}$$

$$+ \int_0^1 n(1-\lambda)^{n-1} \int_{0+}^{t} \binom{p}{n} \big|Z_{.-} + \lambda \Delta Z\big|^\epsilon \big(\operatorname{sgn}\big(Z_{.-} + \lambda \Delta Z\big)\big)^n\, dZ^{[n]}\, d\lambda\,.$$

Writing $|Z_0|^p$ as $\int_{[\![0]\!]} d\,¦Z^{[p]}¦$ produces this useful estimate:

Corollary 4.5.5 *For every L^0-integrator Z, stopping time T, and $p > 1$ let $n = \lfloor p \rfloor$ be the largest integer less than or equal to p and set $\epsilon \stackrel{\text{def}}{=} p - n < 1$.*

$$\textit{Then}\quad |Z|_T^p \le p \int_0^T |Z|_{.-}^{p-1} \cdot \operatorname{sgn} Z_{.-}\, dZ + \sum_{\nu=2}^{n-1} \binom{p}{\nu} \int_0^T |Z|_{.-}^{p-\nu}\, d¦Z^{[\nu]}¦$$

$$+ \int_0^1 n(1-\lambda)^{n-1} \int_{0+}^{T} \binom{p}{n} \big(|Z|_{.-} + \lambda |\Delta Z|\big)^\epsilon\, d¦Z^{[n]}¦\, d\lambda\,. \tag{4.5.6}$$

Proof. This is clear when $p > \lfloor p \rfloor$. In the case that p is an integer, apply this to a sequence of $p_n > p$ that decrease to p and take the limit.

Now thanks to inequality (4.5.5) and theorem 3.8.4, $¦Z^{[\rho]}¦$ is locally integrable and therefore has a Doob–Meyer decomposition when ρ is any real number between 2 and q. We use this observation to define positive increasing previsible processes $Z^{\langle\rho\rangle}$ as follows: $Z^{\langle 1\rangle} = ¦\widehat{Z}¦$; and for $\rho \in [2, q]$, $Z^{\langle\rho\rangle}$ is the previsible part in the Doob–Meyer decomposition of $¦Z^{[\rho]}¦$. For instance, $Z^{\langle 2\rangle} = \langle Z, Z\rangle$. In summary

$$Z^{\langle 1\rangle} \stackrel{\text{def}}{=} ¦\widehat{Z}¦\,,\; Z^{\langle 2\rangle} \stackrel{\text{def}}{=} \langle Z, Z\rangle\,,\ \text{and}\ \ Z^{\langle\rho\rangle} \stackrel{\text{def}}{=} \widehat{¦Z^{[\rho]}¦} \qquad \text{for}\ \ 2 \le \rho \le q\,.$$

Exercise 4.5.6 $(X{*}Z)^{\langle\rho\rangle} = |X|^\rho {*} Z^{\langle\rho\rangle}$ for $X \in \mathcal{P}_b$ and $\rho \in \{1\} \cup [2, q]$.

In the following keep in mind that $Z^{\langle\rho\rangle} = 0$ for $\rho > 2$ if Z is continuous, and $Z^{\langle\rho\rangle} = 0$ for $\rho > 1$ if in addition Z has no martingale component, i.e., if Z is a continuous finite variation process.The desired previsible controller $\Lambda^{\langle q\rangle}[Z]$

will be constructed from the processes $Z^{\langle\rho\rangle}$, which we call the ***previsible higher order brackets***. On the way to the construction and estimate three auxiliary results are needed:

Lemma 4.5.7 *For $2 \le \rho < \sigma < \tau \le q$, we have both $Z^{\langle\sigma\rangle} \le Z^{\langle\rho\rangle} \vee Z^{\langle\tau\rangle}$ and $\big(Z^{\langle\sigma\rangle}\big)^{1/\sigma} \le \big(Z^{\langle\rho\rangle}\big)^{1/\rho} \vee \big(Z^{\langle\tau\rangle}\big)^{1/\tau}$, except possibly on an evanescent set. Also,*

$$\Big\| \big(Z_T^{\langle\sigma\rangle}\big)^{1/\sigma} \Big\|_{L^p} \le \Big\| \big(Z_T^{\langle\rho\rangle}\big)^{1/\rho} \Big\|_{L^p} \vee \Big\| \big(Z_T^{\langle\tau\rangle}\big)^{1/\tau} \Big\|_{L^p} \tag{4.5.7}$$

for any stopping time T and $p \in (0,\infty)$ – the right-hand side is finite for sure if Z^T is $\mathcal{I}^q$-bounded and $p \le q$.

Proof. A little exercise in calculus furnishes the equality

$$\inf\big\{A\lambda^{\rho-\sigma} + B\lambda^{\tau-\sigma} : \lambda > 0\big\} = C \cdot A^{\frac{\tau-\sigma}{\tau-\rho}} B^{\frac{\sigma-\rho}{\tau-\rho}} , \tag{4.5.8}$$

with
$$C = \Big(\frac{\sigma-\rho}{\tau-\sigma}\Big)^{\frac{\rho-\sigma}{\tau-\rho}} + \Big(\frac{\sigma-\rho}{\tau-\sigma}\Big)^{\frac{\tau-\sigma}{\tau-\rho}} .$$

The choice $A = B = 1$ and $\lambda = \big|\Delta Z_s\big|$ gives

$$C \cdot \big|\Delta Z_s\big|^\sigma \le \big|\Delta Z_s\big|^\rho + \big|\Delta Z_s\big|^\tau , \qquad 0 \le s < \infty ,$$

which says $\quad C \cdot d\,¦Z^{[\sigma]}¦ \le d\,¦Z^{[\rho]}¦ + d\,¦Z^{[\tau]}¦$

and implies
$$C \cdot dZ^{\langle\sigma\rangle} \le dZ^{\langle\rho\rangle} + dZ^{\langle\tau\rangle} \tag{4.5.9}$$

and
$$C \cdot Z^{\langle\sigma\rangle} \le Z^{\langle\rho\rangle} + Z^{\langle\tau\rangle} ,$$

except possibly on an evanescent set. Homogeneity produces

$$C \cdot \lambda^\sigma Z^{\langle\sigma\rangle} \le \lambda^\rho Z^{\langle\rho\rangle} + \lambda^\tau Z^{\langle\tau\rangle} , \qquad \lambda > 0 .$$

By changing $Z^{\langle\sigma\rangle}$ on an evanescent set we can arrange things so that this inequality holds at all points of the base space $\boldsymbol{B}$ and for all $\lambda > 0$. Equation (4.5.8) implies

$$C \cdot Z^{\langle\sigma\rangle} \le C \cdot \big(Z^{\langle\rho\rangle}\big)^{\frac{\tau-\sigma}{\tau-\rho}} \cdot \big(Z^{\langle\tau\rangle}\big)^{\frac{\sigma-\rho}{\tau-\rho}} ,$$

i.e.,
$$Z^{\langle\sigma\rangle} \le \big(Z^{\langle\rho\rangle}\big)^{\frac{\tau-\sigma}{\tau-\rho}} \cdot \big(Z^{\langle\tau\rangle}\big)^{\frac{\sigma-\rho}{\tau-\rho}}$$

and
$$\big(Z^{\langle\sigma\rangle}\big)^{1/\sigma} \le \big(Z^{\langle\rho\rangle\,1/\rho}\big)^{\frac{\rho(\tau-\sigma)}{\sigma(\tau-\rho)}} \cdot \big(Z^{\langle\tau\rangle\,1/\tau}\big)^{\frac{\tau(\sigma-\rho)}{\sigma(\tau-\rho)}} . \tag{$*$}$$

The two exponents e_ρ and e_τ on the right-hand side sum to 1, in either of the previous two inequalities, and this produces the first two inequalities of lemma 4.5.7; the third one follows by taking the p^{th} root after applying Hölder's inequality with conjugate exponents $1/e_\rho$ and $1/e_\tau$ to the p^{th} power of $(*)$. ∎

Exercise 4.5.8 Let $\mu^{\langle\rho\rangle}$ denote the Doléans–Dade measure of $Z^{\langle\rho\rangle}$. Then $\mu^{\langle\sigma\rangle} \le \mu^{\langle\rho\rangle} \vee \mu^{\langle\tau\rangle}$ whenever $2 \le \rho < \sigma < \tau \le q$.

Lemma 4.5.9 *At any stopping time T and for all $X \in \mathcal{P}_b$ and all $p \in [2, q]$*

$$\Big\| |X{*}Z|_T^\star \Big\|_{L^p} \le C_p^\diamond \cdot \max_{\rho=1^\diamond,2,p^\diamond} \Big\| \Big(\int_0^T |X|^\rho \, dZ^{\langle\rho\rangle} \Big)^{1/\rho} \Big\|_{L^p} , \tag{4.5.10}$$

$$\le C_p^\diamond \cdot \max_{\rho=1^\diamond,2,q^\diamond} \Big\| \Big(\int_0^T |X|^\rho \, dZ^{\langle\rho\rangle} \Big)^{1/\rho} \Big\|_{L^p} , \tag{4.5.11}$$

with universal constant $C_p^\diamond \le 9.5p$.

Proof. First assume that Z is a global L^q-integrator. Let $n = \lfloor p \rfloor$ be the largest integer less than or equal to p, set $\epsilon = p - n$

and
$$\zeta = \zeta[Z] \stackrel{\text{def}}{=} \max_{\rho=1,\ldots,n,p} \Big\| \big(Z_\infty^{\langle\rho\rangle} \big)^{1/\rho} \Big\|_{L^p} \tag{4.5.12}$$

by inequality (4.5.7):
$$= \max_{\rho=1,2,p} \Big\| \big(Z_\infty^{\langle\rho\rangle} \big)^{1/\rho} \Big\|_{L^p} = \max_{\rho=1^\diamond,2,p^\diamond} \Big\| \big(Z_\infty^{\langle\rho\rangle} \big)^{1/\rho} \Big\|_{L^p}$$

$$\le \max_{\rho=1^\diamond,2,q^\diamond} \Big\| \big(Z_\infty^{\langle\rho\rangle} \big)^{1/\rho} \Big\|_{L^p} . \tag{4.5.13}$$

The last equality follows from the fact that $Z^{\langle\rho\rangle} = 0$ for $\rho > 2$ if Z is continuous and for $\rho = 1$ if it is a martingale, and the previous inequality follows again from (4.5.7). Applying the expectation to inequality (4.5.6) on page 240 produces

$$\mathbb{E}[|Z|_\infty^p] \le p\mathbb{E}\Big[\int_0^\infty |Z|_{\bullet-}^{p-1} \cdot \operatorname{sgn} Z_{\bullet-} \, dZ \Big] + \sum_{\nu=2}^{n-1} \binom{p}{\nu} \mathbb{E}\Big[\int_0^\infty |Z|_{\bullet-}^{p-\nu} \, d\,¦Z^{[\nu]}¦ \Big]$$
$$+ \int_0^1 n(1-\lambda)^{n-1} \binom{p}{n} \mathbb{E}\Big[\int_{0+}^\infty \big(|Z|_{\bullet-} + \lambda |\Delta Z| \big)^\epsilon \, d\,¦Z^{[n]}¦ \Big] \, d\lambda .$$
$$\le \sum_{\nu=1}^{n-1} \binom{p}{\nu} \mathbb{E}\Big[\int_0^\infty |Z|_{\bullet-}^{p-\nu} \, dZ^{\langle\nu\rangle} \Big]$$
$$+ \int_0^1 n(1-\lambda)^{n-1} \binom{p}{n} \mathbb{E}\Big[\int_{0+}^\infty \big(|Z|_{\bullet-} + \lambda |\Delta Z| \big)^\epsilon \, d\,¦Z^{[n]}¦ \Big] \, d\lambda$$
$$= Q_1 + Q_2 . \tag{4.5.14}$$

Let us estimate the expectations in the first quantity Q_1:

$$\mathbb{E}\Big[\int_0^\infty |Z|_{\bullet-}^{p-\nu} \, dZ^{\langle\nu\rangle} \Big] \le \mathbb{E}\Big[|Z_\infty^\star|^{p-\nu} \cdot Z_\infty^{\langle\nu\rangle} \Big]$$

using Hölder's inequality:
$$\le \|Z_\infty^*\|_{L^p}^{p-\nu} \cdot \Big\| \big(Z_\infty^{\langle\nu\rangle} \big)^{1/\nu} \Big\|_{L^p}^\nu$$

by definition (4.5.12):
$$\le \|Z_\infty^\star\|_{L^p}^{p-\nu} \cdot \zeta^\nu .$$

Therefore
$$Q_1 \le \sum_{\nu=1}^{n-1} \binom{p}{\nu} \|Z_\infty^\star\|_{L^p}^{p-\nu} \cdot \zeta^\nu . \tag{4.5.15}$$

To treat Q_2 we assume to start with that $\zeta = 1$. This implies that the measure $X \mapsto \mathbb{E}[\int X \, d\lvert Z^{[p]}\rvert]$ on measurable processes X has total mass

$$\mathbb{E}\Big[\int 1 \, d\lvert Z^{[p]}\rvert\Big] = \mathbb{E}\Big[Z_\infty^{\langle p\rangle}\Big] = \Big\| \big(Z_\infty^{\langle p\rangle}\big)^{1/p} \Big\|_{L^p}^p \le \zeta^p = 1$$

(For the first equality see inequality (4.3.1) on page 221 and exercise 2.5.33 on page 86) and makes the Jensen inequality of exercise A.3.25 applicable to the concave function $\mathbb{R}_+ \ni z \mapsto z^\epsilon$ in $(*)$ below:

$$\begin{aligned}
&\mathbb{E}\Big[\int_{0+}^\infty \big(|Z|_{\text{.}-} + \lambda|\Delta Z|\big)^\epsilon \, d\lvert Z^{[n]}\rvert\Big] \\
&= \mathbb{E}\Big[\int_{0+}^\infty \big(|Z|_{\text{.}-}|\Delta Z|^{-1} + \lambda\big)^\epsilon \, d\lvert Z^{[p]}\rvert\Big] && (*) \\
&\le \Big(\mathbb{E}\Big[\int_{0+}^\infty |Z|_{\text{.}-}|\Delta Z^{-1} \, d\lvert Z^{[p]}\rvert\Big] + \lambda\mathbb{E}\Big[\int_{0+}^\infty d\lvert Z^{[p]}\rvert\Big]\Big)^\epsilon \\
&= \Big(\mathbb{E}\Big[\int_{0+}^\infty |Z|_{\text{.}-} \, d\lvert Z^{[p-1]}\rvert\Big] + \lambda\mathbb{E}\Big[Z_\infty^{\langle p\rangle}\Big]\Big)^\epsilon \\
&\le \Big(\mathbb{E}\Big[Z_\infty^\star Z_\infty^{\langle p-1\rangle}\Big] + \lambda\Big)^\epsilon \\
\text{by Hölder:} \quad &\le \Big(\|Z_\infty^\star\|_{L^p} \cdot \Big\|\big|Z_\infty^{\langle p-1\rangle}\big|^{1/(p-1)}\Big\|_{L^p}^{p-1} + \lambda\Big)^\epsilon \\
\text{as } \zeta = 1: \quad &\le \big(\|Z_\infty^\star\|_{L^p} \cdot \zeta^{p-1} + \lambda\big)^\epsilon = \big(\|Z_\infty^\star\|_{L^p} + \lambda\zeta\big)^\epsilon \cdot \zeta^n \,.
\end{aligned}$$

We now put this and inequality (4.5.15) into inequality (4.5.14) and obtain

$$\begin{aligned}
\mathbb{E}[|Z|_\infty^p] &\le \sum_{\nu=1}^{n-1} \binom{p}{\nu} \|Z_\infty^\star\|_{L^p}^{p-\nu} \cdot \zeta^\nu \\
&\quad + \int_0^1 n(1-\lambda)^{n-1}\binom{p}{n}\big(\|Z_\infty^\star\|_{L^p} + \lambda\zeta\big)^\epsilon \cdot \zeta^n \, d\lambda \\
\text{by A.2.43:} \quad &= \big(\|Z_\infty^\star\|_{L^p} + \zeta\big)^p - \|Z_\infty^\star\|_{L^p}^p \,,
\end{aligned}$$

which we rewrite as

$$\| Z_\infty \|_{L^p}^p + \| Z_\infty^\star \|_{L^p}^p \le \big(\| Z_\infty^\star \|_{L^p} + \zeta[Z]\big)^p \,. \tag{4.5.16}$$

If $\zeta[Z] \ne 1$, we obtain $\zeta[\rho Z] = 1$ and with it inequality (4.5.16) for a suitable multiple ρZ of Z; division by ρ^p produces that inequality for the given Z.

We leave it to the reader to convince herself with the aid of theorem 2.3.6 that (4.5.10) and (4.5.11) hold, with $C_p^\diamond \le 4^p$. Since this constant increases exponentially with p rather than linearly, we go a different, more labor-intensive route:

If Z is a positive increasing process I, then $I = I^\star$ and (4.5.16) gives

$$2^{1/p} \cdot \|I^\star_\infty\|_{L^p} \le \|I^\star_\infty\|_{L^p} + \zeta[I] ,$$

i.e.,
$$\|I^\star_\infty\|_{L^p} \le \Big(2^{1/p} - 1\Big)^{-1} \cdot \zeta[I] . \tag{4.5.17}$$

It is easy to see that $\big(2^{1/p}-1\big)^{-1} \le p/\ln 2 \le 3p/2$. If I instead is predictable and has finite variation, then inequality (4.5.17) still obtains. Namely, there is a previsible process D of absolute value 1 such that $D{*}I$ is increasing. We can choose for D the Radon–Nikodym derivative of the Doléans–Dade measure of I with respect to that of $\vert I \vert$. Since $I^{\langle\rho\rangle} = \vert I \vert^{\langle\rho\rangle}$ for all ρ and therefore $\zeta[I] = \zeta[\vert I \vert]$, we arrive again at inequality (4.5.17):

$$\| I^\star_\infty \|_{L^p} \le \Big\| \vert I \vert^\star_\infty \Big\|_{L^p} \le (3p/2)\, \zeta[\vert I \vert] = (3p/2)\, \zeta[I] . \tag{4.5.18}$$

Next consider the case that Z is a q-integrable martingale M. Doob's maximal theorem 2.5.19, rewritten as

$$\big((1/p')^p + 1\big) \cdot \|M^\star_\infty\|^p_{L^p} \le \|M_\infty\|^p_{L^p} + \|M^\star_\infty\|^p_{L^p} ,$$

turns (4.5.16) into

$$\big((1/p')^p + 1\big)^{1/p} \cdot \|M^\star_\infty\|_{L^p} \le \|M^\star_\infty\|_{L^p} + \zeta[M] ,$$

which reads
$$\|M^\star_\infty\|_{L^p} \le \Big(\big((1/p')^p + 1\big)^{1/p} - 1\Big)^{-1} \cdot \zeta[M] . \tag{4.5.19}$$

We leave as an exercise the estimate $\Big(\big((1/p')^p+1\big)^{1/p}-1\Big)^{-1} \le 5p$ for $p > 2$.

Let us return to the general L^p-integrator Z. Let $Z = \widehat{Z} + \widetilde{Z}$ be its Doob–Meyer decomposition. Due to lemma 4.5.10 below, $\zeta[\widehat{Z}] \le \zeta[Z]$ and $\zeta[\widetilde{Z}] \le 2\zeta[Z]$. Consequently,

$$\|Z^\star_\infty\|_{L^p} \le \Big\|\widehat{Z}^\star_\infty\Big\|_{L^p} + \Big\|\widetilde{Z}^\star_\infty\Big\|_{L^p}$$

by (4.5.18) and (4.5.19):
$$\le (3p/2 + 2 \cdot 5p)\, \zeta[Z] \le 12p \cdot \zeta[Z] .$$

The number 12 can be replaced by 9.5 if slightly more fastidious estimates are used. In view of the definition (4.5.12) of ζ and the bound (4.5.13) for it we have arrived at

$$\| Z^\star_\infty \|_{L^p} \le 9.5p \max_{\rho=1,2,p^\diamond} \Big\| \big(Z^{\langle\rho\rangle}_\infty\big)^{1/\rho} \Big\|_{L^p} \le 9.5p \max_{\rho=1,2,q^\diamond} \Big\| \big(Z^{\langle\rho\rangle}_\infty\big)^{1/\rho} \Big\|_{L^p} .$$

Inequality (4.5.10) follows from an application of this and exercise 4.5.6 to $X{*}Z^{T\wedge T_n}$, for which the quantities ζ, etc., are finite if T_n reduces Z to a global L^q-integrator, and letting $T_n \uparrow \infty$. This establishes (4.5.10) and (4.5.11). ∎

Lemma 4.5.10 *Let $Z = \widehat{Z} + \widetilde{Z}$ be the Doob–Meyer decomposition of Z. Then*

$$\widehat{Z}^{\langle\rho\rangle} \le Z^{\langle\rho\rangle} \quad \textit{and} \quad \widetilde{Z}^{\langle\rho\rangle} \le 2^\rho Z^{\langle\rho\rangle} , \qquad \rho \in \{1\} \cup [2, q] .$$

Proof. We clearly may reduce this to the case that Z is a *global* L^q-integrator.

First the case $\rho = 1$. Since $\widehat{Z}^{[1]} = \widehat{Z}$ is previsible, $\widehat{Z}^{\langle 1\rangle} = ¦\widehat{Z}¦ = Z^{\langle 1\rangle}$.

Next let $2 \le \rho \le q$. Then $¦\widehat{Z}^{[\rho]}¦_t = \sum_{s\le t} \big|\Delta\widehat{Z}_s\big|^\rho$ is increasing and predictable on the grounds (cf. 4.3.3) that it jumps only at predictable stopping times S and there has the jump (see exercise 4.3.6)

$$\Delta ¦\widehat{Z}^{[\rho]}¦_S = \big|\Delta\widehat{Z}_S\big|^\rho = \big|\mathbb{E}[\Delta Z_S|\mathcal{F}_{S-}]\big|^\rho ,$$

which is measurable on the strict past $\mathcal{F}_{S-}$. Now this jump is by Jensen's inequality (A.3.10) less than

$$\mathbb{E}\big[|\Delta Z_S|^\rho \big|\mathcal{F}_{S-}\big] = \mathbb{E}\big[\Delta ¦Z^{[\rho]}¦_S \big|\mathcal{F}_{S-}\big] = \Delta Z^{\langle\rho\rangle}_S .$$

That is to say, the predictable increasing process $\widehat{Z}^{\langle\rho\rangle} = ¦\widehat{Z}^{[\rho]}¦$, which has no continuous part, jumps only at predictable times S and there by less than the predictable increasing process $Z^{\langle\rho\rangle}$. Consequently (see exercise 4.3.7) $d\widehat{Z}^{\langle\rho\rangle} \le dZ^{\langle\rho\rangle}$ and the first inequality is established.

Now to the martingale part. Clearly $\widetilde{Z}^{\langle 1\rangle} = 0$. At $\rho = 2$ we observe that $[Z, Z]$ and $[\widetilde{Z}, \widetilde{Z}] + [\widehat{Z}, \widehat{Z}]$ differ by the local martingale $2[\widehat{Z}, \widetilde{Z}]$ – see exercise 3.8.24(iii) – and therefore

$$\widetilde{Z}^{\langle 2\rangle} = \widehat{[\widetilde{Z}, \widetilde{Z}]} \le \widehat{[Z, Z]} = \langle Z, Z\rangle = Z^{\langle 2\rangle} .$$

If $\rho > 2$, then $\qquad \big|\Delta\widetilde{Z}_s\big|^\rho \le 2^{\rho-1}\big(\big|\Delta Z_s\big|^\rho + \big|\Delta\widehat{Z}_s\big|^\rho\big)$,

which reads $\qquad d¦\widetilde{Z}^{[\rho]}¦ \le 2^{\rho-1}\big(d¦Z^{[\rho]}¦ + d¦\widehat{Z}^{[\rho]}¦\big)$

by part 1: $\qquad \le 2^{\rho-1}\big(d¦Z^{[\rho]}¦ + dZ^{\langle\rho\rangle}\big)$.

The predictable parts of the Doob–Meyer decomposition are thus related by

$$\widetilde{Z}^{\langle\rho\rangle} \le 2^{\rho-1}\big(Z^{\langle\rho\rangle} + Z^{\langle\rho\rangle}\big) = 2^\rho Z^{\langle\rho\rangle} .$$

∎

Proof of Theorem 4.5.1 for a Single Integrator. While lemma 4.5.9 affords pathwise and solid control of the indefinite integral by previsible processes of finite variation, it is still bothersome to have to contend with two or three different previsible processes $Z^{\langle\rho\rangle}$. Fortunately it is possible to reduce their number to only one. Namely, for each of the $Z^{\langle\rho\rangle}$, $\rho = 1^\diamond, 2, q^\diamond$, let $\mu^{\langle\rho\rangle}$ denote its Doléans–Dade measure. To this collection add (artificially) the measure $\mu^{\langle 0\rangle} \stackrel{\text{def}}{=} \alpha \cdot dt \times \mathbb{P}$. Since the measures on $\mathcal{P}$ form a vector lattice (page 406), there is a least upper bound $\nu \stackrel{\text{def}}{=} \mu^{\langle 0\rangle} \vee \mu^{\langle 1\rangle} \vee \mu^{\langle 2\rangle} \vee \mu^{\langle q^\diamond\rangle}$. If Z is a martingale, then $\mu^{\langle 1\rangle} = 0$, so that $\nu = \mu^{\langle 0\rangle} \vee \mu^{\langle 1^\diamond\rangle} \vee \mu^{\langle 2\rangle} \vee \mu^{\langle q^\diamond\rangle}$, always. Let $\Lambda^{\langle q\rangle}[Z]$ denote the Doléans–Dade process of ν. It provides the pathwise and solid control of the indefinite integral $X{*}Z$ promised in theorem 4.5.1. Indeed, since by exercise 4.5.8

$$dZ^{\langle\rho\rangle} \le d\Lambda^{\langle q\rangle}[Z]\;, \qquad\qquad \rho\in\{1^\diamond,2,p^\diamond,q^\diamond\}\;,$$

each of the $Z^{\langle\rho\rangle}$ in (4.5.10) and (4.5.11) can be replaced by $\Lambda^{\langle q\rangle}[Z]$ without disturbing the inequalities; with exercise A.8.5, inequality (4.5.1) is then immediate from (4.5.10).

Except for inequality (4.5.2), which we save for later, the proof of theorem 4.5.1 is complete in the case $d=1$. ∎

Exercise 4.5.11 Assume that Z is a continuous integrator. Then Z is a local L^q-integrator for any $q>0$. Then $\Lambda=\Lambda^{\langle q\rangle}=\Lambda^{\langle 2\rangle}$ is a controller for $[Z,Z]$. Next let f be a function with two continuous derivatives, both bounded by L. Then Λ also controls $f(Z)$. In fact for all $T\in\mathfrak{T}$, $X\in\mathcal{P}$, and $p\in[2,q]$

$$\big\|\,|X{*}f(Z)|^\star_T\big\|_{L^p} \le (C^\diamond_p+1)L\cdot\max_{\rho=1,2}\Big\|\Big(\int_0^T |X|^\rho_s\,d\Lambda_s\Big)^{1/\rho}\Big\|_{L^p}\;.$$

Previsible Control of Vectors of Integrators

A stochastic differential equation frequently is driven by not one or two but by a whole slew $\boldsymbol{Z}=(Z^1,Z^2,\ldots,Z^d)$ of integrators – see equation (1.1.9) on page 8 or equation (5.1.3) on page 271, and page 56. Its solution requires a single previsible control for all the Z^η simultaneously. This can of course simply be had by adding the $\Lambda^{\langle q\rangle}[Z^\eta]$; but that introduces their number d into the estimates, sacrificing sharpness of estimates and rendering them inapplicable to random measures. So we shall go a different if slightly more labor-intensive route.

We are after control of $\boldsymbol{Z}$ as expressed in inequality (4.5.1); the problem is to find and estimate a suitable previsible controller $\Lambda=\Lambda^{\langle q\rangle}[\boldsymbol{Z}]$ as in the scalar case. The idea is simple. Write $\boldsymbol{X}=|\boldsymbol{X}|\cdot\boldsymbol{X}'$, where $\boldsymbol{X}'$ is a vector field of previsible processes with $|\boldsymbol{X}'_s|(\omega)\overset{\text{def}}{=}\sup_\eta|X'_{\eta\,s}(\omega)|\le 1$ for all $(s,\omega)\in\boldsymbol{B}$. Then $\boldsymbol{X}{*}\boldsymbol{Z}=|\boldsymbol{X}|{*}(\boldsymbol{X}'{*}\boldsymbol{Z})$, and so in view of inequality (4.5.11)

$$\|\,\boldsymbol{X}{*}\boldsymbol{Z}^\star_T\|_{L^p}\le C^\diamond_p\cdot\max_{\rho=1^\diamond,2,q^\diamond}\Big\|\Big(\int_0^T|\boldsymbol{X}|^\rho\,d(\boldsymbol{X}'{*}\boldsymbol{Z})^{\langle\rho\rangle}\Big)^{1/\rho}\Big\|_{L^p}$$

whenever $2\le p\le q$. It turns out that there are increasing previsible processes $\boldsymbol{Z}^{\langle\rho\rangle}$, $\rho=1^\diamond,2,q^\diamond$, that satisfy

$$d(\boldsymbol{X}'{*}\boldsymbol{Z})^{\langle\rho\rangle}\le d\boldsymbol{Z}^{\langle\rho\rangle}$$

simultaneously for all predictable $\boldsymbol{X}'=(X'_1,\ldots,X'_d)$ with $|\boldsymbol{X}'|\le 1$. Then

$$\|\,|\boldsymbol{X}{*}\boldsymbol{Z}|^\star_T\|_{L^p}\le C^\diamond_p\cdot\max_{\rho=1^\diamond,2,q^\diamond}\Big\|\Big(\int_0^T|\boldsymbol{X}|^\rho\,d\boldsymbol{Z}^{\langle\rho\rangle}\Big)^{1/\rho}\Big\|_{L^p}\;. \tag{4.5.20}$$

The $\boldsymbol{Z}^{\langle\rho\rangle}$ can take the role of the $Z^{\langle\rho\rangle}$ in lemma 4.5.9. They can in fact be chosen to be of the form $\boldsymbol{Z}^{\langle\rho\rangle} = ({}^\rho\boldsymbol{X}*\boldsymbol{Z})^{\langle\rho\rangle}$ with ${}^\rho\boldsymbol{X}$ predictable and having $|{}^\rho\boldsymbol{X}| \le 1$; this latter fact will lead to the estimate (4.5.2).

1) To find $\boldsymbol{Z}^{\langle 1\rangle}$ we look at the Doob–Meyer decomposition $\boldsymbol{Z} = \widehat{\boldsymbol{Z}} + \widetilde{\boldsymbol{Z}}$, in obvious notation. Clearly

$$d(\boldsymbol{X}'*\boldsymbol{Z})^{\langle 1\rangle} = \textstyle\sum_\eta X'_\eta\, d\widehat{Z^\eta} \le \sum_\eta \big|X'_\eta\big|\, d\big\lceil\widehat{Z^\eta}\big\rceil \le d\boldsymbol{Z}^{\langle 1\rangle} \qquad (4.5.21)$$

for all $\boldsymbol{X}' \in \mathcal{P}^d$ having $|\boldsymbol{X}'| \le 1$, provided that we define

$$\boldsymbol{Z}^{\langle 1\rangle} \stackrel{\text{def}}{=} \sum_{1\le\eta\le d} \big\lceil\widehat{Z}^\eta\big\rceil\ .$$

To estimate the size of this controller let G^η be a previsible Radon–Nikodym derivative of the Doléans–Dade measure of $\widehat{Z}^\eta$ with respect to that of $\big\lceil\widehat{Z}^\eta\big\rceil$. These are previsible processes of absolute value 1, which we assemble into a d-tuple to make up the vector field ${}^1\boldsymbol{X}$. Then

$$\mathbb{E}\Big[\boldsymbol{Z}^{\langle 1\rangle}_\infty\Big] = \mathbb{E}\Big[\int {}^1\boldsymbol{X}\, d\boldsymbol{Z}\Big] \le \big\lceil {}^1\boldsymbol{X}*\widehat{\boldsymbol{Z}}\big\rceil_{\mathcal{I}^1}$$

by inequality (4.3.1):
$$\le \big\lceil {}^1\boldsymbol{X}*\boldsymbol{Z}\big\rceil_{\mathcal{I}^1} \le \big\lceil \boldsymbol{Z}\big\rceil_{\mathcal{I}^1}\ . \qquad (4.5.22)$$

Exercise 4.5.12 Assume $\boldsymbol{Z}$ is a global L^q-integrator. Then the Doléans–Dade measure of $\boldsymbol{Z}^{\langle 1\rangle}$ is the maximum in the vector lattice $\mathfrak{M}^*[\mathcal{P}]$ (see page 406) of the Doléans–Dade measures of the processes $\big\{\boldsymbol{X}'*\boldsymbol{Z} : \boldsymbol{X}' \in (\mathcal{E}^d)^\sigma, |\boldsymbol{X}'| \le 1\big\}$.

2) To find next the previsible controller $\boldsymbol{Z}^{\langle 2\rangle}$, consider the equality

$$d(\boldsymbol{X}'*\boldsymbol{Z})^{\langle 2\rangle} = \sum_{1\le\eta,\theta\le d} X'_\eta\, X'_\theta\, d\langle Z^\eta, Z^\theta\rangle\ .$$

Let $\mu^{\eta,\theta}$ be the Doléans–Dade measure of the previsible bracket $\langle Z^\eta, Z^\theta\rangle$. There exists a positive σ-additive measure μ on the previsibles with respect to which every one of the $\mu^{\eta,\theta}$ is absolutely continuous, for instance, the sum of their variations. Let $G^{\eta,\theta}$ be a previsible Radon–Nikodym derivative of $\mu^{\eta,\theta}$ with respect to μ, and V the Doléans–Dade process of μ. Then

$$\langle Z^\eta, Z^\theta\rangle = G^{\eta,\theta}*V\ .$$

On the product of the vector space $\mathcal{G}$ of $d\times d$-matrices g with the unit ball (box) of $\ell^\infty(d)$ define the function Φ by $\Phi(g,y) \stackrel{\text{def}}{=} \sum_{\eta,\theta} y_\eta y_\theta\, g^{\eta,\theta}$ and the function σ by $\sigma(g) \stackrel{\text{def}}{=} \sup\{\Phi(g,y) : y \in \ell^\infty_1\}$. This is a continuous function of $g \in \mathcal{G}$, so the process $\sigma(G)$ is previsible. The previous equality gives

$$d(\boldsymbol{X}'*\boldsymbol{Z})^{\langle 2\rangle} = \sum_{\eta,\theta} X'_\eta X'_\theta\, G^{\eta,\theta}\, dV \le \sigma(G)\, dV = d\boldsymbol{Z}^{\langle 2\rangle}\ ,$$

for all $\boldsymbol{X}' \in \mathcal{P}^d$ with $|\boldsymbol{X}'| \le 1$, provided we define

$$\boldsymbol{Z}^{\langle 2\rangle} \stackrel{\text{def}}{=} \sigma(G)*V\ .$$

To estimate the size of $\boldsymbol{Z}^{\langle 2\rangle}$, we use the Borel function $\gamma : \mathcal{G} \to \ell_1^\infty$ with $\sigma(g) = \Phi\big(g, \gamma(g)\big)$ that is provided by lemma A.2.21 (b). Since ${}^2\boldsymbol{X} \stackrel{\text{def}}{=} \gamma \circ G$ is a previsible vector field with $|{}^2\boldsymbol{X}| \le 1$,

$$\mathbb{E}\Big[\boldsymbol{Z}^{\langle 2\rangle}_\infty\Big] = \mathbb{E}\Big[\int_0^\infty \sigma(G)\, dV\Big] = \int \sigma(G)\, d\mu$$

$$= \int \sum_{\eta,\theta} {}^2X_\eta {}^2X_\theta\, G^{\eta,\theta}\, d\mu = \mathbb{E}\Big[\int_0^\infty \sum_{\eta,\theta} {}^2X_\eta {}^2X_\theta\, d\langle Z^\eta, Z^\theta\rangle\Big]$$

$$= \mathbb{E}[\langle {}^2\boldsymbol{X}{*}\boldsymbol{Z}, {}^2\boldsymbol{X}{*}\boldsymbol{Z}\rangle_\infty] = \mathbb{E}\big[[{}^2\boldsymbol{X}{*}\boldsymbol{Z}, {}^2\boldsymbol{X}{*}\boldsymbol{Z}]_\infty\big] \tag{4.5.23}$$

$$= \mathbb{E}\Big[\big(S_\infty[{}^2\boldsymbol{X}{*}\boldsymbol{Z}]\big)^2\Big] \le \lceil\!\lceil {}^2\boldsymbol{X}{*}\boldsymbol{Z} \rceil\!\rceil^2_{\mathcal{I}^2} \le \lceil\!\lceil \boldsymbol{Z} \rceil\!\rceil^2_{\mathcal{I}^2}\ . \tag{4.5.24}$$

Exercise 4.5.13 Assume $\boldsymbol{Z}$ is a global L^q-integrator, $q \ge 2$. Then the Doléans–Dade measure of $\boldsymbol{Z}^{\langle 2\rangle}$ is the maximum in the vector lattice $\mathfrak{M}^*[\mathcal{P}]$ of the Doléans–Dade measures of the brackets $\big\{[\boldsymbol{X}'{*}\boldsymbol{Z}, \boldsymbol{X}'{*}\boldsymbol{Z}] : \boldsymbol{X}' \in (\mathcal{E}^d)^\sigma, |\boldsymbol{X}'| \le 1\big\}$.

q) To find a useful previsible controller $\boldsymbol{Z}^{\langle q\rangle}$ now that $\boldsymbol{Z}^{\langle 1\rangle}$ and $\boldsymbol{Z}^{\langle 2\rangle}$ have been identified, we employ the Doob–Meyer decomposition of the jump measure $\jmath_{\boldsymbol{Z}}$ from page 232. According to it,

$$\mathbb{E}\Big[\int_0^\infty |X|^q\, d\lceil\!\lceil \boldsymbol{X}'{*}\boldsymbol{Z}^{[q]}\rceil\!\rceil\Big] = \mathbb{E}\Big[\int_{\mathbb{R}^d_*\times[\![0,\infty)\!)} |X_s|^q \big|\langle \boldsymbol{X}'_s|\boldsymbol{y}\rangle\big|^q\, \jmath_{\boldsymbol{Z}}(d\boldsymbol{y}, ds)\Big]$$

$$= \mathbb{E}\Big[\int_{[\![0,\infty)\!)} |X_s|^q \int_{\mathbb{R}^d_*} \big|\langle \boldsymbol{X}'_s|\boldsymbol{y}\rangle\big|^q\, \nu_s(d\boldsymbol{y})\, dV_s\Big]\ .$$

Now the process $\sigma^{\langle q\rangle}$ defined by

$$\sigma^{\langle q\rangle}_s = \sup_{|\boldsymbol{x}'|\le 1} \int \big|\langle \boldsymbol{x}'|\boldsymbol{y}\rangle\big|^q\, \nu_s(d\boldsymbol{y}) = \sup_{|\boldsymbol{x}'|\le 1} \big\|\boldsymbol{x}'\big\|^q_{L^q(\nu_s)}$$

is previsible, inasmuch as it suffices to extend the supremum over $\boldsymbol{x}' \in \ell_1^\infty(d)$ with rational components. Therefore

$$\mathbb{E}\Big[\int_0^\infty |X|^q\, d(\boldsymbol{X}'{*}\boldsymbol{Z})^{\langle q\rangle}\Big] = \mathbb{E}\Big[\int_0^\infty |X|^q_s \int_{\mathbb{R}^d_*} \big|\langle \boldsymbol{X}'_s|\boldsymbol{y}\rangle\big|^q\, \nu_s(d\boldsymbol{y})\, dV_s\Big]$$

$$\le \mathbb{E}\Big[\int_0^\infty |X|^q\, \sigma^{\langle q\rangle}_s\, dV_s\Big]\ .$$

From this inequality we read off the fact that $d(\boldsymbol{X}'{*}\boldsymbol{Z})^{\langle q\rangle} \le d\boldsymbol{Z}^{\langle q\rangle}$ for all $\boldsymbol{X}' \in \mathcal{P}^d$ with $|\boldsymbol{X}'| \le 1$, provided we define

$$\boldsymbol{Z}^{\langle q\rangle} \stackrel{\text{def}}{=} \sigma^{\langle q\rangle}{*}V\ .$$

To estimate $\boldsymbol{Z}^{\langle q\rangle}$ observe that the supremum in the definition of $\sigma^{\langle q\rangle}$ is assumed in one of the 2^d extreme points (corners) of $\ell_1^\infty(d)$, on the grounds that the function

$$\boldsymbol{x} \mapsto \phi_\varpi(\boldsymbol{x}) \stackrel{\text{def}}{=} \int \big|\langle \boldsymbol{x}|\boldsymbol{y}\rangle\big|^q\, \nu_\varpi(d\boldsymbol{y})$$

is convex.[13] Enumerate the corners: $\boldsymbol{c}_1, \boldsymbol{c}_2, \ldots$ and consider the previsible sets $P_k \stackrel{\text{def}}{=} \{\varpi : \phi_\varpi(\boldsymbol{c}_k) = \sigma(\varpi)\}$ and $P'_k \stackrel{\text{def}}{=} P_k \setminus \bigcup_{i<k} P_i$, $k = 1, \ldots, 2^d$. The vector field ${}^q\boldsymbol{X}$ which on P'_k has the value $\boldsymbol{c}_k$ clearly satisfies

$$\boldsymbol{Z}^{\langle q\rangle} = ({}^q\boldsymbol{X}*\boldsymbol{Z})^{\langle q\rangle} .$$

Thanks to inequality (4.5.5) and theorem 3.8.4,

$$\mathbb{E}\Big[\boldsymbol{Z}^{\langle q\rangle}_\infty\Big] = \mathbb{E}\Big[({}^q\boldsymbol{X}*\boldsymbol{Z})^{\langle q\rangle}_\infty\Big] = \mathbb{E}[\,\vert({}^q\boldsymbol{X}*\boldsymbol{Z})^{[q]}\vert_\infty]$$

by 3.8.21: $$= \mathbb{E}\Big[\sum_{s<\infty} |\langle {}^q\boldsymbol{X}_s|\Delta\boldsymbol{Z}_s\rangle|^q\Big] = \mathbb{E}\Big[\int_{\check{\boldsymbol{B}}} |\langle {}^q\boldsymbol{X}_s|\boldsymbol{y}\rangle|^q \, \jmath_{\boldsymbol{Z}}(d\boldsymbol{y}, ds)\Big] \tag{4.5.25}$$

$$\le \mathbb{E}\big[S^q_\infty[{}^q\boldsymbol{X}*\boldsymbol{Z}]\big] \le \lceil\!\lceil {}^q\boldsymbol{X}*\boldsymbol{Z} \rceil\!\rceil^q_{\mathcal{I}^q} \le \lceil\!\lceil \boldsymbol{Z} \rceil\!\rceil^q_{\mathcal{I}^q} . \tag{4.5.26}$$

Proof of Theorem 4.5.1. We now define the desired previsible controller $\Lambda^{\langle q\rangle}[\boldsymbol{Z}]$ as before to be the Doléans–Dade process of the supremum of $\mu^{\langle 0\rangle}$ and the Doléans–Dade measures of $\boldsymbol{Z}^{\langle 1\rangle}$, $\boldsymbol{Z}^{\langle 2\rangle}$, and $\boldsymbol{Z}^{\langle q\rangle}$ and continue as in the proof of theorem 4.5.1 on page 245, replacing $Z^{\langle\rho\rangle}$ by $\boldsymbol{Z}^{\langle\rho\rangle}$ for $\rho = 1, 2, q$.

To establish the estimate (4.5.2) of $\Lambda^{\langle q\rangle}[\boldsymbol{Z}]$ observe that

$$\Lambda^{\langle q\rangle}_T[\boldsymbol{Z}] \le \alpha\cdot T + \boldsymbol{Z}^{\langle 1\rangle}_T + \boldsymbol{Z}^{\langle 2\rangle}_T + \boldsymbol{Z}^{\langle q\rangle}_T ,$$

so that $$\mathbb{E}\Big[\Lambda^{\langle q\rangle}_T[\boldsymbol{Z}]\Big] \le \alpha\cdot\mathbb{E}[T] + \mathbb{E}\Big[\boldsymbol{Z}^{\langle 1\rangle}_T\Big] + \mathbb{E}\Big[\boldsymbol{Z}^{\langle 2\rangle}_T\Big] + \mathbb{E}\Big[\boldsymbol{Z}^{\langle q\rangle}_T\Big]$$

by (4.5.22), (4.5.24), and (4.5.26): $$\le \alpha\cdot\mathbb{E}[T] + \lceil\!\lceil \boldsymbol{Z}^T \rceil\!\rceil_{\mathcal{I}^1} + \lceil\!\lceil \boldsymbol{Z} \rceil\!\rceil^2_{\mathcal{I}^2} + \lceil\!\lceil \boldsymbol{Z}^T \rceil\!\rceil^q_{\mathcal{I}^q}$$

$$\le \alpha\cdot\mathbb{E}[T] + \lceil\!\lceil \boldsymbol{Z}^T \rceil\!\rceil_{\mathcal{I}^q} + \lceil\!\lceil \boldsymbol{Z}^T \rceil\!\rceil^2_{\mathcal{I}^q} + \lceil\!\lceil \boldsymbol{Z}^T \rceil\!\rceil^q_{\mathcal{I}^q}$$

$$\le \alpha\cdot\mathbb{E}[T] + 3\Big(\lceil\!\lceil \boldsymbol{Z} \rceil\!\rceil_{\mathcal{I}^q} \vee \lceil\!\lceil \boldsymbol{Z} \rceil\!\rceil^q_{\mathcal{I}^q}\Big) .$$ ∎

Exercise 4.5.14 Assume $\boldsymbol{Z}$ is a global L^q-integrator, $q > 2$. Then the Doléans–Dade measure of $\boldsymbol{Z}^{\langle q\rangle}$ is the maximum in the vector lattice $\mathfrak{M}^*[\mathcal{P}]$ of the Doléans–Dade measures of $\big\{|\check{H}|^q*\jmath_{\boldsymbol{Z}} : \check{H}(\boldsymbol{y}, s) \stackrel{\text{def}}{=} \langle \boldsymbol{X}'_s|\boldsymbol{y}\rangle ,\ \boldsymbol{X}' \in (\mathcal{E}^d)^\sigma, |\boldsymbol{X}'| \le 1\big\}$.

Exercise 4.5.15 Repeat exercise 4.5.11 for a slew $\boldsymbol{Z}$ of continuous integrators: let Λ be its previsible controller as constructed above. Then Λ is a controller for the slew $[Z^\eta, Z^\theta]$, $\eta, \theta = 1, \ldots, d$. Next let f be a function with continuous first and second partial derivatives:[14]

$$|f_{;\eta}(x)u^\eta| \le L\cdot|u|_p \ \text{ and } \ |f_{;\eta\theta}u^\eta u^\theta| \le L\cdot|u|_p^2 , \qquad u \in \mathbb{R}^d .$$

Then Λ also controls $f(\boldsymbol{Z})$. In fact, for all $T \in \mathfrak{T}$ and $X \in \mathcal{P}$

$$\big\|\,|X*f(\boldsymbol{Z})|^\star_T\big\|_{L^p} \le (C^\diamond_p + 1)L\cdot\max_{\rho=1,2}\Big\|\Big(\int_0^T |X|^\rho_s \, d\Lambda_s\Big)^{1/\rho}\Big\|_{L^p} .$$

---

[13] Recall that $\varpi = (s, \omega)$. Thus ν_ϖ is ν_s with ω in evidence.

[14] Subscripts after semicolons denote partial derivatives, e.g., $\Phi_{;\eta} \stackrel{\text{def}}{=} \frac{\partial\Phi}{\partial x^\eta}$, $\Phi_{;\eta\theta} \stackrel{\text{def}}{=} \frac{\partial^2\Phi}{\partial x^\eta \partial x^\theta}$.

Exercise 4.5.16 If $\boldsymbol{Z}$ is quasi-left-continuous (${}^p\boldsymbol{Z} = 0$), then $\Lambda^{\langle q\rangle}[\boldsymbol{Z}]$ is continuous. Conversely, if $\Lambda^{\langle q\rangle}[\boldsymbol{Z}]$ is continuous, then the jump of $\boldsymbol{X}*\boldsymbol{Z}$ at any predictable stopping time is negligible, for every $\boldsymbol{X} \in \mathcal{P}^d$.

Exercise 4.5.17 Inequality (4.5.1) extends to $0 < p \le 2$ with

$$C_p^\diamond \le 2^{0\vee(1-p)/p}(1 + C_p^{(4.3.6)}) .$$

Exercise 4.5.18 In the case $d = 1$ and when Z is an L^p-integrator (not merely a local one, $p \ge 2$) then the functional $\|\ \|_{p\text{–}Z}^\diamond$ on processes $F : \boldsymbol{B} \to \overline{\mathbb{R}}$ defined by

$$\|F\|_{p\text{–}Z}^\diamond \stackrel{\text{def}}{=} C_p^\diamond \cdot \max_{\rho=1^\diamond,p^\diamond} \left\| \left(\int^* |F|^\rho \, d\Lambda \right)^{1/\rho} \right\|_{L^p}^*$$

is a mean on $\mathcal{E}$ that majorizes the elementary dZ-integral in the sense of L^p.

If Z is a continuous local martingale, then $1^\diamond = p^\diamond = 2$ and, up to the factor $C_p^\diamond \le p\sqrt{e/2}$, $\|\ \|_{p\text{–}Z}^\diamond$ agrees with the Hardy mean of definition (4.2.9); thus $\|\ \|_{p\text{–}Z}^\diamond$ is an extension to general integrators of the Hardy mean when $2 \le p < \infty$.

Exercise 4.5.19 For a d-dimensional Wiener process $\boldsymbol{W}$ and $q \ge 2$,

$$\Lambda_t^{\langle q\rangle}[\boldsymbol{W}] = \left\lceil \boldsymbol{W}^t \right\rceil_{\mathcal{I}^2}^2 = d\cdot t .$$

Exercise 4.5.20 Suppose $\boldsymbol{Z}$ is continuous and use the previsible controller Λ of remark 4.5.2 on page 238 to define THE time transformation (4.5.4). Let $0 \le g_\eta \in \mathcal{F}_{T^\kappa}$, and $1 \le \eta, \theta, \iota \le d$. Set $\|\mathbf{g}\|_{L^p} \stackrel{\text{def}}{=} \|\sum_{\eta=1}^d |g_\eta|\,\|_{L^p}$.

(i) For $\ell = 0, 1, \ldots$ there are constants $C_\ell \le \ell!(C_p^\diamond)^\ell$ such that,

for $\kappa < \mu < \kappa{+}1$,
$$\left\| g_\eta \cdot |Z^\eta - Z^{\eta T^\kappa}|_{T^\mu}^{\star\ell} \right\|_{L^p} \le C_\ell\,(\mu{-}\kappa)^{\ell/2} \cdot \|\mathbf{g}\|_{L^p} \tag{4.5.27}$$

and
$$\left\| \int_{T^\kappa}^{T^\mu} |g_\iota| \cdot |Z^\iota {-} Z^\iota_{T^\kappa}|_s^{\star\ell} \, d\left\lceil [Z^\eta, Z^\theta] \right\rceil_s \right\|_{L^p} \le \frac{C_\ell}{\ell/2+1}\,(\mu{-}\kappa)^{\ell/2+1} \cdot \|\mathbf{g}\|_{L^p} . \tag{4.5.28}$$

(ii) For $\ell = 0, 1, \ldots$ there are polynomials P_ℓ such that for any $\mu > \kappa$

$$\left\| g_\eta \cdot |Z^\eta - Z^{\eta T^\kappa}|_{T^\mu}^{\star\ell} \right\|_{L^p} \le P_\ell(\sqrt{\mu{-}\kappa}) \cdot \|\mathbf{g}\|_{L^p} . \tag{4.5.29}$$

Exercise 4.5.21 (Emery) Let Y, A be positive, adapted, increasing, and right-continuous processes, with A also previsible. If $\mathbb{E}[Y_T] \le \mathbb{E}[A_T]$ for all finite stopping times T, then for all $y, a > 0$

$$\mathbb{P}[Y_\infty \ge y, A_\infty \le a] \le \mathbb{E}[A_\infty \wedge a]/y ; \tag{4.5.30}$$

in particular,
$$[Y_\infty = \infty] \subseteq [A_\infty = \infty] \quad \mathbb{P}\text{-almost surely.} \tag{4.5.31}$$

Exercise 4.5.22 (Yor) Let Y, A be positive random variables satisfying the inequality $\mathbb{P}[Y \ge y, A \le a] \le \mathbb{E}[A \wedge a]/y$ for $y, a > 0$. Next let $\phi, \psi : \mathbb{R}_+ \to \mathbb{R}_+$ be càdlàg increasing functions, and set

$$\Phi(x) \stackrel{\text{def}}{=} \phi(x) + x \int_{(x}^\infty \frac{d\phi(x)}{x} .$$

Then
$$\mathbb{E}[\phi(Y) \cdot \psi(\frac{1}{A})] \le \mathbb{E}\Big[\big(\Phi(A) {+} \phi(A)\big) \cdot \psi(\frac{1}{A}) + \int_{\frac{1}{A}}^\infty \Phi(\frac{1}{y})\, d\psi(y) \Big] . \tag{4.5.32}$$

In particular, for $0 \le \alpha < \beta < 1$,

$$\mathbb{E}[Y^\beta / A^\alpha] \le \Big(\frac{2}{(1-\beta)(\beta-\alpha)} \Big) \cdot \mathbb{E}[A^{\beta-\alpha}] \tag{4.5.33}$$

and
$$\mathbb{E}[Y^\beta] \le \frac{2-\beta}{1-\beta} \cdot \mathbb{E}[A^\beta] . \tag{4.5.34}$$

Exercise 4.5.23 From THE previsible controller Λ of the L^q-integrator $\boldsymbol{Z}$ define

$$A \stackrel{\text{def}}{=} C_q^{\diamond q} \cdot (\Lambda^{1/1^\diamond} \vee \Lambda^{1/q^\diamond})$$

and show that $$\mathbb{E}\Big[|\boldsymbol{Z}|_T^{\star p\beta}/A_T^{p\alpha}\Big] \le \frac{2}{(1-\beta)(\beta-\alpha)} \cdot \mathbb{E}\Big[A_T^{p(\beta-\alpha)}\Big]$$

for all finite stopping times T, all $p \le q$, and all $0 \le \alpha < \beta < 1$. Use this to estimate from below Λ at the first time $\boldsymbol{Z}$ leaves the ball of a given radius r about its starting point $\boldsymbol{Z}_0$. Deduce that the first time a Wiener process leaves an open ball about the origin has moments of all (positive and negative) orders.

Previsible Control of Random Measures

Our definition 3.10.1 of a random measure was a straightforward generalization of a d-tuple of integrators; we would simply replace the auxiliary space $\{1,\ldots,d\}$ of indices by a locally compact space $\boldsymbol{H}$, and regard a random measure as an $\boldsymbol{H}$-tuple of (infinitesimal) integrators. This view has already paid off in a simple proof of the Doob–Meyer decomposition 4.3.24 for random measures. It does so again in a straightforward generalization of the control theorem 4.5.1 to random measures. On the way we need a small technical result, from which the desired control follows with but a little soft analysis:

Exercise 4.5.24 Let $\boldsymbol{Z}$ be a global L^q-integrator of length d, view it as a column vector, let $C : \ell^1(d) \to \ell^1('d)$ be a contractive linear map, and set $'\boldsymbol{Z} \stackrel{\text{def}}{=} C\boldsymbol{Z}$. Then $\jmath_{'\boldsymbol{Z}} = C[\jmath_{\boldsymbol{Z}}]$. Next, let μ be the Doléans–Dade measure for any one of the previsible controllers $\boldsymbol{Z}^{\langle 1\rangle}, \boldsymbol{Z}^{\langle 2\rangle}, \boldsymbol{Z}^{\langle q\rangle}, \Lambda^{\langle q\rangle}[\boldsymbol{Z}]$ and $'\mu$ the Doléans–Dade measure for the corresponding controller of $'\boldsymbol{Z}$. Then $'\mu \le \mu$.

Theorem 4.5.25 (Revesz [94]) *Let $q \ge 2$ and suppose ζ is a spatially bounded L^q-random measure with auxiliary space $\boldsymbol{H}$. There exist a previsible increasing process $\Lambda = \Lambda^{\langle q\rangle}[\zeta]$ and a universal constant $C_p^\diamond$ that control ζ in the following sense: for every $\check{X} \in \check{\mathcal{P}}$, every stopping time T, and every $p \in [2,q]$*

$$\Big\| (\check{X}{*}\zeta)_T^\star \Big\|_{L^p} \le C_p^\diamond \cdot \max_{\rho=1^\diamond,p^\diamond} \Big\| \Big(\int^* [\![0,T]\!] \cdot |\check{X}_s|_\infty^\rho \, d\Lambda_s \Big)^{1/\rho} \Big\|_{L^p}^* . \qquad (4.5.35)$$

Here $|\check{X}_s|_\infty \stackrel{\text{def}}{=} \sup_{\eta\in\boldsymbol{H}} |\check{X}(\eta,s)|$ is $\mathcal{P}$-analytic and hence universally $\mathcal{P}$-measurable. The meaning of $1^\diamond, p^\diamond$ is mutatis mutandis *as in theorem 4.5.1 on page 238. Part of the claim is that the left-hand side makes sense, i.e., that $[\![0,T]\!]\cdot\check{X}$ is ζ–p-integrable, whenever the right-hand side is finite.*

Proof. Denote by $\mathcal{K}$ the paving of $\boldsymbol{H}$ by compacta. Let (K_ν) be a sequence in $\mathcal{K}$ whose interiors cover $\boldsymbol{H}$, cover each K_ν by a finite collection B_ν of balls of radius less than $1/\nu$, and let P_n denote the collection of atoms of the algebra of sets generated by $B_1 \cup \ldots \cup B_n$. This yields a sequence (P^n) of partitions of $\boldsymbol{H}$ into mutually disjoint Borel subsets such that P^n refines P^{n-1} and such that $\mathcal{B}^\bullet(\boldsymbol{H})$ is generated by $\bigcup_n P^n$. Suppose P^n has d_n members $B_1^n, \ldots, B_{d_n}^n$. Then $Z_i^n \stackrel{\text{def}}{=} B_i^n{*}\zeta$ defines a vector $\boldsymbol{Z}^n$ of L^q-integrators of length d_n, of integrator size less than $\lceil\zeta\rceil_{\mathcal{I}^q}$, and controllable by a previsible increasing process $\Lambda^n \stackrel{\text{def}}{=} \Lambda^{\langle q\rangle}[\boldsymbol{Z}^n]$. Collapsing P^n to P^{n-1} gives rise to a contractive linear map $C^n_{n-1} : \ell^1(d_n) \to \ell^1(d_{n-1})$ in an obvious way. By exercise 4.5.24, the Doléans–Dade measures of the Λ^n increase with n. They have a least upper bound μ in the order-complete vector lattice $\mathfrak{M}^*[\mathcal{P}]$, and the Doléans–Dade process Λ of μ will satisfy the description.

To see that it does, let $\check{\mathcal{E}}'$ denote the collection of finite sums of the form $\sum h_i \otimes X_i$, with h_i step functions over the algebra generated by $\bigcup P^n$ and $X_i \in \mathcal{P}_{00}$. It is evident that (4.5.35) is satisfied when $\check{X} \in \check{\mathcal{E}}'$, with $C^\diamond_p = C^{\diamond(4.5.1)}_p$. Since both sides depend continuously on their arguments in the topology of confined uniform convergence, the inequality will stay even for $\check{X}$ in the confined uniform closure of $\check{\mathcal{E}}' = \check{\mathcal{E}}'_{00}$, which contains $C_{00}[\boldsymbol{H}]\otimes\mathcal{P}$ and is both an algebra and a vector lattice (theorem A.2.2). Since the right-hand side is not a mean in its argument $\check{X}$, in view of the appearance of the sup-norm $|\ |_\infty$, it is not possible to extend to $\check{X} \in \check{\mathcal{P}}$ by the usual sequential closure argument and we have to go a more circuitous route.

To begin with, let us show that $|\check{X}|_\infty$ is measurable on the universal completion $\mathcal{P}^*$ whenever $\check{X} \in \check{\mathcal{P}}$. Indeed, for any $a \in \mathbb{R}$, $[|\check{X}|_\infty > a]$ is the projection on $\boldsymbol{B}$ of the $\mathcal{K}\times\mathcal{P}$-analytic (see A.5.3) set $[|\check{X}| > a]$ of $\mathcal{B}^\bullet[\boldsymbol{H}]\otimes\mathcal{P}$, and is therefore $\mathcal{P}$-analytic (proposition A.5.4) and $\mathcal{P}^*$-measurable (apply theorem A.5.9 to every outer measure on $\mathcal{P}$). Hence $|\check{X}|_\infty \in \mathcal{P}^*$. In fact, this argument shows that $|\check{X}|_\infty$ is measurable for any mean on $\mathcal{P}$ that is continuous along arbitrary increasing sequences, since such a mean is a $\mathcal{P}$-capacity. In particular (see proposition 3.6.5 or equation (A.3.2)), $|\check{X}|_\infty$ is measurable for the mean $\|\ \|^\diamond$ that is defined on $F : \boldsymbol{B} \to \overline{\mathbb{R}}$ by

$$\|F\|^\diamond \stackrel{\text{def}}{=} C^\diamond_p \cdot \max_{\rho=1^\diamond,p^\diamond} \left\| \left(\int^* |F|^\rho \, d\Lambda \right)^{1/\rho} \right\|^*_{L^p} .$$

Next let $g \in L^{p'}$ be an element of the unit ball of the Banach-space dual $L^{p'}$ of L^p, and define θ on $\check{\mathcal{P}}_{00}$ by $\theta(\check{X}) \stackrel{\text{def}}{=} \mathbb{E}[g\cdot\zeta(\check{X})]$. This is a σ-additive measure of finite variation: $\lceil\theta\rceil(|\check{X}|) \leq \|\check{X}\|_{\zeta-p}$. There is a $\check{\mathcal{P}}$-measurable Radon–Nikodym derivative $\check{G} = d\theta/d\lceil\theta\rceil$ with $|\check{G}| = 1$. Also, $\lceil\theta\rceil$ has a disintegration (see corollary A.3.42), so that

$$\theta(\check{X}) = \int_{\boldsymbol{B}} \int_{\boldsymbol{H}} \check{X}(\eta,\varpi)\check{G}(\eta,\varpi)\nu_\varpi(d\eta)\,\mu(d\varpi) \leq \big\| |\check{X}|_\infty \big\|^\diamond . \qquad (4.5.36)$$

The equality in (4.5.36) holds for all $\check{X} \in \check{\mathcal{P}} \cap \mathcal{L}^1(\theta)$ (ibidem), while the inequality so far is known only for $\check{X} \in \overline{\check{\mathcal{E}}'}_{00}$. Now there exists a sequence $(\check{G}_n)$ in $\check{\mathcal{E}}'$ that converges in $\|\ \|^*_\theta$-mean to $\check{G} = 1/\check{G}$ and has $|\check{G}_n| \leq 1$. Replacing $\check{X}$ by $\check{X}\cdot\check{G}_n$ in (4.5.36) and taking the limit produces

$$\lceil\theta\rceil(\check{X}) = \int_{\boldsymbol{B}} \int_{\boldsymbol{H}} \check{X}(\eta,\varpi)\nu_\varpi(d\eta)\,\mu(d\varpi) \leq \big\| |\check{X}|_\infty \big\|^\diamond , \qquad \check{X} \in \overline{\check{\mathcal{E}}'}_{00} .$$

In particular, when $\check{X}$ does not depend on $\eta \in \boldsymbol{H}$, $\check{X} = 1_{\boldsymbol{H}} \otimes X$ with $X \in \mathcal{P}_{00}$, say, then this inequality, in view of $\nu_\varpi(\boldsymbol{H}) = 1$, results in

$$\int_{\boldsymbol{B}} X(\varpi)\, \mu(d\varpi) \le \big\| |X|_\infty \big\|^\diamond = \|X\|^\diamond ,$$

and by exercise 3.6.16 in:
$$\int^* |X(\varpi)|\, \mu(d\varpi) \le \|X\|^\diamond \qquad \forall\, X \in \mathcal{P}^* .$$

Thus for $X' \in \mathcal{P}$ with $|X'| \le 1$ and $\check{X} \in \check{\mathcal{P}} \cap \mathcal{L}^1[\zeta]$ we have

$$\mathbb{E}\Big[g \cdot \int X'\, d(\check{X} {*} \zeta)\Big] = \theta(X' \cdot \check{X}) \le \,¦\theta¦\, \big(|X' \cdot \check{X}|\big)$$

$$= \int_{\boldsymbol{B}} \int_{\boldsymbol{H}} |X'(\varpi)|\, |\check{X}(\eta, \varpi)| \nu_\varpi(d\eta)\, \mu(d\varpi)$$

as $|\check{X}'| \le 1$:
$$\le \int_{\boldsymbol{B}} \int_{\boldsymbol{H}} |\check{X}(\eta, \varpi)| \nu_\varpi(d\eta)\, \mu(d\varpi)$$

as $\nu_\varpi(\boldsymbol{H}) = 1$:
$$\le \int_{\boldsymbol{B}}^* |\check{X}|_\infty(\varpi)\, \mu(d\varpi) \le \big\| |\check{X}|_\infty \big\|^\diamond .$$

Taking the supremum over $g \in L_1^{p'}$ and $X' \in \mathcal{E}_1'$ gives

$$¦\check{X} {*} \zeta¦_{\mathcal{I}^p} \le \big\| |\check{X}|_\infty \big\|^\diamond$$

and, finally,
$$\Big\| (\check{X} {*} \zeta)^\star_\infty \Big\|_{L^p} \le C_p^{\star(2.3.5)} \cdot \big\| |\check{X}|_\infty \big\|^\diamond .$$

This inequality was established under the assumption that $\check{X} \in \check{\mathcal{P}}$ is ζ–p-integrable. It is left to be shown that this is the case whenever the right-hand side is finite. Now by corollary 3.6.10, $\| \check{X} \|^*_{\zeta - p}$ is the supremum of $\| \int \check{Y}\, d\zeta \|_{L^p}$, taken over $\check{Y} \in \mathcal{L}^1[\zeta{-}p]$ with $|\check{Y}| \le |\check{X}|$. Such $\check{Y}$ have $\| \check{Y} \|^\diamond \le \| \check{X} \|^\diamond$, whence $\| \check{X} \|^*_{\zeta - p} \le \| \check{X} \|^\diamond < \infty$.

Now replace $\check{X}$ by $[\![0, T]\!] \cdot \check{X}$ to obtain inequality (4.5.35). ▮

Project 4.5.26 *Make a theory of time transformations.*

4.6 Lévy Processes

Let $(\Omega, \mathcal{F}_., \mathbb{P})$ be a measured filtration and $\boldsymbol{Z}_.$ an adapted $\mathbb{R}^d$-valued process that is right-continuous in probability. $\boldsymbol{Z}_.$ is a ***Lévy process*** on $\mathcal{F}_.$ if it has independent identically distributed and stationary increments and $\boldsymbol{Z}_0 = 0$. To say that the increments of $\boldsymbol{Z}$ are independent means that for any $0 \le s < t$ the increment $\boldsymbol{Z}_t - \boldsymbol{Z}_s$ is independent of $\mathcal{F}_s$; to say that the increments of $\boldsymbol{Z}$ are stationary and identically distributed means that the increments $\boldsymbol{Z}_t - \boldsymbol{Z}_s$ and $\boldsymbol{Z}_{t'} - \boldsymbol{Z}_{s'}$ have the same law whenever the elapsed times $t - s$ and $t' - s'$ are the same. If the filtration is not specified, a Lévy process $\boldsymbol{Z}$ is understood to be a Lévy process on its own basic filtration $\mathcal{F}^0_{\bullet}[\boldsymbol{Z}]$. Here are a few simple observations:

Exercise 4.6.1 A Lévy process $\boldsymbol{Z}$ on $\mathcal{F}_{\bullet}$ is a Lévy process both on its basic filtration $\mathcal{F}^0_{\bullet}[\boldsymbol{Z}]$ and on the natural enlargement of $\mathcal{F}_{\bullet}$. At any instant s, $\mathcal{F}_s$ and $\mathcal{F}^0_{\infty}[\boldsymbol{Z}-\boldsymbol{Z}^s]$ are independent. At any finite $\mathcal{F}_{\bullet}$-stopping time T, $\boldsymbol{Z}'_{\bullet} \stackrel{\text{def}}{=} \boldsymbol{Z}_{T+\bullet} - \boldsymbol{Z}_T$ is independent of $\mathcal{F}_T$; in fact $\boldsymbol{Z}'_{\bullet}$ is a Lévy process (on its basic and natural filtrations). Here $Z'_{\bullet}$ is the map $t \mapsto \boldsymbol{Z}'_t$, etc.

Exercise 4.6.2 If $\boldsymbol{Z}_{\bullet}$ and $\boldsymbol{Z}'_{\bullet}$ are $\mathbb{R}^d$-valued Lévy processes on the measured filtration $(\Omega, \mathcal{F}_{\bullet}, \mathbb{P})$, then so is any linear combination $\alpha \boldsymbol{Z} + \beta \boldsymbol{Z}'$ with constant coefficients. If $\boldsymbol{Z}^{(n)}$ are Lévy processes on $(\Omega, \mathcal{F}_{\bullet}, \mathbb{P})$ and $|\boldsymbol{Z}^{(n)} - \boldsymbol{Z}|^{\star}_t \xrightarrow[n\to\infty]{} 0$ in probability at all instants t, then $\boldsymbol{Z}$ is a Lévy process.

Exercise 4.6.3 If the Lévy process $\boldsymbol{Z}$ is an L^1-integrator, then the previsible part $\widehat{\boldsymbol{Z}}_{\bullet}$ of its Doob–Meyer decomposition has $\widehat{\boldsymbol{Z}}_t = \boldsymbol{A} \cdot t$, with $\boldsymbol{A} = \mathbb{E}[\boldsymbol{Z}_1]$; thus then both $\widehat{\boldsymbol{Z}}$ and $\widetilde{\boldsymbol{Z}}$ are Lévy processes.

We now have sufficiently many tools at our disposal to analyze this important class of processes. The idea is to look at them as integrators. The stochastic calculus developed so far eases their analysis considerably, and, on the other hand, Lévy processes provide fine examples of various applications and serve to illuminate some of the previously developed notions.

In view of exercise 4.6.1 we may and do assume the natural conditions.

Let us denote the inner product on $\mathbb{R}^d$ variously by juxtaposition or by $\langle \,|\, \rangle$: for $\boldsymbol{\zeta} \in \mathbb{R}^d$,

$$\boldsymbol{\zeta}\boldsymbol{Z}_t = \langle \boldsymbol{\zeta}|\boldsymbol{Z}_t \rangle = \sum_{\eta=1}^{d} \zeta_\eta Z^\eta_t \ .$$

It is convenient to start by analyzing the characteristic functions of the distributions μ_t of the $\boldsymbol{Z}_t$. For $\boldsymbol{\zeta} \in \mathbb{R}^d$ and $s, t \geq 0$

$$\begin{aligned} \widehat{\mu_{s+t}}(\boldsymbol{\zeta}) &\stackrel{\text{def}}{=} \mathbb{E}\Big[e^{i\langle \boldsymbol{\zeta}|\boldsymbol{Z}_{s+t}\rangle}\Big] = \mathbb{E}\Big[e^{i\langle \boldsymbol{\zeta}|\boldsymbol{Z}_{s+t}-\boldsymbol{Z}_s\rangle} e^{i\langle \boldsymbol{\zeta}|\boldsymbol{Z}_s\rangle}\Big] \\ &= \mathbb{E}\Big[\mathbb{E}\Big[e^{i\langle \boldsymbol{\zeta}|\boldsymbol{Z}_{s+t}-\boldsymbol{Z}_s\rangle}\Big|\mathcal{F}^0_s[\boldsymbol{Z}]\Big] \cdot e^{i\langle \boldsymbol{\zeta}|\boldsymbol{Z}_s\rangle}\Big] \\ \text{by independence:} \quad &= \mathbb{E}\Big[e^{i\langle \boldsymbol{\zeta}|\boldsymbol{Z}_{s+t}-\boldsymbol{Z}_s\rangle}\Big]\mathbb{E}\Big[e^{i\langle \boldsymbol{\zeta}|\boldsymbol{Z}_s\rangle}\Big] \\ \text{by stationarity:} \quad &= \mathbb{E}\Big[e^{i\langle \boldsymbol{\zeta}|\boldsymbol{Z}_t\rangle}\Big]\mathbb{E}\Big[e^{i\langle \boldsymbol{\zeta}|\boldsymbol{Z}_s\rangle}\Big] = \widehat{\mu_s}(\boldsymbol{\zeta}) \cdot \widehat{\mu_t}(\boldsymbol{\zeta}) \ . \end{aligned} \tag{4.6.1}$$

From (A.3.16), $\mu_{s+t} = \mu_s \star \mu_t$.

That is to say, $\{\mu_t : t \geq 0\}$ is a convolution semigroup. Equation (4.6.1) says that $t \mapsto \widehat{\mu_t}(\boldsymbol{\zeta})$ is multiplicative. As this function is evidently right-continuous in t, it is of the form $\widehat{\mu_t}(\boldsymbol{\zeta}) = e^{t\cdot\psi(\boldsymbol{\zeta})}$ for some number $\psi(\boldsymbol{\zeta}) \in \mathbb{C}$:

$$\widehat{\mu_t}(\boldsymbol{\zeta}) = \mathbb{E}\Big[e^{i\langle \boldsymbol{\zeta}|\boldsymbol{Z}_t\rangle}\Big] = e^{t\cdot\psi(\boldsymbol{\zeta})} \ , \qquad 0 \leq t < \infty \ .$$

But then $t \mapsto \widehat{\mu_t}(\boldsymbol{\zeta})$ is even continuous, so by the Continuity Theorem A.4.3 the convolution semigroup $\{\mu_t : 0 \leq t\}$ is weakly continuous.

Since $\widehat{\mu_t}(\boldsymbol{\zeta})$ depends continuously on $\boldsymbol{\zeta}$, so does ψ; and since $\widehat{\mu_t}$ is bounded, the real part of $\psi(\boldsymbol{\zeta})$ is negative. Also evidently $\psi(0) = 0$. Equation (4.6.1) generalizes immediately to

$$\mathbb{E}\Big[e^{i\langle\boldsymbol{\zeta}|\boldsymbol{Z}_t-\boldsymbol{Z}_s\rangle}\big|\mathcal{F}_s\Big] = e^{(t-s)\cdot\psi(\boldsymbol{\zeta})} ,$$

or, equivalently,
$$\mathbb{E}\Big[e^{i\langle\boldsymbol{\zeta}|\boldsymbol{Z}_t\rangle}A\Big] = e^{(t-s)\cdot\psi(\boldsymbol{\zeta})}\mathbb{E}\Big[e^{i\langle\boldsymbol{\zeta}|\boldsymbol{Z}_s\rangle}A\Big] \tag{4.6.2}$$

for $0 \le s \le t$ and $A \in \mathcal{F}_s$. Consider now the process $M^{\boldsymbol{\zeta}}$ that is defined by

$$e^{i\langle\boldsymbol{\zeta}|\boldsymbol{Z}_t\rangle} = M_t^{\boldsymbol{\zeta}} + \psi(\boldsymbol{\zeta})\int_0^t e^{i\langle\boldsymbol{\zeta}|\boldsymbol{Z}_s\rangle}\, ds \tag{4.6.3}$$

– read the integral as the Bochner integral of a right-continuous $L^1(\mathbb{P})$-valued curve. $M^{\boldsymbol{\zeta}}$ is a martingale. Indeed, for $0 \le s < t$ and $A \in \mathcal{F}_s$, repeated applications of (4.6.2) and Fubini's theorem A.3.18 give

$$\mathbb{E}\Big[\big(M_t^{\boldsymbol{\zeta}}-M_s^{\boldsymbol{\zeta}}\big)\cdot A\Big]=\mathbb{E}\Big[e^{i\langle\boldsymbol{\zeta}|\boldsymbol{Z}_s\rangle}A\Big]\Big(e^{(t-s)\cdot\psi(\boldsymbol{\zeta})}-1-\psi(\boldsymbol{\zeta})\int_s^t e^{(\tau-s)\cdot\psi(\boldsymbol{\zeta})}\, d\tau\Big)=0 .$$

Since $|M_t^{\boldsymbol{\zeta}}| \le 1 + t|\psi(\boldsymbol{\zeta})|$, the real and imaginary parts of these martingales are L^p-integrators of (stopped) sizes less than $C_p^{(2.5.6)}\big(1+t|\psi(\boldsymbol{\zeta})|\big)$, for $p > 1$, $t < \infty$, and $\boldsymbol{\zeta} \in \mathbb{R}^d$, and the processes $e^{i\langle\boldsymbol{\zeta}|\boldsymbol{Z}\rangle}$ are L^p-integrators of (stopped) sizes

$$\big\lceil e^{i\langle\boldsymbol{\zeta}|\boldsymbol{Z}^t\rangle}\big\rceil_{\mathcal{I}^p} \le 2C_p^{(2.5.6)}\big(1 + 2t|\psi(\boldsymbol{\zeta})|\big) . \tag{4.6.4}$$

Each of the processes $e^{i\langle\boldsymbol{\zeta}|\boldsymbol{Z}\rangle}$ has therefore a càdlàg modification. Let us show that $\boldsymbol{Z}$ itself does as well. To this end consider now the set

$$\mathcal{B}^u \stackrel{\text{def}}{=} \Big\{(\omega,\boldsymbol{\zeta}) \in \Omega\times\mathbb{R}^d : \mathbb{Q} \ni q \mapsto e^{i\langle\boldsymbol{\zeta}|\boldsymbol{Z}_q^u\rangle} \text{ has oscillatory discontinuities}\Big\} .$$

First observe that $\mathcal{B}^u$ belongs to $\mathcal{F}_u \otimes Borel([0,u])$, by viewing the function $\mathbb{Q}\cap[0,u] \ni q \mapsto e^{i\langle\boldsymbol{\zeta}|\boldsymbol{Z}_q(\omega)\rangle}$ as a process with underlying measurable space $\big(\Omega\times\mathbb{R}^d, \mathcal{F}_u \otimes Borel([0,u])\big)$ and by analyzing the events that the real or imaginary part of this process upcrosses some rational interval (a,b) infinitely often, with the help of the stopping times T_k of the proof of lemma 2.3.1. In terms of $\mathcal{B}^u$ the existence of a càdlàg modification of the $e^{i\langle\boldsymbol{\zeta}|\boldsymbol{Z}\rangle}$ can be expressed as $\int \mathcal{B}^u(\omega,\boldsymbol{\zeta})\, \mathbb{P}(d\omega) = 0$ for all $\boldsymbol{\zeta} \in \mathbb{R}^d$. Integrating over $\boldsymbol{\zeta} \in \mathbb{R}^d$, applying Fubini's theorem, and discarding a suitable nearly empty subset of Ω, leads to this situation: for every $\omega \in \Omega$, the functions $\mathbb{Q} \ni q \mapsto e^{i\langle\boldsymbol{\zeta}|\boldsymbol{Z}_q\rangle}$ have right and left limits, with the possible exception of a $d\boldsymbol{\zeta}$-negligible set of points in $\mathbb{R}^d$. According to exercise A.3.34 on page 412, the paths $\mathbb{Q} \ni q \mapsto \boldsymbol{Z}_q$ themselves have right and left limits, for every $\omega \in \Omega$. We use this to define a càdlàg modification $\boldsymbol{Z}'$ *via* $\boldsymbol{Z}'_t \stackrel{\text{def}}{=} \lim_{\mathbb{Q}\ni q\downarrow t} \boldsymbol{Z}_q$, which is adapted to the natural enlargement of the underlying filtration, and is plainly a Lévy process again. We rename this modification to $\boldsymbol{Z}$, and arrive at the following situation: we may, and therefore shall, henceforth assume that ***a Lévy process is càdlàg and bounded on bounded intervals.*** In the remainder of this section $\boldsymbol{Z}$ is a fixed Lévy process on a filtration that satisfies the natural conditions.

Lemma 4.6.4 *(i) $\boldsymbol{Z}$ is an L^0-integrator. (ii) For any bounded continuous function $\boldsymbol{F}:[0,\infty)\to\mathbb{R}^d$ whose components have finite variation and compact support*

$$\mathbb{E}\Big[e^{i\int_0^\infty\langle\boldsymbol{Z}|d\boldsymbol{F}\rangle}\Big]=e^{\int_0^\infty\psi(-\boldsymbol{F}_s)\,ds}\;. \tag{4.6.5}$$

*(iii) The **logcharacteristic function** $\psi_{\boldsymbol{Z}}\stackrel{\text{def}}{=}\psi:\boldsymbol{\zeta}\mapsto t^{-1}\ln\big(\widehat{\mu_t}(\boldsymbol{\zeta})\big)$ therefore determines the law of $\boldsymbol{Z}$.*

Proof. (i) The stopping times $T_n\stackrel{\text{def}}{=}\inf\{t:|\boldsymbol{Z}|_t\geq n\}$ increase without bound, since $\boldsymbol{Z}\in\mathfrak{D}$. For $|\boldsymbol{\zeta}|<1/n$, the process $e^{i\langle\boldsymbol{\zeta}|\boldsymbol{Z}\rangle}\cdot[\![0,T_n)\!)+[\![T_n,\infty)\!)$ is an L^0-integrator whose values lie in a disk of radius 1 about $1\in\mathbb{C}$. Applying the main branch of the logarithm produces $i\langle\boldsymbol{\zeta}|\boldsymbol{Z}\rangle\cdot[\![0,T_n)\!)$. By Itô's theorem 3.9.1 this is an L^0-integrator for all such $\boldsymbol{\zeta}$. Then so is $\boldsymbol{Z}\cdot[\![0,T_n)\!)$. Since $T_n\uparrow\infty$, $\boldsymbol{Z}$ is a local L^0-integrator. The claim follows from proposition 2.1.9 on page 52.

(ii) Assume for the moment that F is a left-continuous step function with steps at $0=s_0<s_1<\cdots<s_K$, and let $t>0$. Then, with $\sigma_k=s_k\wedge t$,

$$\begin{aligned}\mathbb{E}\Big[e^{i\int_0^t\langle\boldsymbol{F}|d\boldsymbol{Z}\rangle}\Big]&=\mathbb{E}\Big[e^{i\sum_{1\leq k\leq K}\langle\boldsymbol{F}_{s_{k-1}}|\boldsymbol{Z}_{\sigma_k}-\boldsymbol{Z}_{\sigma_{k-1}}\rangle}\Big]\\&=\prod_{k=1}^K\mathbb{E}\Big[e^{i\langle\boldsymbol{F}_{s_{k-1}}|\boldsymbol{Z}_{\sigma_k}-\boldsymbol{Z}_{\sigma_{k-1}}\rangle}\Big]\\&=\prod_{k=1}^K e^{\psi(\boldsymbol{F}_{s_{k-1}})(\sigma_k-\sigma_{k-1})}=e^{\int_0^t\psi(\boldsymbol{F}_s)\,ds}\;.\end{aligned}$$

Now the class of bounded functions $\boldsymbol{F}:[0,\infty)\to\mathbb{R}^d$ for which the equality

$$\mathbb{E}\Big[e^{i\int_0^t\langle\boldsymbol{F}|d\boldsymbol{Z}\rangle}\Big]=e^{\int_0^t\psi(\boldsymbol{F}_s)\,ds}\qquad(t\geq 0)$$

holds is closed under pointwise limits of dominated sequences. This follows from the Dominated Convergence Theorem, applied to the stochastic integral with respect to $d\boldsymbol{Z}$ and to the ordinary Lebesgue integral with respect to ds. So this class contains all bounded $\mathbb{R}^d$-valued Borel functions on the half-line. We apply this equality to a continuous function $\boldsymbol{F}$ of finite variation and compact support. Then $\int_0^\infty\langle\boldsymbol{Z}|d\boldsymbol{F}\rangle=\int_0^\infty\langle-\boldsymbol{F}|d\boldsymbol{Z}\rangle$ and (4.6.5) follows.

(iii) The functions $\mathscr{D}^d\ni\boldsymbol{\zeta}_.\mapsto e^{i\int_0^\infty\langle\boldsymbol{\zeta}|d\boldsymbol{F}\rangle}$, where $\boldsymbol{F}:[0,\infty)\to\mathbb{R}^d$ is continuously differentiable and has compact support, say, form a multiplicative class $\mathcal{M}$ that generates a σ-algebra $\mathcal{F}$ on path space $\mathscr{D}^d$. Any two measures that agree on $\mathcal{M}$ agree on $\mathcal{F}$. ∎

Exercise 4.6.5 (The Zero-One Law) The regularization of the basic filtration $\mathcal{F}^0_.[\boldsymbol{Z}]$ is right-continuous and thus equals $\mathcal{F}_.[\boldsymbol{Z}]$.

The Lévy–Khintchine Formula

This formula – equation (4.6.12) on page 259 – is a description of the logcharacteristic function ψ. We approach it by analyzing the jump measure $\jmath_{\boldsymbol{Z}}$ of $\boldsymbol{Z}$ (see page 180). The finite variation process

$$V^{\boldsymbol{\zeta}} \stackrel{\text{def}}{=} \left[e^{i\langle\boldsymbol{\zeta}|\boldsymbol{Z}\rangle}, e^{-i\langle\boldsymbol{\zeta}|\boldsymbol{Z}\rangle}\right]$$

has continuous part $${}^{c}V^{\boldsymbol{\zeta}} = {}^{c}\!\left[e^{i\langle\boldsymbol{\zeta}|\boldsymbol{Z}\rangle}, e^{-i\langle\boldsymbol{\zeta}|\boldsymbol{Z}\rangle}\right] = {}^{c}[\langle\boldsymbol{\zeta}|\boldsymbol{Z}\rangle, \langle\boldsymbol{\zeta}|\boldsymbol{Z}\rangle] \tag{4.6.6}$$

and jump part $${}^{j}V_t^{\boldsymbol{\zeta}} = \sum_{s\le t}\left|\Delta e_s^{i\langle\boldsymbol{\zeta}|\boldsymbol{Z}\rangle}\right|^2 = \sum_{s\le t}\left|e^{i\langle\boldsymbol{\zeta}|\Delta\boldsymbol{Z}_s\rangle} - 1\right|^2$$
$$= \int_{[\![0,t]\!]} \left|e^{i\langle\boldsymbol{\zeta}|\boldsymbol{y}\rangle} - 1\right|^2 \jmath_{\boldsymbol{Z}}(d\boldsymbol{y}, ds) . \tag{4.6.7}$$

Taking the L^p-norm, $1 < p < \infty$, in (4.6.7) results in

$$\left\|{}^{j}V_t^{\boldsymbol{\zeta}}\right\|_{L^p} \le 2K_p^{(3.8.9)}C_p^{(4.6.4)}\big(1 + 2t|\psi(\boldsymbol{\zeta})|\big) .$$

By A.3.29, $\displaystyle\left\|\int_{[|\boldsymbol{\zeta}|\le 1]} {}^{j}V_t^{\boldsymbol{\zeta}}\, d\boldsymbol{\zeta}\right\|_{L^p} \le \int_{[|\boldsymbol{\zeta}|\le 1]} \left\|{}^{j}V_t^{\boldsymbol{\zeta}}\right\|_{L^p} d\boldsymbol{\zeta} < \infty$.

Setting $$h_0'(\boldsymbol{y}) \stackrel{\text{def}}{=} \int_{[|\boldsymbol{\zeta}|\le 1]} \left|e^{i\langle\boldsymbol{\zeta}|\boldsymbol{y}\rangle} - 1\right|^2 d\boldsymbol{\zeta} ,$$

we obtain $$\left\lceil\!\left\lceil \big(h_0'{*}\jmath_{\boldsymbol{Z}}\big)_t \right\rceil\!\right\rceil_{\mathcal{I}^p} < \infty , \qquad \forall\, t < \infty, \quad \forall\, p < \infty .$$

That is to say, $h_0'{*}\jmath_{\boldsymbol{Z}}$ is an L^p-integrator for all p. Now h_0' is a prototypical Hunt function (see exercise 3.10.11). From this we read off the next result.

Lemma 4.6.6 *The indefinite integral $H{*}\jmath_{\boldsymbol{Z}}$ is an L^p-integrator, for every previsible Hunt function H and every $p \in (0,\infty)$.*

Now fix a *sure and time-independent* Hunt function $h : \mathbb{R}^d \to \mathbb{R}$. Then the indefinite integral $h{*}\jmath_{\boldsymbol{Z}}$ has increment

$$(h{*}\jmath_{\boldsymbol{Z}})_t - (h{*}\jmath_{\boldsymbol{Z}})_s = \sum_{s<\sigma\le t} h(\Delta\boldsymbol{Z}_\sigma) = \sum_{\sigma\le t} h\big(\Delta[\boldsymbol{Z} - \boldsymbol{Z}_s]_\sigma\big) , \qquad s < t ,$$

which in view of exercises 1.3.21 and 4.6.1 is $\mathcal{F}_t$-measurable and independent of $\mathcal{F}_s$; and $h{*}\jmath_{\boldsymbol{Z}}$ clearly has stationary increments. Now exercise 4.6.3 says that $\widehat{h{*}\jmath_{\boldsymbol{Z}}} = h{*}\widehat{\jmath_{\boldsymbol{Z}}}$ has at the instant t the value

$$\big(\widehat{h{*}\jmath_{\boldsymbol{Z}}}\big)_t = \nu(h)\cdot t , \quad \text{with} \ \ \nu(h) \stackrel{\text{def}}{=} \mathbb{E}\Big[\sum_{\sigma\le 1} h(\Delta\boldsymbol{Z}_\sigma)\Big] .$$

Clearly $\nu(h)$ depends linearly and increasingly on h: ν is a Radon measure on punctured d-space $\mathbb{R}_*^d \stackrel{\text{def}}{=} \mathbb{R}^d \setminus \{0\}$ and integrates the Hunt function $h_0 : \boldsymbol{y} \mapsto |\boldsymbol{y}|^2 \wedge 1$. An easy consequence of this is the following:

Lemma 4.6.7 *For any sure and time-independent Hunt function* h *on* $\mathbb{R}^d$*,* $h{*}\jmath_{\boldsymbol{Z}}$ *is a Lévy process. The previsible part* $\widehat{\jmath_{\boldsymbol{Z}}}$ *of* $\jmath_{\boldsymbol{Z}}$ *has the form*

$$\widehat{\jmath_{\boldsymbol{Z}}}(d\boldsymbol{y}, ds) = \nu(d\boldsymbol{y}) \times ds \;, \tag{4.6.8}$$

where ν*, the* ***Lévy measure*** *of* Z*, is a measure on punctured d-space that integrates* h_0*. Consequently, the jumps of* $\boldsymbol{Z}$*, if any, occur only at totally inaccessible stopping times; that is to say,* $\boldsymbol{Z}$ *is quasi-left-continuous (see proposition 4.3.25).*

In the terms of page 231, equation (4.6.8) says that the jump intensity measure $\widehat{\jmath_{\boldsymbol{Z}}}$ has the disintegration[15]

$$\int_{[\![0,\infty)\!)} h_s(d\boldsymbol{y})\, \widehat{\jmath_{\boldsymbol{Z}}}(d\boldsymbol{y}, ds) = \int_0^\infty \int_{\mathbb{R}^d_*} h_s(\boldsymbol{y})\, \nu(d\boldsymbol{y})\; ds \;,$$

with the jump intensity rate ν independent of $\varpi \in \boldsymbol{B}$. Therefore

$$(H{*}\widehat{\jmath_{\boldsymbol{Z}}})_t = \int_0^t \int_{\mathbb{R}^d_*} H_s(\boldsymbol{y})\, \nu(\boldsymbol{y})\; ds$$

for any random Hunt function H. Next, since $e^{i\langle\boldsymbol{\zeta}|\boldsymbol{Z}\rangle} \cdot e^{-i\langle\boldsymbol{\zeta}|\boldsymbol{Z}\rangle} = 1$, equation (3.8.10) gives

$$-dV^{\boldsymbol{\zeta}} = e_-^{i\langle\boldsymbol{\zeta}|\boldsymbol{Z}\rangle} de^{-i\langle\boldsymbol{\zeta}|\boldsymbol{Z}\rangle} + e_-^{-i\langle\boldsymbol{\zeta}|\boldsymbol{Z}\rangle} de^{i\langle\boldsymbol{\zeta}|\boldsymbol{Z}\rangle}$$

by (4.6.3):

$$= e_-^{i\langle\boldsymbol{\zeta}|\boldsymbol{Z}\rangle}\; dM^{-\boldsymbol{\zeta}} + e_-^{-i\langle\boldsymbol{\zeta}|\boldsymbol{Z}\rangle}\; dM^{\boldsymbol{\zeta}} + \;\psi(-\boldsymbol{\zeta})\, dt \;+\; \psi(\boldsymbol{\zeta})\, dt \;,$$

whence

$$\widehat{V^{\boldsymbol{\zeta}}_t} = -t \cdot \big(\psi(-\boldsymbol{\zeta}) + \psi(\boldsymbol{\zeta})\big) \;.$$

From (4.6.7)

$$\widehat{{}^jV^{\boldsymbol{\zeta}}_t} = t \cdot \int \Big|e^{i\boldsymbol{\zeta}\boldsymbol{y}} - 1\Big|^2 \; \nu(d\boldsymbol{y}) \;.$$

By (4.6.6) $\quad {}^c\big[\langle\boldsymbol{\zeta}|\boldsymbol{Z}\rangle, \langle\boldsymbol{\zeta}|\boldsymbol{Z}\rangle\big] = \widehat{{}^cV^{\boldsymbol{\zeta}}} = \widehat{V^{\boldsymbol{\zeta}}} - \widehat{{}^jV^{\boldsymbol{\zeta}}} = t \cdot g(\boldsymbol{\zeta}) \;,$

where the constant $g(\boldsymbol{\zeta})$ must be of the form $\boldsymbol{\zeta}_\eta \boldsymbol{\zeta}_\theta B^{\eta\theta}$, in view of the bilinearity of $\boldsymbol{\zeta} \mapsto {}^c\big[\langle\boldsymbol{\zeta}|\boldsymbol{Z}\rangle, \langle\boldsymbol{\zeta}|\boldsymbol{Z}\rangle\big]$. To summarize:

Lemma 4.6.8 *There is a constant symmetric positive semidefinite matrix* B *with*

$${}^c\big[\langle\boldsymbol{\zeta}|\boldsymbol{Z}\rangle, \langle\boldsymbol{\zeta}|\boldsymbol{Z}\rangle\big]_t = \sum_{1\le\eta,\theta\le d} \boldsymbol{\zeta}_\eta \boldsymbol{\zeta}_\theta B^{\eta\theta} \cdot t \;,$$

which is to say,

$${}^c[Z^\eta, Z^\theta] = B^{\eta\theta} \cdot t \;. \tag{4.6.9}$$

Consequently the continuous martingale part ${}^{\tilde c}\boldsymbol{Z}$ *of* $\boldsymbol{Z}$ *(proposition 4.4.7) is a Wiener process with covariance matrix* B *(exercise 3.9.6).*

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[15] $[\![0,\infty)\!) \stackrel{\text{def}}{=} \mathbb{R}^d_* \times [\![0,\infty)\!)$ and $[\![0,t]\!] \stackrel{\text{def}}{=} \mathbb{R}^d_* \times [\![0,t]\!]$.

We are now in position to establish the renowned Lévy–Khintchine formula.

Since
$$e^{i\langle\boldsymbol{\zeta}|\boldsymbol{Z}_t\rangle} - 1 = \int_{0+}^{t} i e_{s-}^{i\langle\boldsymbol{\zeta}|\boldsymbol{Z}\rangle}\, d\langle\boldsymbol{\zeta}|\boldsymbol{Z}\rangle_s$$
$$- 1/2 \int_{0+}^{t} e_{s-}^{i\langle\boldsymbol{\zeta}|\boldsymbol{Z}\rangle}\, \zeta_\eta \zeta_\theta d{}^{c}[Z^\eta, Z^\theta]_s$$
$$+ \sum_{0<s\le t} e_{s-}^{i\langle\boldsymbol{\zeta}|\boldsymbol{Z}\rangle}\Big(e^{i\langle\boldsymbol{\zeta}|\Delta\boldsymbol{Z}_s\rangle} - 1 - i\langle\boldsymbol{\zeta}|\Delta\boldsymbol{Z}_s\rangle\Big)\,,$$
$$\Big(e_{-\boldsymbol{.}}^{-i\langle\boldsymbol{\zeta}|\boldsymbol{Z}\rangle} * e^{i\langle\boldsymbol{\zeta}|\boldsymbol{Z}\rangle}\Big)_t = \Big(i\langle\boldsymbol{\zeta}|\boldsymbol{Z}\rangle_t - \int_0^t\!\!\int i\langle\boldsymbol{\zeta}|\boldsymbol{y}\rangle\cdot[|\boldsymbol{y}|>1]\,\jmath_{\boldsymbol{Z}}(d\boldsymbol{y},ds)\Big)$$
$$- 1/2\zeta_\eta\zeta_\theta {}^{c}[Z^\eta, Z^\theta]_t$$
$$+ \int_0^t\!\!\int \Big(e^{i\langle\boldsymbol{\zeta}|\boldsymbol{y}\rangle} - 1 - i\langle\boldsymbol{\zeta}|\boldsymbol{y}\rangle\,[|\boldsymbol{y}|\le 1]\Big)\jmath_{\boldsymbol{Z}}(d\boldsymbol{y},ds)\,,$$

the integrand of the previous line being a Hunt function. Now from (4.6.3)

$$Mart_t + t\,\psi(\boldsymbol{\zeta}) = i\langle\boldsymbol{\zeta}|{}^{s}\boldsymbol{Z}_t\rangle - \frac{t}{2}\,\boldsymbol{\zeta}_\eta\boldsymbol{\zeta}_\theta B^{\eta\theta}$$
$$+ t\int \Big(e^{i\langle\boldsymbol{\zeta}|\boldsymbol{y}\rangle} - 1 - i\langle\boldsymbol{\zeta}|\boldsymbol{y}\rangle\,[|\boldsymbol{y}|\le 1]\Big)\nu(d\boldsymbol{y})\,,$$

where
$${}^{s}\boldsymbol{Z}_t \stackrel{\text{def}}{=} \Big(\boldsymbol{Z}_t - \int_0^t\!\!\int \boldsymbol{y}\cdot[|\boldsymbol{y}|>1]\,\jmath_{\boldsymbol{Z}}(d\boldsymbol{y},ds)\Big)\,. \tag{4.6.10}$$

Taking previsible parts of this L^2-integrator results in

$$t\cdot\psi(\boldsymbol{\zeta}) = i\langle\boldsymbol{\zeta}|\widehat{{}^{s}\boldsymbol{Z}}_t\rangle - \frac{t}{2}\cdot\zeta_\eta\zeta_\theta B^{\eta\theta}$$
$$+ t\cdot\int \Big(e^{i\langle\boldsymbol{\zeta}|\boldsymbol{y}\rangle} - 1 - i\langle\boldsymbol{\zeta}|\boldsymbol{y}\rangle\,[|\boldsymbol{y}|\le 1]\Big)\,\nu(d\boldsymbol{y})\,.$$

Now three of the last four terms are of the form $t\cdot const$. Then so must be the fourth; that is to say,
$$\widehat{{}^{s}\boldsymbol{Z}}_t = t\,\boldsymbol{A} \tag{4.6.11}$$
for some constant vector $\boldsymbol{A}$. We have finally arrived at the promised description of ψ:

Theorem 4.6.9 (The Lévy–Khintchine Formula) *There are a vector $\boldsymbol{A}\in\mathbb{R}^d$, a constant positive semidefinite matrix B, and on punctured d-space $\mathbb{R}^d_*$ a positive measure ν that integrates $h_0 : \boldsymbol{y}\mapsto|\boldsymbol{y}|^2\wedge 1$, such that*

$$\psi(\boldsymbol{\zeta}) = i\langle\boldsymbol{\zeta}|\boldsymbol{A}\rangle - \frac{1}{2}\,\zeta_\eta\zeta_\theta B^{\eta\theta} + \int \Big(e^{i\langle\boldsymbol{\zeta}|\boldsymbol{y}\rangle} - 1 - i\langle\boldsymbol{\zeta}|\boldsymbol{y}\rangle\,[|\boldsymbol{y}|\le 1]\Big)\nu(d\boldsymbol{y})\,. \tag{4.6.12}$$

$(\boldsymbol{A}, B, \nu)$ is called the ***characteristic triple*** of the Lévy process $\boldsymbol{Z}$. According to lemma 4.6.4, it determines the law of $\boldsymbol{Z}$.

We now embark on a little calculation that forms the basis for the next two results. Let $\boldsymbol{F} = (F_\eta)_{\eta=1\ldots d} : [0,\infty) \to \mathbb{R}^d$ be a right-continuous function whose components have finite variation and vanish after some fixed time, and let h be a Borel function of relatively compact carrier on $\mathbb{R}^d_* \times [0,\infty)$. Set

$$M = -\boldsymbol{F} {*}^{\tilde{c}}\boldsymbol{Z} \ , \quad \text{i.e.,} \quad M_t \stackrel{\text{def}}{=} -\int_0^t \langle \boldsymbol{F}_s | d^{\tilde{c}}\boldsymbol{Z}_s \rangle \ ,$$

which is a continuous martingale with square function

$$v_t \stackrel{\text{def}}{=} [M,M]_t = {}^c[M,M]_t = \int_0^t F_{\eta s} F_{\theta s} B^{\eta\theta} \, ds \ , \tag{4.6.13}$$

and set[15] $\quad V \stackrel{\text{def}}{=} h {*} \jmath_{\boldsymbol{Z}}$, i.e., $V_t = \int_{[\![0,t]\!]} h_s(\boldsymbol{y}) \, \jmath_{\boldsymbol{Z}}(d\boldsymbol{y}, ds)$.

Since the carrier $[h \neq 0]$ is relatively compact, there is an $\epsilon > 0$ such that $h_s(\boldsymbol{y}) = 0$ for $|\boldsymbol{y}| < \epsilon$. Therefore V is a finite variation process without continuous component and is constant between its jumps

$$\Delta V_s = h_s(\Delta \boldsymbol{Z}_s) \ , \tag{4.6.14}$$

which have size at least $\epsilon > 0$. We compute:

$$E_t \stackrel{\text{def}}{=} \exp\left(iM_t \ + \ v_t/2 \ + \ iV_t\right)$$

by Itô:
$$= 1 + i\int_0^t E_{s-} \, dM_s + i^2/2 \int_0^t E_{s-} \, d^c[M,M]_s$$
$$+ 1/2 \int_0^t E_{s-} \, dv_s$$
$$+ i \int_{[\![0,t]\!]} E_{s-} \, dV_s + \sum_{0<s\le t} E_s - E_{s-} - iE_{s-}\Delta V_s$$

by (4.6.13):
$$= 1 + i\int_0^t E_{s-} \, dM_s$$

and (4.6.14):
$$+ i \sum_{0<s\le t} E_{s-}\Delta V_s + \sum_{0<s\le t} E_{s-}\Big(e^{i\Delta V_s} - 1 - i\Delta V_s\Big)$$
$$= 1 + i\int_0^t E_{s-} \, dM_s + \sum_{0<s\le t} E_{s-}\Big(e^{i\Delta V_s} - 1\Big) .$$

Thus
$$E_t = 1 - i\int_0^t E_{s-} \, \langle \boldsymbol{F} | d^{\tilde{c}}\boldsymbol{Z} \rangle + \int_{[\![0,t]\!]} E_{s-} \cdot \Big(e^{ih_s(\boldsymbol{y})} - 1\Big) \, \jmath_{\boldsymbol{Z}}(d\boldsymbol{y}, ds) \ . \tag{4.6.15}$$

The Martingale Representation Theorem

The martingale representation theorem 4.2.15 for Wiener processes extends to Levy processes. It comes as a first application of equation (4.6.15): take $t = \infty$ and multiply (4.6.15) by the complex constant

$$\exp\Big(-v_\infty/2 \ - \ i\int_{[\![0,\infty)\!)\!)} h_s(\boldsymbol{y})\, \widehat{\jmath_{\boldsymbol{Z}}}(d\boldsymbol{y}, ds)\Big)$$

to obtain
$$\exp\Big(i\Big(\int_0^\infty \langle {}^{\tilde{c}}\boldsymbol{Z}|d\boldsymbol{F}\rangle + \int_{[\![0,\infty)\!)\!)} h_s(\boldsymbol{y})\, \widetilde{\jmath_{\boldsymbol{Z}}}(d\boldsymbol{y}, ds)\Big)\Big) \tag{4.6.16}$$

$$= c + \int_0^\infty \langle \boldsymbol{X}_s|d^{\tilde{c}}\boldsymbol{Z}_s\rangle + \int_{[\![0,\infty)\!)\!)} H_s(\boldsymbol{y})\, \widetilde{\jmath_{\boldsymbol{Z}}}(d\boldsymbol{y}, ds)\ , \tag{4.6.17}$$

where c is a constant, $\boldsymbol{X}$ is some bounded previsible $\mathbb{C}^d$-valued process that vanishes after some instant, and $H : [\![0,\infty)\!)\!) \to \mathbb{C}$ is some bounded previsible random function that vanishes on $[|\boldsymbol{y}| < \epsilon]$ for some $\epsilon > 0$. Now observe that the exponentials in (4.6.16), with $\boldsymbol{F}$ and h as specified above, form a multiplicative class[16] $\mathcal{M}$ of bounded $\mathcal{F}^0_\infty[\boldsymbol{Z}]$-measurable random variables on Ω. As $\boldsymbol{F}$ and h vary, the random variables

$$\int_0^\infty \langle {}^{\tilde{c}}\boldsymbol{Z}|d\boldsymbol{F}\rangle + \int_{[\![0,\infty)\!)\!)} h_s(\boldsymbol{y})\, \widetilde{\jmath_{\boldsymbol{Z}}}(d\boldsymbol{y}, ds)$$

form a vector space Γ that generates precisely the same σ-algebra as $\mathcal{M} = e^{i\Gamma}$ (page 410); it is nearly evident that this σ-algebra is $\mathcal{F}^0_\infty[\boldsymbol{Z}]$. The point of these observations is this: *$\mathcal{M}$ is a multiplicative class that generates $\mathcal{F}^0_\infty[\boldsymbol{Z}]$ and consists of stochastic integrals of the form appearing in* (4.6.17).

Theorem 4.6.10 *Suppose $\boldsymbol{Z}$ is a Lévy process. Every random variable $F \in L^2\big(\mathcal{F}^0_\infty[\boldsymbol{Z}]\big)$ is the sum of the constant $c = \mathbb{E}[F]$ and a stochastic integral of the form*

$$\int_0^\infty \langle \boldsymbol{X}_s|d^{\tilde{c}}\boldsymbol{Z}_s\rangle + \int_{[\![0,\infty)\!)\!)} H_s(\boldsymbol{y})\, \widetilde{\jmath_{\boldsymbol{Z}}}(d\boldsymbol{y}, ds)\ , \tag{4.6.18}$$

where $\boldsymbol{X} = (X_\eta)_{\eta=1\ldots d}$ is a vector of predictable processes and H a predictable random function on $[\![0,\infty)\!)\!) = \mathbb{R}^d_* \times [\![0,\infty)\!)$, with the pair $(\boldsymbol{X}, H)$ satisfying

$$\|\,\boldsymbol{X}, H\,\|^*_2 \stackrel{\text{def}}{=} \Big(\mathbb{E}\Big[\int^* X_{\eta s}\overline{X_{\theta}}_s B^{\eta\theta}\, ds + \int^* |H|^2_s(\boldsymbol{y})\,\nu(d\boldsymbol{y})\, ds\Big]\Big)^{1/2} < \infty\ .$$

Proof. Let us denote by ${}^cM$ and ${}^jM$ the martingales whose limits at infinity appear as the second and third entries of equation (4.6.17), by M their sum,

[16] A multiplicative class is by (our) definition closed under complex conjugation. The complex-conjugate $\overline{X_\eta}$ equals X_η, of course, when X_η is real.

and let us compute $\mathbb{E}[|M|^2_\infty]$: since $[{}^cM, {}^jM] = 0$,

$$\mathbb{E}[|M|^2_\infty] = \mathbb{E}\Big[[M, \overline{M}]_\infty\Big] = \mathbb{E}\Big[[{}^cM, {}^c\overline{M}]_\infty + [{}^jM, {}^j\overline{M}]_\infty\Big]$$

$$= \mathbb{E}\left[{}^c[{}^cM, {}^c\overline{M}]_\infty + \int_{[\![0,\infty)\!)} |H|^2_s(\boldsymbol{y})\jmath_{\boldsymbol{Z}}(d\boldsymbol{y}, ds)\right]$$

as $\widehat{\jmath_{\boldsymbol{Z}}} = \nu \times \lambda$:

$$= \mathbb{E}\left[\int X_{\eta_s}\overline{X_{\theta}}_s B^{\eta\theta}\, ds + \int |H|^2_s(d\boldsymbol{y})\, \nu(d\boldsymbol{y})ds\right]$$

$$= \left(\|\boldsymbol{X}, H\|^*_2\right)^2 .$$

Now the vector space of previsible pairs $(\boldsymbol{X}, H)$ with $\|\,\boldsymbol{X}, H\,\|^*_2 < \infty$ is evidently complete – it is simply the cartesian product of two $\mathcal{L}^2$-spaces – and the linear map $\mathcal{U}$ that associates with every pair $(\boldsymbol{X}, H)$ the stochastic integral (4.6.18) is an isometry of that set into $L^2_{\mathbb{C}}\big(\mathcal{F}^0_\infty[\boldsymbol{Z}]\big)$; its image $\mathcal{I}$ is therefore a closed subspace of $L^2_{\mathbb{C}}\big(\mathcal{F}^0_\infty[\boldsymbol{Z}]\big)$ and contains the multiplicative class $\mathcal{M}$, which generates $\mathcal{F}^0_\infty[\boldsymbol{Z}]$. We conclude with exercise A.3.5 on page 393 that $\mathcal{I}$ contains all bounded complex-valued $\mathcal{F}^0_\infty[\boldsymbol{Z}]$-measurable functions and, as it is closed, is all of $L^2_{\mathbb{C}}\big(\mathcal{F}^0_\infty[\boldsymbol{Z}]\big)$. The restriction of $\mathcal{U}$ to real integrands will exhaust all of $L^2_{\mathbb{R}}\big(\mathcal{F}^0_\infty[\boldsymbol{Z}]\big)$. ∎

Corollary 4.6.11 *For $H \in L^2(\nu\times\lambda)$, $H{*}\widetilde{\jmath_{\boldsymbol{Z}}}$ is a square integrable martingale.*

Project 4.6.12 [94] *Extend theorem 4.6.10 to exponents p other than 2.*

The Characteristic Function of the Jump Measure, in fact of the pair $({}^{\tilde{c}}\boldsymbol{Z}, \jmath_{\boldsymbol{Z}})$, can be computed from equation (4.6.15). We take the expectation:

$$e_t \stackrel{\text{def}}{=} \mathbb{E}[E_t] = 1 + \mathbb{E}\Big[\int_{[\![0,t]\!]} E_{s-}\cdot\big(e^{ih_s(\boldsymbol{y})}{-}1\big)\ \jmath_{\boldsymbol{Z}}(d\boldsymbol{y}, ds)\Big]$$

as $\widehat{\jmath_{\boldsymbol{Z}}} = \nu \times \lambda$:

$$= 1 + \int_0^t e_s \int_{\mathbb{R}^d_*} \big(e^{ih_s(\boldsymbol{y})}{-}1\big)\nu(d\boldsymbol{y})\ ds\ ,$$

whence $\quad e'_t = e_t{\cdot}\phi_t \;$ with $\; \phi_t = \int_{\mathbb{R}^d_*} \big(e^{ih_t(\boldsymbol{y})}{-}1\big)\nu(d\boldsymbol{y})\ ,$

and so $\quad e_t = e^{\int_0^t \phi_s\, ds} = \exp\Big(\int_0^t \int_{\mathbb{R}^d_*} \big(e^{ih_s(\boldsymbol{y})}{-}1\big)\nu(d\boldsymbol{y})\ ds\Big)\ .$

Evaluating this at $t = \infty$ and multiplying with $\exp\big(-v_\infty/2\big)$ gives

$$\mathbb{E}\Big[\exp\Big(i\int {}^{\tilde{c}}\boldsymbol{Z}\, d\boldsymbol{F} + i\int h_s(\boldsymbol{y})\jmath_{\boldsymbol{Z}}(d\boldsymbol{y}, ds)\Big)\Big]$$

$$= \exp\Big(\frac{-1}{2}\int F_{\eta_s}F_{\theta_s}B^{\eta\theta}\, ds\Big) \times \exp\Big(\int\!\!\int\big(e^{ih_s(\boldsymbol{y})}{-}1\big)\nu(d\boldsymbol{y})\ ds\Big)\ . \quad (4.6.19)$$

In order to illuminate this equation, let $\mathcal{V}$ denote the cartesian product of the path space $\mathscr{C}^d$ with the space $\mathfrak{M}^{\bullet} \stackrel{\text{def}}{=} \mathfrak{M}^{\bullet}\big(\mathbb{R}^d_* \times [0,\infty)\big)$ of Radon measures on $\mathbb{R}^d_* \times [0,\infty)$, the former given the topology of uniform convergence on compacta and the latter the weak* topology $\sigma\big(\mathfrak{M}^{\bullet}, C_{00}(\mathbb{R}^d_* \times [0,\infty))\big)$ (see A.2.32). We equip $\mathcal{V}$ with the product of these two locally convex topologies, which is metrizable and locally convex and, in fact, Fréchet. For every pair $(\boldsymbol{F}, h)$, where $\boldsymbol{F}$ is a vector of distribution functions on $[0,\infty)$ that vanish ultimately and $h : \mathbb{R}^d_* \times [0,\infty) \to \mathbb{R}$ is continuous and of compact support, consider the function

$$\gamma^{\boldsymbol{F},h} : (\boldsymbol{z}, \mu) \mapsto \int_0^\infty \langle \boldsymbol{z} | d\boldsymbol{F} \rangle + \int_{\mathbb{R}^d_* \times [0,\infty)} h_s(\boldsymbol{y})\, \mu(d\boldsymbol{y}, ds) .$$

The $\gamma^{\boldsymbol{F},h}$ are evidently continuous linear functionals on $\mathcal{V}$, and their collection forms a vector space[17] Γ that generates the Borels on $\mathcal{V}$. If we consider the pair $({}^{\tilde{c}}\boldsymbol{Z}, \jmath_{\boldsymbol{Z}})$ as a $\mathcal{V}$-valued random variable, then equation (4.6.19) simply says that the law $\mathbb{L}$ of this random variable has the characteristic function $\widehat{\mathbb{L}}^{\Gamma}(\gamma^{\boldsymbol{F},h})$ given by the right-hand side of equation (4.6.19). Now this is the product of the characteristic function of the law of ${}^{\tilde{c}}\boldsymbol{Z}$ with that of $\jmath_{\boldsymbol{Z}}$. We conclude that *the Wiener process* ${}^{\tilde{c}}\boldsymbol{Z}$ *and the random measure* $\jmath_{\boldsymbol{Z}}$ *are independent.* The same argument gives

Proposition 4.6.13 *Suppose* $\boldsymbol{Z}^{(1)}, \ldots, \boldsymbol{Z}^{(K)}$ *are* $\mathbb{R}^d$*-valued Lévy processes on* $(\Omega, \mathcal{F}_{\bullet}, \mathbb{P})$. *If the brackets* $[Z^{(k)\eta}, Z^{(l)\theta}]$ *are evanescent whenever* $1 \le k \ne l \le K$ *and* $1 \le \eta, \theta \le d$, *then the* $\boldsymbol{Z}^{(1)}, \ldots, \boldsymbol{Z}^{(K)}$ *are independent.*

Proof. Denote by $(\boldsymbol{A}^{(k)}, B^{(k)}, \nu^{(k)})$ the characteristic triple of $\boldsymbol{Z}^{(k)}$ and by $\mathbb{L}^{(k)}$ its law on $\mathcal{V}$, and define $\boldsymbol{Z} = \sum_k \boldsymbol{Z}^{(k)}$. This is a Lévy process, whose characteristic triple shall be denoted by $(\boldsymbol{A}, B, \nu)$. Since by assumption no two of the $\boldsymbol{Z}^{(k)}$ ever jump at the same time, we have

$$\sum_{s \le t} H_s(\Delta \boldsymbol{Z}_s) = \sum_k \sum_{s \le t} H_s(\Delta \boldsymbol{Z}^{(k)}_s)$$

for any Hunt function H, which signifies that $\jmath_{\boldsymbol{Z}} = \sum_k \jmath_{\boldsymbol{Z}^{(k)}}$

and implies that $\qquad \nu = \sum_k \nu^{(k)} .$

From (4.6.9) $\qquad B = \sum_k B^{(k)} ,$

since ${}^c\big[\boldsymbol{Z}^{(k)}, \boldsymbol{Z}^{(l)}\big] = 0$. Let $\boldsymbol{F}^{(k)} : [0,\infty) \to \mathbb{R}^d$, $1 \le k, l \le K$, be right-continuous functions whose components have finite variation and vanish after some fixed time, and let $h^{(k)}$ be Borel functions of relatively compact carrier on $\mathbb{R}^d_* \times [0,\infty)$. Set

$$M = -\sum_{1 \le k \le K} \boldsymbol{F}^{(\boldsymbol{k})} {*}\, {}^{\tilde{c}}\boldsymbol{Z}^{(\boldsymbol{k})} ,$$

[17] In fact, Γ is the whole dual $\mathcal{V}^*$ of $\mathcal{V}$ (see exercise A.2.33).

which is a continuous martingale with square function

$$v_t \stackrel{\text{def}}{=} [M,M]_t = {}^c[M,M]_t = \sum_k \int_0^t F^{(k)}_{\eta s} F^{(k)}_{\theta s} B^{(k)\eta\theta}\, ds\ ,$$

and set $\quad V = \sum_k h^{(k)} * \jmath_{\boldsymbol{Z}^{(k)}}$, i.e., $V_t = \sum_k \int_{[\![0,t]\!]} h^{(k)}_s(\boldsymbol{y})\, \jmath_{\boldsymbol{Z}^{(k)}}(d\boldsymbol{y}, ds)$.

A straightforward repetition of the computation leading to equation (4.6.15) on page 260 produces

$$\begin{aligned} E_t &\stackrel{\text{def}}{=} \exp\left(iM_t \;+\; v_t/2 \;+\; iV_t\right) \\ &= 1 - i\int_0^t E_{s-}\, dM_s + \sum_k \int_{[\![0,t]\!]} E_{s-}\cdot\Big(e^{ih^{(k)}_s(\boldsymbol{y})} - 1\Big)\, \jmath_{\boldsymbol{Z}^{(k)}}(d\boldsymbol{y}, ds) \end{aligned}$$

and
$$\begin{aligned} e_t &\stackrel{\text{def}}{=} \mathbb{E}[E_t] = 1 + \mathbb{E}\Big[\sum_k \int_{[\![0,t]\!]} E_{s-}\cdot\big(e^{ih^{(k)}_s(\boldsymbol{y})} - 1\big)\, \jmath_{\boldsymbol{Z}^{(k)}}(d\boldsymbol{y}, ds)\Big] \\ &= 1 + \int_0^t e_s \phi_s\, ds \quad \text{with} \quad \phi_s = \sum_k \int_{\mathbb{R}^d_*} \big(e^{ih^{(k)}_s(\boldsymbol{y})} - 1\big)\nu^{(k)}(d\boldsymbol{y})\ , \end{aligned}$$

whence $\quad e_t = e^{\int_0^t \phi_s\, ds} = \exp\Big(\sum_k \int_0^t \int_{\mathbb{R}^d_*} \big(e^{ih^{(k)}_s(\boldsymbol{y})} - 1\big)\nu^{(k)}(d\boldsymbol{y})\, ds\Big)$.

Evaluating this at $t = \infty$ and dividing by $\exp(v_\infty/2)$ gives

$$\begin{aligned} &\mathbb{E}\Big[\exp\Big(i\sum_k\Big(\int {}^{\tilde c}\boldsymbol{Z}^{(k)}\, d\boldsymbol{F}^{(k)} + \int h^{(k)}_s(\boldsymbol{y}) \jmath_{\boldsymbol{Z}^{(k)}}(d\boldsymbol{y}, ds)\Big)\Big)\Big] \\ &= \prod_k \exp\Big(\frac{-1}{2}\int F^{(k)}_{\eta s} F^{(k)}_{\theta s} B^{(k)\eta\theta}\, ds\Big) \times \exp\Big(\int\!\!\int \big(e^{ih^{(k)}_s(\boldsymbol{y})} - 1\big)\nu^{(k)}(d\boldsymbol{y})\, ds\Big) \\ &= \prod_k \widehat{\mathbb{L}}^{(k)}\big(\gamma^{\boldsymbol{F}^{(k)}, h^{(k)}}\big)\ : \end{aligned} \tag{4.6.20}$$

the characteristic function of the k-tuple $\big(({}^{\tilde c}\boldsymbol{Z}^{(k)}, \jmath_{\boldsymbol{Z}^{(k)}})\big)_{1\le k\le K}$ is the product of the characteristic functions of the components. Now apply A.3.36. ∎

Exercise 4.6.14 (i) Suppose $A^{(k)}$ are mutually disjoint relatively compact Borel subsets of $\mathbb{R}^d_* \times [0,\infty)$. Applying equation (4.6.20) with $h^{(k)} \stackrel{\text{def}}{=} \alpha^{(k)} A^{(k)}$, show that the random variables $\jmath_{\boldsymbol{Z}}(A^{(k)})$ are independent Poisson with means $(\nu\times\lambda)(A^{(k)})$. In other words, the jump measure of our Lévy process is a Poisson point process on $\mathbb{R}^d_*$ with intensity rate ν, in the sense of definition 3.10.19 on page 185.

(ii) Suppose next that $h^{(k)} : \mathbb{R}^d_* \to \mathbb{R}$ are time-independent sure Hunt functions with disjoint carriers. Then the indefinite integrals $\boldsymbol{Z}^{(k)} \stackrel{\text{def}}{=} h^{(k)} * \widetilde{\jmath_{\boldsymbol{Z}}}$ are independent Lévy processes with characteristic triples $\big(0, 0, h^{(k)}\nu\big)$.

Exercise 4.6.15 If ζ is a Poisson point process with intensity rate ν and $f \in \mathcal{L}^1(\nu)$, then $f * \zeta$ is a Lévy process with values in $\mathbb{R}$ and with characteristic triple $\big(\int f[|f| \le 1] d\nu, 0, f[\nu]\big)$.

Canonical Components of a Lévy Process

Since our Lévy process $\boldsymbol{Z}$ with characteristic triple $(\boldsymbol{A}, B, \nu)$ is quasi-left-continuous, the sparsely and previsibly supported part ${}^{p}Z$ of proposition 4.4.5 vanishes and the decomposition (4.4.3) of exercise 4.4.10 boils down to

$$\boldsymbol{Z} = {}^{\hat{v}}\boldsymbol{Z} + {}^{\tilde{c}}\boldsymbol{Z} + {}^{\tilde{s}}\boldsymbol{Z} + {}^{l}\boldsymbol{Z} \ . \tag{4.6.21}$$

The following table shows the features of the various parts.

Part	Given by	Char. Triple	Prev. Control
${}^{\hat{v}}\boldsymbol{Z}_t$	$t\boldsymbol{A} = \widehat{{}^{s}\boldsymbol{Z}_t}$, see (4.6.10)	$(\boldsymbol{A}, 0, 0)$	via (4.6.22)
${}^{\tilde{c}}\boldsymbol{Z}_t$	$\sigma \boldsymbol{W}$, see lemma 4.6.8	$(0, B, 0)$	via (4.6.23)
${}^{\tilde{s}}\boldsymbol{Z}_t$	$\int \boldsymbol{y} \cdot [\lvert\boldsymbol{y}\rvert \le 1] \times [\![0,t]\!] \, \widetilde{\jmath_{\boldsymbol{Z}}}(d\boldsymbol{y}, ds)$	$(0, 0, {}^{s}\nu)$	via (4.6.24)
${}^{l}\boldsymbol{Z}_t$	$\int \boldsymbol{y} \cdot [\lvert\boldsymbol{y}\rvert > 1] \times [\![0,t]\!] \, \jmath_{\boldsymbol{Z}}(d\boldsymbol{y}, ds)$	$(0, 0, {}^{l}\nu)$	via (4.6.29)

To discuss the items in this table it is convenient to introduce some notation. We write $|\ |$ for the sup-norm $|\ |_\infty$ on vectors or sequences and $|\ |_1$ for the ℓ^1-norm: $|\boldsymbol{x}|_1 = \sum_\eta |x_\eta|$. Furthermore we write

$${}^{s}\nu(d\boldsymbol{y}) \stackrel{\text{def}}{=} [|y| \le 1]\, \nu(d\boldsymbol{y}) \qquad \text{for the small-jump intensity rate,}$$

$${}^{l}\nu(d\boldsymbol{y}) \stackrel{\text{def}}{=} [|y| > 1]\, \nu(d\boldsymbol{y}) \qquad \text{for the large-jump intensity rate,}$$

and

$$|\mu|_\rho \stackrel{\text{def}}{=} \sup_{|\boldsymbol{x}'| \le 1} \Big(\int \big|\langle \boldsymbol{x}' | \boldsymbol{y} \rangle\big|^\rho \, \mu(d\boldsymbol{y}) \Big)^{1/\rho}, \qquad 0 < \rho < \infty ,$$

for any positive measure μ on $\mathbb{R}^d_*$. Now the previsible controllers $\boldsymbol{Z}^{\langle\rho\rangle}$ of inequality (4.5.20) on page 246 are given in terms of the characteristic triple $(\boldsymbol{A}, B, \nu)$ by

$$\boldsymbol{Z}_t^{\langle\rho\rangle} = t \cdot \begin{cases} \sup\limits_{|\boldsymbol{x}'| \le 1} \Big(\langle \boldsymbol{x}' | \boldsymbol{A} \rangle + \int \langle \boldsymbol{x}' | \boldsymbol{y} \rangle \cdot [|\boldsymbol{y}| > 1]\, \nu(d\boldsymbol{y}) \Big) \le |\boldsymbol{A}|_1 + |{}^{l}\nu|_1^1 , & \rho = 1, \\ \sup\limits_{|\boldsymbol{x}'| \le 1} \Big(x'_\eta x'_\theta B^{\eta\theta} + \int \big|\langle \boldsymbol{x}' | \boldsymbol{y} \rangle\big|^2 \nu(d\boldsymbol{y}) \Big) \le |B| + |\nu|_2^2 , & \rho = 2, \\ \sup\limits_{|\boldsymbol{x}'| \le 1} \Big(\int \big|\langle \boldsymbol{x}' | \boldsymbol{y} \rangle\big|^\rho \nu(d\boldsymbol{y}) \Big) = |\nu|_\rho^\rho , & \rho > 2, \end{cases}$$

provided of course that $\boldsymbol{Z}$ is a local L^q-integrator for some $q \ge 2$ and $\rho \le q$. Here $|B| \stackrel{\text{def}}{=} \sup\{x'_\eta x'_\theta B^{\eta\theta} : |\boldsymbol{x}'|_\infty \le 1\}$. Now the first three Lévy processes ${}^{\hat{v}}\boldsymbol{Z}$, ${}^{\tilde{c}}\boldsymbol{Z}$, and ${}^{\tilde{s}}\boldsymbol{Z}$ all have bounded jumps and thus are L^q-integrators for all $q < \infty$. Inequality (4.5.20) and the second column of the table above therefore result in the following inequalities: for any previsible $\boldsymbol{X}$, $2 \le p \le q < \infty$, and instant t

$$\left\|\left(\boldsymbol{X}*^{\hat{v}}\boldsymbol{Z}\right)^\star_t\right\|_{L^p} \le \left(|\boldsymbol{A}|_1 + |^l\nu|_1\right)\left\|\int_0^t |\boldsymbol{X}|_s\, ds\right\|_{L^p}, \tag{4.6.22}$$

$$\left\|\left(\boldsymbol{X}*^{\tilde{c}}\boldsymbol{Z}\right)^\star_t\right\|_{L^p} \le C_p^{\diamond(4.5.11)} \cdot |B| \cdot \left\|\left(\int_0^t |\boldsymbol{X}|_s^2\, ds\right)^{1/2}\right\|_{L^p}, \tag{4.6.23}$$

and
$$\left\|\left(\boldsymbol{X}*^{\tilde{s}}\boldsymbol{Z}\right)^\star_t\right\|_{L^p} \le C_p^{\diamond} \cdot \max_{\rho=2,p} |^s\nu|_\rho \cdot \left\|\left(\int_0^t |\boldsymbol{X}|_s^\rho\, ds\right)^{1/\rho}\right\|_{L^p}. \tag{4.6.24}$$

Lastly, let us estimate the large-jump part $^l\boldsymbol{Z}$, first for $0 < p \le 1$. To this end let $\boldsymbol{X}$ be previsible and set $Y \overset{\text{def}}{=} \boldsymbol{X}*^l\boldsymbol{Z}$. Then

$$|Y_t^\star|^p \le \sum_{s\le t} \left(|Y_{s-}| + |\Delta Y_s|\right)^p - |Y_{s-}|^p$$

as $p \le 1$:
$$\le \sum_{s\le t} |\Delta Y_s|^p = \sum_{s\le t} \left|\langle \boldsymbol{X}_s | \Delta^l\boldsymbol{Z}_s\rangle\right|^p$$

$$= \int_{[\![0,t]\!]} |\langle \boldsymbol{X}_s|\boldsymbol{y}\rangle|^p \cdot [|\boldsymbol{y}| > 1]\, \jmath_{\boldsymbol{Z}}(d\boldsymbol{y}, ds)\,.$$

Hence
$$\left\|\left(\boldsymbol{X}*^l\boldsymbol{Z}\right)^\star_t\right\|_{L^p} \le |^l\nu|_p \cdot \left(\int\int_0^t |\boldsymbol{X}|_s^p\, ds\, d\mathbb{P}\right)^{1/p} \quad \text{for } 0 < p \le 1\,. \tag{4.6.25}$$

This inequality of course says nothing at all unless every linear functional $\boldsymbol{x}'$ on $\mathbb{R}^d$ has p^{th} moments with respect to $^l\nu$, so that $|^l\nu|_p$ is finite. Let us next address the case that $|^l\nu|_p < \infty$ for some $p \in [1,2]$. Then $^l\boldsymbol{Z}$ is an L^p-integrator with Doob–Meyer decomposition

$$^l\boldsymbol{Z} = t \cdot \int \boldsymbol{y}\; ^l\nu(d\boldsymbol{y}) + \int_{[\![0,t]\!]} \boldsymbol{y} \cdot [|\boldsymbol{y}| > 1]\, \widetilde{\jmath_{\boldsymbol{Z}}}(d\boldsymbol{y}, ds)\,.$$

With $Y \overset{\text{def}}{=} \boldsymbol{X}*^l\boldsymbol{Z}$, a little computation produces

$$\left\|\widehat{Y}_t^\star\right\|_{L^p} \le |^l\nu|_p \cdot \left(\int\int_0^t |\boldsymbol{X}|_s^p\, ds\, d\mathbb{P}\right)^{1/p}, \tag{4.6.26}$$

and theorem 4.2.12 gives

$$\begin{aligned}
\left\|\widetilde{Y}_t^\star\right\|_{L^p} &\le C_p^{(4.2.4)} \cdot \left\|S_t[\widetilde{Y}]\right\|_{L^p} = C_p \cdot \left\|\left(\sum_{s\le t}\left|\langle \boldsymbol{X}_s|\Delta^l\widetilde{\boldsymbol{Z}}_s\rangle\right|^2\right)^{1/2}\right\|_{L^p} \\
&\le C_p \cdot \left\|\left(\sum_{s\le t}\left|\langle \boldsymbol{X}_s|\Delta^l\widetilde{\boldsymbol{Z}}_s\rangle\right|^p\right)^{1/p}\right\|_{L^p} \\
&= C_p \cdot \left(\mathbb{E}\left[\int_{[\![0,t]\!]} |\langle \boldsymbol{X}_s|\boldsymbol{y}\rangle|^p \cdot [|\boldsymbol{y}| > 1]\, \jmath_{\boldsymbol{Z}}(d\boldsymbol{y}, ds)\right]\right)^{1/p} \\
&\le C_p\, |^l\nu|_p \cdot \left(\int\int_0^t |\boldsymbol{X}|_s^p\, ds\, d\mathbb{P}\right)^{1/p}.
\end{aligned} \tag{4.6.27}$$

Putting (4.6.26) and (4.6.27) together yields, for $1 \le p \le 2$,

$$\left\| \left(\boldsymbol{X} {*} {}^{l}\boldsymbol{Z} \right)^{\star}_{t} \right\|_{L^p} \le \left(1 + C_p\right) |{}^{l}\nu|_p \cdot \left(\int \int_0^t |\boldsymbol{X}|^p_s \, ds \, d\mathbb{P} \right)^{1/p} . \tag{4.6.28}$$

If $p \ge 2$ and $|{}^{l}\nu|_p < \infty$, then we use inequality (4.6.26) to estimate the previsible finite variation part ${}^{l}\widehat{\boldsymbol{Z}}$, and inequality (4.5.20) for the martingale part ${}^{l}\widetilde{\boldsymbol{Z}}$. Writing down the resulting inequality together with (4.6.25) and (4.6.28) gives

$$\left\| \left(\boldsymbol{X} {*} {}^{l}\boldsymbol{Z} \right)^{\star}_{t} \right\|_{L^p} \le \begin{cases} C_p \cdot |{}^{l}\nu|_p \cdot \left\| \left(\int_0^t |\boldsymbol{X}|^p_s \, ds \right)^{1/p} \right\|_{L^p} & \text{for } 0 < p \le 2, \\ C_p \cdot \max\limits_{\rho = 2, p} |{}^{l}\nu|_\rho \cdot \left\| \left(\int_0^t |\boldsymbol{X}|^\rho_s \, ds \right)^{1/\rho} \right\|_{L^p} & \text{for } 2 \le p \le q, \end{cases} \tag{4.6.29}$$

where $$C_p = \begin{cases} 1 & \text{for } 0 < p \le 1, \\ C_p^{(4.2.4)} + 1 & \text{for } 1 < p \le 2, \\ C^{\diamond (4.5.11)} & \text{for } 2 \le p. \end{cases}$$

We leave to the reader the proof of the necessity part and the estimation of the universal constants $C_p^{(\rho)}(\mathfrak{t})$ in the following proposition – the sufficiency has been established above.

Proposition 4.6.16 *Let $\boldsymbol{Z}$ be a Lévy process with characteristic triple $\mathfrak{t} = (\boldsymbol{A}, B, \nu)$ and let $0 < q < \infty$. Then $\boldsymbol{Z}$ is an L^q-integrator if and only if its Lévy measure ν has q^{th} moments away from zero:*

$$|{}^{l}\nu|_q \overset{\text{def}}{=} \sup_{|\boldsymbol{x}'| \le 1} \left(\int \left| \langle \boldsymbol{x}' | \boldsymbol{y} \rangle \right|^q \cdot [|\boldsymbol{y}| > 1] \, \nu(d\boldsymbol{y}) \right)^{1/q} < \infty .$$

If so, then there is the following estimate for the stochastic integral of a previsible integrand $\boldsymbol{X}$ with respect to $\boldsymbol{Z}$: for any stopping time T and any $p \in (0, q)$

$$\| |\boldsymbol{X} {*} \boldsymbol{Z}|^{\star}_T \|_{L^p} \le \max_{\rho = 1^\diamond, 2, p^\diamond} C_p^{(\rho)}(\mathfrak{t}) \cdot \left\| \left(\int_0^T |\boldsymbol{X}_s|^\rho \, ds \right)^{1/\rho} \right\|_{L^p} . \tag{4.6.30}$$

Construction of Lévy Processes

Do Lévy processes exist? For all we know so far we could have been investigating the void situation in the previous 11 pages.

Theorem 4.6.17 *Let $(\boldsymbol{A}, B, \nu)$ be a triple having the properties spelled out in theorem 4.6.9 on page 259. There exist a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and on it a Lévy process $\boldsymbol{Z}$ with characteristic triple $(\boldsymbol{A}, B, \nu)$; any two such processes have the same law.*

Proof. The uniqueness of the law has been shown in lemma 4.6.4. We prove the existence piecemeal.

First the continuous martingale part ${}^{\tilde c}\boldsymbol{Z}$. By exercise 3.9.6 on page 161, there is a probability space $({}^{\tilde c}\Omega, {}^{\tilde c}\mathcal{F}, {}^{\tilde c}\mathbb{P})$ on which there lives a d-dimensional Wiener process with covariance matrix B. This Lévy process is clearly a good candidate for the continuous martingale part ${}^{\tilde c}\boldsymbol{Z}$ of the prospective Lévy process $\boldsymbol{Z}$.

Next, the idea leading to the "jump part" ${}^{j}\boldsymbol{Z} \stackrel{\text{def}}{=} {}^{\tilde s}\boldsymbol{Z} + {}^{l}\boldsymbol{Z}$ is this: we construct the jump measure $\jmath_{\boldsymbol{Z}}$ and, roughly (!), write ${}^{j}\boldsymbol{Z}_t$ as $\int_{[\![0,t]\!]} \boldsymbol{y}\, \jmath_{\boldsymbol{Z}}(d\boldsymbol{y}, ds)$. Exercise 4.6.14 (i) shows that $\jmath_{\boldsymbol{Z}}$ should be a Poisson point process with intensity rate ν on $\mathbb{R}^d_*$. The construction following definition (3.10.9) on page 184 provides such a Poisson point process π. The proof of the theorem will be finished once the following facts have been established – they are left as an exercise in bookkeeping:

Exercise 4.6.18 π has intensity $\nu\times\lambda$ and is independent of ${}^{\tilde c}\boldsymbol{Z}$.

$$ {}^{k}\boldsymbol{Z}_t \stackrel{\text{def}}{=} \int_{[\![0,t]\!]} \boldsymbol{y}\cdot[2^k < |\boldsymbol{y}| \le 2^{k+1}]\,\pi(d\boldsymbol{y}, ds)\,, \qquad k\in\mathbb{Z}\,, $$

has independent stationary increments and is continuous in q-mean for all $q<\infty$; it is an L^q-integrator, càdlàg after suitable modification. The sum

$$ {}^{\tilde s}\boldsymbol{Z} \stackrel{\text{def}}{=} \sum\nolimits_{k<0} \widetilde{{}^{k}\boldsymbol{Z}} $$

converges in the $\mathcal{I}^q$-norm and is a martingale and a Lévy process with characteristic triple $(0,0,{}^{s}\nu)$, càdlàg after modification. Since the set $[1<|\boldsymbol{y}|]$ is ν-integrable, $\pi\big([1<|\boldsymbol{y}|]\times[0,t]\big)<\infty$ a.s. Now as π is a sum of point masses, this implies that

$$ {}^{l}\boldsymbol{Z}_t \stackrel{\text{def}}{=} \int_{[\![0,t]\!]} \boldsymbol{y}\cdot[1<|\boldsymbol{y}|]\,\pi(d\boldsymbol{y}, ds) = \sum\nolimits_{0\le k} {}^{k}\boldsymbol{Z}_t $$

is almost surely a finite sum of points in $\mathbb{R}^d_*$ and defines a Lévy process ${}^{l}\boldsymbol{Z}$ with characteristic triple $(0,0,{}^{l}\nu)$. Finally, setting ${}^{\hat v}\boldsymbol{Z}_t \stackrel{\text{def}}{=} \boldsymbol{A}\cdot t$,

$$ \boldsymbol{Z}_t \stackrel{\text{def}}{=} {}^{\hat v}\boldsymbol{Z}_t + {}^{\tilde c}\boldsymbol{Z}_t + {}^{\tilde s}\boldsymbol{Z}_t + {}^{l}\boldsymbol{Z}_t $$

is a Lévy process with the given characteristic triple $(\boldsymbol{A}, B, \nu)$, written in its canonical decomposition (4.6.21). ∎

Exercise 4.6.19 For $d=1$ and $\nu=\delta_1$ the point mass at $1\in\mathbb{R}$, Z_t is a Poisson process and has Doob–Meyer decomposition $Z_t = t + \widetilde{Z}_t$.

Feller Semigroup and Generator

We continue to consider the Lévy process $\boldsymbol{Z}$ in $\mathbb{R}^d$ with convolution semigroup $\mu_{\boldsymbol{.}}$ of distributions. We employ them to define bounded linear operators[18]

$$ \phi \mapsto T_t\phi\,(\boldsymbol{y}) \stackrel{\text{def}}{=} \mathbb{E}\big[\phi(\boldsymbol{y}+\boldsymbol{Z}_t)\big] = \int_{\mathbb{R}^d} \phi(\boldsymbol{y}+\boldsymbol{z})\,\mu_t(d\boldsymbol{z}) = \big(\overset{*}{\mu}_t\star\phi\big)(\boldsymbol{y})\,. \tag{4.6.31} $$

[18] $\overset{*}{\phi}(x) \stackrel{\text{def}}{=} \phi(-x)$ and $\overset{*}{\mu}(\phi) \stackrel{\text{def}}{=} \mu(\overset{*}{\phi})$ define the ***reflections through the origin*** $\overset{*}{\phi}$ and $\overset{*}{\mu}$.

They are easily seen to form a conservative Feller semigroup on $C_0(\mathbb{R}^d)$ (see definition A.9.5 on page 465). Here are a few straightforward observations. $T_.$ is ***translation-invariant*** or ***commutes with translation***. This means the following: for any $\boldsymbol{a} \in \mathbb{R}^d$ and $\phi \in C_0(\mathbb{R}^d)$ define the ***translate*** $\phi_{\boldsymbol{a}}$ by $\phi_{\boldsymbol{a}}(\boldsymbol{x}) = \phi(\boldsymbol{x} - \boldsymbol{a})$; then $T_t(\phi_{\boldsymbol{a}}) = (T_t\phi)_{\boldsymbol{a}}$.

The dissipative generator $\mathcal{A} \stackrel{\text{def}}{=} dT_t/dt_{|t=0}$ of $T_.$ obeys the positive maximum principle (see item A.9.8) and is again translation-invariant: for $\phi \in \operatorname{dom}(\mathcal{A})$ we have $\phi_a \in \operatorname{dom}(\mathcal{A})$ and $\mathcal{A}\phi_a = (\mathcal{A}\phi)_a$. Let us compute $\mathcal{A}$. For instance, on a function ϕ in ***Schwartz space***[19] $\mathcal{S}$, Itô's formula (3.10.6) gives[4], with $\boldsymbol{Z}^{\boldsymbol{z}}_. \stackrel{\text{def}}{=} \boldsymbol{z} + \boldsymbol{Z}_.$,

$$\begin{aligned}
\phi(\boldsymbol{Z}^{\boldsymbol{z}}_t) &= \phi(\boldsymbol{z}) + \int_{0+}^t \phi_{;\eta}(\boldsymbol{Z}^{\boldsymbol{z}}_{s-})\, d^s Z^\eta_s + \frac{1}{2}\int_{0+}^t \phi_{;\eta\theta}(\boldsymbol{Z}^{\boldsymbol{z}}_{.-})\, d^c[Z^\eta, Z^\theta]_s \\
&\quad + \int_0^t \Big(\phi(\boldsymbol{Z}^{\boldsymbol{z}}_{s-} + \boldsymbol{y}) - \phi(\boldsymbol{Z}^{\boldsymbol{z}}_{s-}) - \phi_{;\eta}(\boldsymbol{Z}^{\boldsymbol{z}}_{s-}) \cdot y^\eta[|\boldsymbol{y}| \le 1]\Big)\, \jmath_{\boldsymbol{Z}}(d\boldsymbol{y}, ds) \\
&= Mart_t + \int_0^t \boldsymbol{A}^\eta \phi_{;\eta}(\boldsymbol{Z}^{\boldsymbol{z}}_s)\, ds + \frac{1}{2}\int_0^t B^{\eta\theta}\phi_{;\eta\theta}(\boldsymbol{Z}^{\boldsymbol{z}}_s)\, ds \\
&\quad + \int_0^t \int_{\mathbb{R}^d_*} \Big(\phi(\boldsymbol{Z}^{\boldsymbol{z}}_s + \boldsymbol{y}) - \phi(\boldsymbol{Z}^{\boldsymbol{z}}_s) - \phi_{;\eta}(\boldsymbol{Z}^{\boldsymbol{z}}_s) \cdot y^\eta[|\boldsymbol{y}| \le 1]\Big)\, \nu(d\boldsymbol{y})\, ds\,.
\end{aligned}$$

Here part of the jump-measure integral was shifted into the first term using definition (4.6.10). The second equality comes from the identifications (4.6.11), (4.6.8), and (4.6.9). Since ϕ is bounded, the martingale part $Mart_.$ is an integrable martingale; so taking the expectation is permissible, and differentiating the result in t at $t = 0$ yields[4]

$$\begin{aligned}
\mathcal{A}\phi(\boldsymbol{z}) &= \boldsymbol{A}^\eta \phi_{;\eta}(\boldsymbol{z}) + \frac{1}{2} B^{\eta\theta}\phi_{;\eta\theta}(\boldsymbol{z}) \\
&\quad + \int \Big(\phi(\boldsymbol{z}+\boldsymbol{y}) - \phi(\boldsymbol{z}) - \phi_{;\eta}(\boldsymbol{z})\, y^\eta[|\boldsymbol{y}| \le 1]\Big)\, \nu(d\boldsymbol{y})\,.
\end{aligned} \tag{4.6.32}$$

Example 4.6.20 Suppose the Lévy measure ν has finite mass, $|\nu| \stackrel{\text{def}}{=} \nu(1) < \infty$,

and, with $$\underline{\boldsymbol{A}} \stackrel{\text{def}}{=} \boldsymbol{A} - \int \boldsymbol{y}\, [|\boldsymbol{y}| \le 1]\, \nu(d\boldsymbol{y})\,,$$

set[4] $$\mathcal{D}\phi(\boldsymbol{z}) \stackrel{\text{def}}{=} \underline{A}^\eta \frac{\partial \phi(\boldsymbol{z})}{\partial z^\eta} + \frac{B^{\eta\theta}}{2} \frac{\partial^2 \phi(\boldsymbol{z})}{\partial z^\eta \partial z^\theta}$$

and $$\mathcal{J}\phi(\boldsymbol{z}) \stackrel{\text{def}}{=} \int \Big(\phi(\boldsymbol{z}+\boldsymbol{y}) - \phi(\boldsymbol{z})\Big)\, \nu(d\boldsymbol{y}) = \overset{*}{\nu} \star \phi\,(\boldsymbol{z}) - |\nu| \cdot \phi(\boldsymbol{z})\,.$$

Then evidently $\mathcal{A} = \mathcal{D} + \mathcal{J}$. (4.6.33)

[19] $\mathcal{S} = \{\phi \in C_b^\infty(\mathbb{R}^n) : \sup_x |x|^k \cdot |\phi^{(l)}(x)| < \infty \quad \forall k \in \mathbb{N}$ and for all partial derivatives $\phi^{(l)}\}$.

In the case that ν is atomic, $\mathcal{J}$ is but a linear combination of difference operators, and in the general case we view it as a "continuous superposition" of difference operators. Now $\mathcal{J}$ is also a bounded linear operator on $C_0(\mathbb{R}^d)$, and so

$$T_t^{\mathcal{J}} = e^{t\mathcal{J}} \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} (t\mathcal{J})^k / k!$$

is a contractive, in fact Feller, semigroup. With $\nu^0 \stackrel{\text{def}}{=} \delta_0$ and $\nu^k \stackrel{\text{def}}{=} \nu^{k-1} \star \nu$ denoting the k-fold convolution of ν with itself, it can be written

$$T_t^{\mathcal{J}} \phi = e^{-t|\nu|} \sum_{k=0}^{\infty} \frac{t^k \overset{*}{\nu}^k \star \phi}{k!} .$$

If $\mathcal{D} = 0$, then the characteristic triple is $(0, 0, \nu)$, with $\nu(1) < \infty$. A Lévy process with such a special characteristic triple is a ***compound Poisson process***. In the even more special case that $\mathcal{D} = 0$ and ν is the Dirac measure $\delta_{\boldsymbol{a}}$ at $\boldsymbol{a}$, the Lévy process is the ***Poisson process*** with jump $\boldsymbol{a}$. Then equation (4.6.33) reads $\mathcal{A}\phi(\boldsymbol{z}) = \phi(\boldsymbol{z} + \boldsymbol{a}) - \phi(\boldsymbol{z})$ and integrates to

$$T_t \phi(\boldsymbol{z}) = e^{-t} \sum_{k \geq 0} \frac{t^k \phi(\boldsymbol{z} + k\boldsymbol{a})}{k!} .$$

$\mathcal{D}$ can be exponentiated explicitely as well: with γ_{tB} denoting the Gaussian with covariance matrix tB from definition A.3.50 on page 420, we have

$$T_t^{\mathcal{D}} \phi(\boldsymbol{z}) = \int \phi(\boldsymbol{z} + \underline{\boldsymbol{A}} t + \boldsymbol{y}) \gamma_{tB}(d\boldsymbol{y}) = \overset{*}{\gamma}_{tB} \star \phi_{-\underline{\boldsymbol{A}}t}(\boldsymbol{z}) .$$

In general, the operators $\mathcal{D}$ and $\mathcal{J}$ commute on Schwartz space, which is a core for either operator, and then so do the corresponding semigroups. Hence

$$T_t^{\mathcal{A}}[\phi] = T_t^{\mathcal{J}}\left[T_t^{\mathcal{D}}[\phi]\right] = T_t^{\mathcal{D}}\left[T_t^{\mathcal{J}}[\phi]\right] , \qquad \phi \in C_0(\mathbb{R}^d) .$$

Exercise 4.6.21 (A Special Case of the Hille–Yosida Theorem) Let $(\boldsymbol{A}, B, \nu)$ be a triple having the properties spelled out in theorem 4.6.9 on page 259. Then the operator $\mathcal{A}$ on $\mathcal{S}$ defined in equation (4.6.32) from this triple is conservative and dissipative on $C_0(\mathbb{R}^d)$. $\mathcal{A}$ is closable and $T_\bullet$ is the unique Feller semigroup on $C_0(\mathbb{R}^d)$ with generator $\overline{\mathcal{A}}$, which has $\mathcal{S}$ for a core.

Repeated Footnotes: 196[3] 203[4] 235[10] 239[12] 249[14] 258[15] 261[16] 268[18]