HIGHER WEIGHT HEEGNER POINTS

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Abstract. In this paper we formulate a conjecture which partially generalizes the Gross-Kohnen-Zagier theorem to higher weight modular forms. For $f \in S_{2k}(N)$ satisfying certain conditions, we construct a map from the Heegner points of level N to a complex torus, $\mathbb{C}/L_f$, defined by f. We define higher weight analogues of Heegner divisors on $\mathbb{C}/L_f$. We conjecture they all lie on a line, and their positions are given by the coefficients of a certain Jacobi form corresponding to f. In weight 2, our map is the modular parametrization map (restricted to Heegner points), and our conjectures are implied by Gross-Kohnen-Zagier. For any weight, we expect that our map is the Abel-Jacobi map on a certain modular variety, and so our conjectures are consistent with the conjectures of Beilinson-Bloch. We have verified our map is the Abel-Jacobi for weight 4. We provide numerical evidence to support our conjecture for a variety of examples.

1. Introduction

For integers $N, k \geq 1$, let $S_{2k}(N)$ denote the cusp forms of weight $2k$ on the congruence group $\Gamma_0(N)$. Let $X_0(N)$ be the usual modular curve and $J_0(N)$ its Jacobian. By D we will always mean a negative fundamental discriminant which is a square modulo $4N$. For each D, one can construct a Heegner divisor y_D in $J_0(N)$ and defined over $\mathbb{Q}$. Suppose $f \in S_2(N)$ is any normalized newform whose sign in the functional equation of $L(f,s)$ is -1. Then the celebrated theorem of Gross, Kohnen, and Zagier [Gross et al. 87, Theorem C] says that, as D varies, the f-eigencomponents of the Heegner divisors y_D all 'lie on a line[1]' in the quotient $J_0(N)_f$. Furthermore it says their positions on this line are given by the coefficients of a certain Jacobi form. In particular when N is prime, the positions are the coefficients of a half-integer weight modular form in Shimura correspondence with f.

Now suppose $f \in S_{2k}(N)$ is a normalized newform of weight $2k$ and level N. In addition, assume the coefficients in its Fourier series are rational, and the sign in the functional equation of $L(f,s)$ is -1. Let $\mathcal{H}_N/\Gamma_0(N) \subset X_0(N)$ denote the Heegner points of level N. In this paper we construct a map,

$$\alpha : \mathcal{H}_N/\Gamma_0(N) \to \mathbb{C}/L_f,$$

where $\mathbb{C}/L_f$ is a complex torus defined by the periods of f. Let $h(D)$ denote the class number of the imaginary quadratic field of discriminant D. For each D and fixed choice of its square root (mod$2N$), we get precisely $h(D)$ distinct representatives $\tau_1, \ldots, \tau_{h(D)}$ of $\mathcal{H}_N/\Gamma_0(N)$. Define $(\mathcal{Y}_D)_f = \alpha(\tau_1) + \cdots + \alpha(\tau_{h(D)})$

1991 *Mathematics Subject Classification.*

Key words and phrases. modular forms, elliptic curves, Gross-Kohnen-Zagier formula, Shimura-Kohnen correspondence.

[1] We will say a subset X of an abelian group J lies on a line if $X \subseteq \mathbb{Z} \cdot x_0$ for some $x_0 \in J$.

and define $(y_D)_f = (\mathfrak{y}_D)_f + \overline{(\mathfrak{y}_D)_f}$ in $\mathbb{C}/L_f$. When $k = 1$, α is the usual modular parametrization map restricted to Heegner points, and $(y_D)_f$ is equal to the f-eigencomponent of y_D in $J_0(N)$ as described in the first paragraph. For $k \geq 1$ we formulate conjectures similar to Gross-Kohnen-Zagier. We predict the $(y_D)_f$ all lie in a line in $\mathbb{C}/L_f$, that is, there exists a point $y_f \in \mathbb{C}/L_f$ such that

$$(y_D)_f = m_D y_f,$$

up to torsion, with $m_D \in \mathbb{Z}$. Furthermore we predict the positions m_D on the line are coefficients of a certain Jacobi form corresponding to f. In the case when N is prime and k is odd, the m_D should be the coefficients of a weight $(k+1/2)$ modular form in Shimura correspondence with f.

We expect our map is equivalent to the Abel Jacobi map on Kuga-Sato varieties in the following sense. Let $Y = Y^k$ be the Kuga-Sato variety associated to weight $2k$ forms on $\Gamma_0(N)$. (See [Zhang 97, p.117] for details.) This is a smooth projective variety over $\mathbb{Q}$ of dimension $2k-1$. Set $\mathcal{Z}^k(Y)_{\text{hom}}$ to be the nullhomologous codimension k algebraic cycles, and $\mathrm{CH}^k(Y)_{\text{hom}}$ the group of $\mathcal{Z}^k(Y)_{\text{hom}}$ modulo rational equivalence. Let Φ^k be the usual k-th Abel-Jacobi map,

$$\Phi^k : \mathrm{CH}^k(Y)_{\text{hom}} \to J^k(Y),$$

where $J^k(Y)$ is the k-th intermediate Jacobian of Y. Given any normalized newform $f = \sum_{n\geq 1} a_n q^n \in S_{2k}(N)$ with rational coefficients, there exists an f-isotypical component $J^k_f(Y)$ of $J^k(Y)$, and thus an induced map,

$$\begin{tikzcd}
\mathrm{CH}^k(Y)_{\text{hom}} \arrow[r, "\Phi^k"] \arrow[dr, dashed, "\Phi^k_f"'] & J^k(Y) \arrow[d] \\
 & J^k_f(Y)
\end{tikzcd}$$

Our expectation is that the image of Φ^k_f on classes of CM cycles in $\mathrm{CH}^k(Y)_{\text{hom}}$ is equal (up to a constant) to the image of our map α on Heegner points in $X_0(N)$. If we assume this is the case, then our conjectures are consistent with the conjectures of Beilinson and Bloch. In this setting they predict

$$\operatorname{rank}_{\mathbb{Z}} \mathrm{CH}^k(Y_F)_{\text{hom}} = \operatorname{ord}_{s=k} L_F(H^{2k-1}(Y), s).$$

If we assume $\operatorname{ord}_{s=k} L(f,s) = 1$, then a refinement of their conjecture predicts the image of Φ^k_f on CM divisors in $Y_{\mathbb{Q}}$ should have rank at most 1 in $J^k_f(Y)$.

We have verified the equivalence of α and Φ^2_f in the case of weight 4. For this we used an explicit description of Φ^2_f on CM cycles given by Schoen in [Schoen 86]. In fact, in [Schoen 93] Schoen uses this map to investigate a consequence of Beilinson-Bloch similar to the one described above. For a specific $Y = Y^4$ and f he computes Φ_f on certain CM divisors in Y defined over the quadratic number field $\mathbb{Q}(i)$. From this he finds numerical evidence that the images lie on a line and their positions are given by a certain weight $5/2$ form corresponding to f.

The sections of this paper are divided as follows. In Section 2 we describe our map and its lattice of periods. In Section 3 we give explicit statements of our conjectures. In Section 4 we describe the algorithm we created to numerically verify the conjectures in a variety of examples. Note our algorithm could be applied to compute coefficients of half-integer weight modular forms. In sections 5 and 6 we compute some examples and use them to verify our conjectures in two different ways.

2. Higher Weight Heegner Points

Let $\mathfrak{h}$ denote the upper half-plane. Suppose f is a normalized newform in $S_{2k}(N)$ having a Fourier expansion of the form,

$$f(\tau) = \sum_{n=1}^{\infty} a_n q^n, \qquad q = \exp(2\pi i \tau),\ \tau \in \mathfrak{h}$$

with $a_n \in \mathbb{Q}$.

Recall the L-function of f is defined by the Dirichlet series,

$$L(f,s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}, \qquad \mathrm{Re}(s) > k + 1/2,$$

and has an analytic continuation to all of $\mathbb{C}$. Moreover the function $\Lambda(f,s) = N^{s/2}(2\pi)^{-s}\Gamma(s)L(f,s)$ satisfies the functional equation,

$$\Lambda(f,s) = \varepsilon\Lambda(f, 2k-s),$$

where $\varepsilon = \pm 1$ is the sign of the functional equation of $L(f,s)$.

For each prime divisor p of N, let $q = p^\ell$, $\ell \in \mathbb{N}$ such that $\gcd(q, N/q) = 1$ and set $\omega_q = \left(\begin{smallmatrix} qx_0 & 1 \\ Ny_0 & q \end{smallmatrix}\right)$, for some $x_0, y_0 \in \mathbb{Z}$, with $qx_0 - (N/q)y_0 = 1$. Define $\Gamma_0^*(N)$ to be the group generated by $\Gamma_0(N)$ and each ω_q. Let S be a set of generators for $\Gamma_0^*(N)$. Define the period integrals of f for the set S by,

$$\mathcal{P} = \left\{ (2\pi i)^k \int_{i\infty}^{\gamma(i\infty)} f(z) z^m dz \ :\ m \in \{0, \ldots, 2k-2\},\ \gamma \in S \right\} \subseteq \mathbb{C}.$$

These are sometimes referred to as Shimura integrals. It is straightforward to see that every integral of the form,

$$(2\pi i)^k \int_{i\infty}^{\gamma(i\infty)} f(z) z^m dz, \qquad \gamma \in \Gamma_0^*(N),\ 0 \le m \le 2k-2$$

is in an integral linear combination of elements in $\mathcal{P}$. (See [Shimura 73, Section 8.2], for example). In fact, the $\mathbb{Z}$-module generated by $\mathcal{P}$ forms a lattice,

Lemma 2.1. *$L := \mathit{Span}_{\mathbb{Z}}(\mathcal{P})$ is a lattice in $\mathbb{C}$.*

Proof. By theorems of Razar [Razar 77, Theorem 4] and Šokurov [Šokurov 80, Lemma 5.6], the set $\mathcal{P}$ is contained in some lattice. Hence L is of rank ≤ 2. To show its rank is 2, it suffices to show there exist nonzero complex numbers $u^+, u^- \in L$ with $u^+ \in \mathbb{R}$ and $u^- \in i\mathbb{R}$.

Suppose m is a prime not dividing N, and χ a primitive Dirichlet character modulo m. Define $(f \otimes \chi) := \sum_{n \ge 1} \chi(n) a_n q^n$, and $L(f \otimes \chi, s)$ to be its Dirichlet series. Let $\Lambda(f \otimes \chi, s) = (2\pi)^{-s}(Nm^2)^{s/2}\Gamma(s)L(f \otimes \chi, s)$. Then for $\mathrm{Re}(s) > k + 1/2$, we have

$$i^s (Nm^2)^{-s/2} \Lambda(f \otimes \chi, s) = \int_0^{i\infty} (f \otimes \chi)(z) z^s \frac{dz}{z}. \tag{2.1}$$

Let $g(\chi)$ denote the Gauss sum associated to χ. Then an expression for χ in terms of the additive characters is given by,

$$\chi(n) = m^{-1} g(\chi) \sum_{u \bmod m} \bar{\chi}(-u) e^{2\pi i n u/m}.$$

So

$$(f \otimes \chi)(\tau) = m^{-1} g(\chi) \sum_{u \bmod m} \bar{\chi}(-u) f(z + u/m).$$

Substituting this into (2.1) gives

$$i^s (Nm^2)^{-s/2} \Lambda(f \otimes \chi, s) = m^{-1} g(\chi) \sum_{u \bmod m} \bar{\chi}(-u) \int_0^{i\infty} f(z + u/m) z^s \frac{dz}{z},$$

and replacing z by $z - u/m$ and rearranging implies

$$i^{-s} g(\chi)^{-1} N^{-s/2} \Lambda(f \otimes \chi, s) = (-1)^{s-1} \sum_{u \bmod m} \bar{\chi}(-u) \int_{i\infty}^{u/m} f(z)(mz - u)^{s-1} dz.$$

Now let $s = 2k - 1$ in the above equation, and multiply both sides by $(2\pi i)^k$. In addition suppose χ is a quadratic Dirichlet character modulo m. If $m \equiv 3 \bmod 4$, then $g(\chi) = i\sqrt{m}$, and if $m \equiv 1 \bmod 4$ then $g(\chi) = \sqrt{m}$. Hence since $\Lambda(f \otimes \chi, 2k-1)$ is real-valued and nonzero, the right hand side of this equation is either purely real or purely imaginary depending on the choice of m. Then this proves the lemma since the right hand side is in L for any m.

□

Let $D < 0$ be a fundamental discriminant, and assume D is a square modulo $4N$. Fix a residue class $r \bmod 2N$ satisfying $D \equiv r^2 \bmod 4N$. Then

$$\mathcal{Q}_N^D(r) := \left\{[A, B, C] : A > 0, B, C \in \mathbb{Z}, D = B^2 - 4AC, A \equiv 0 \bmod N, B \equiv r \bmod 2N\right\}.$$

corresponds to a subset of the positive definite binary quadratic forms of discriminant D. We define $\mathcal{H}_N^D(r)$ to be the roots in $\mathfrak{h}$ of $\mathcal{Q}_N^D(r)$,

$$\mathcal{H}_N^D(r) := \left\{\tau = \frac{-B + \sqrt{D}}{2A} : [A, B, C] \in \mathcal{Q}_N^D(r), C = \frac{|D| + B^2}{4A}\right\}.$$

$\Gamma_0(N)$ preserves $\mathcal{H}_N^D(r)$, and the classes of $\mathcal{H}_N^D(r)/\Gamma_0(N)$ are in bijection with the classes of reduced binary quadratic forms of discriminant D. We will call $\mathcal{H}_N^D(r)/\Gamma_0(N)$ the set of Heegner points of level N, discriminant D, and root r. Define $\mathcal{H}_N$ to be the union of $\mathcal{H}_N^D(r)$ over all D, r, and so $\mathcal{H}_N/\Gamma_0(N)$ are the Heegner points of level N.

For each $\tau = \frac{-B+\sqrt{D}}{2A} \in \mathcal{H}_N^D(r)$, set $Q_\tau(z) := Az^2 + Bz + C$. We now define a function $\alpha = \alpha_f : \mathcal{H}_N \to \mathbb{C}$ by

$$\alpha(\tau) := (2\pi i)^k \int_{i\infty}^{\tau} f(z) Q_\tau(z)^{k-1} dz.$$

Lemma 2.2. *The map α induces a well-defined map (which we will also denote by α),*

$$\alpha : \mathcal{H}_N/\Gamma_0(N) \to \mathbb{C}/L.$$

Proof. For any $\tau \in \mathcal{H}_N$ of discriminant D and $\gamma \in \Gamma_0(N)$, we will show

$$\alpha(\gamma\tau) - \alpha(\tau) = (2\pi i)^k \cdot \int_{i\infty}^{\gamma(i\infty)} f(z) Q_{\gamma\tau}(z)^{k-1} dz.$$

Since $Q_{\gamma\tau}(z)$ has integer coefficients, this will imply $\alpha(\gamma\tau) - \alpha(\tau) \in L$ for all $\gamma \in \Gamma_0(N)$.

Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$. Then

$$\begin{aligned}
&\alpha(\gamma\tau) - (2\pi i)^k \cdot \int_{i\infty}^{\gamma(i\infty)} f(z)\, Q_{\gamma\tau}(z)^{k-1} dz \\
&= (2\pi i)^k \cdot \int_{\gamma(i\infty)}^{\gamma\tau} f(z) Q_{\gamma\tau}(z)^{k-1} dz \\
&= (2\pi i)^k \cdot \int_{i\infty}^{\tau} f(\gamma z) Q_{\gamma\tau}(\gamma z)^{k-1} d(\gamma z) \\
&= \alpha(\tau),
\end{aligned}$$

where in the last equality we used $f(\gamma z) = (cz+d)^{2k} f(z)$, $Q_{\gamma\tau}(z) = (-cz + a)^2 Q_\tau(z)$, and $d(\gamma z) = (cz+d)^{-2} dz$.

□

3. Conjectures

Let $\{\tau_1, \ldots, \tau_{h(D)}\} \in \mathcal{H}_N^D(r)$ be any set of distinct class representatives of $\mathcal{H}_N^D(r)/\Gamma_0(N)$. Define

$$P_{D,r} := \sum_{i=1}^{h(D)} \tau_i \in \operatorname{Div}(X_0(N)),$$

where $\operatorname{Div}(X_0(N))$ denotes the group of divisors on $X_0(N)$. If $D = -3$ (resp. $D = -4$), scale $P_{D,r}$ by 1/3 (resp. 1/2). Extend α to P_D by linearity and define

$$(y_{D,r})_f = \alpha(P_{D,r}) + \overline{\alpha(P_{D,r})} \in \mathbb{C}/L.$$

Here, bar denotes complex conjugation in $\mathbb{C}$. We write $y_{D,r}$ or y_D for $(y_{D,r})_f$, and P_D for $P_{D,r}$ when the context of f, r is clear.

By the actions of complex conjugation and Atkin-Lehner on $\mathcal{H}_N$, we have

$$\overline{\alpha(P_{D,r})} = -\varepsilon\alpha(P_{D,r}),$$

where ε is the sign of the functional equation of $L(f,s)$. Thus if $\varepsilon = +1$, then $y_{D,r}$ are in L for all D, r. This is, in some sense, the trivial case. Hence we restrict our attention to the case when $\varepsilon = -1$.

Conjectures 3.1 and 3.3 give a partial generalization of the Gross-Kohnen-Zagier theorem to higher weights.

Conjecture 3.1. *Let* $f = \sum_{n\geq 1} a_n q^n \in S_{2k}(N)$ *be a normalized newform with rational coefficients, and assume* $\varepsilon = -1$ *and* $L'(f,k) \neq 0$. *Then for all fundamental* *discriminants* $D < 0$ *and* $r \bmod 2N$ *with* $D \equiv r^2 \bmod 4N$*, there exist integers* $m_{D,r}$ *such that*

$$t y_{D,r} = m_{D,r} y_f \qquad \text{in } \mathbb{C}/L,$$

where $y_f \in \mathbb{C}/L$ *and* $t \in \mathbb{Z}$ *are both nonzero and independent of* D *and* r*.*

Remark 3.2. Equivalently we could say $y_{D,r} = m_{D,r} y_f$ up to a t-torsion element in $\mathbb{C}/L$.

To state the second conjecture we will need to use Jacobi forms. (See [Eichler and Zagier 85] for background). Let $J_{2k,N}$ denote the set of all Jacobi forms of weight $2k$ and index N. Then such a $\phi \in J_{2k,N}$ is a function $\phi : \mathfrak{h} \times \mathbb{C} \to \mathbb{C}$, which satisfies the transformation law

$$\phi\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right) = (c\tau + d)^{2k} e^{2\pi i N \frac{cz^2}{c\tau + d}} \phi(\tau, z),$$

for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$, and has a Fourier expansion of the form

$$\phi(\tau, z) = \sum_{\substack{n,r \in \mathbb{Z} \\ r^2 \leq 4Nn}} c(n,r) q^n \zeta^r, \qquad q = e^{2\pi i \tau},\ \zeta = e^{2\pi i z}. \tag{3.1}$$

The coefficient $c(n,r)$ depends only on $r^2 - 4Nn$ and on the class $r \bmod 2N$.

Suppose $f \in S_{2k}(N)$ is a normalized newform with $\varepsilon = -1$. Then by [Skoruppa and Zagier 88], there exists a non-zero Jacobi cusp form $\phi_f \in J_{k+1,N}$ which is unique up to scalar multiple and has the same eigenvalues as f under the Hecke operators T_m for m, N coprime. We predict that the coefficients of ϕ_f are related to the m_D from above in the following way,

Conjecture 3.3. *Let* $f = \sum_{n \geq 1} a_n q^n \in S_{2k}(N)$ *be a normalized newform with rational coefficients, and assume* $\varepsilon = -1$ *and* $L'(f,k) \neq 0$*. Assume Conjecture 3.1. Then*

$$m_{D,r} = c(n,r)$$

where $n = \frac{|D| + r^2}{4N}$ *and* $c(n,r)$ *is the* (n,r)*-th coefficient of the Jacobi form* $\phi_f \in J_{k+1,N}$*.*

Remark 3.4. When $k = 2$, the points $(y_{D,r})_f$ and y_f are the same as those defined in [Gross et al. 87], and both of our conjectures are implied by Theorem C of their paper. (Actually their theorem is only for D coprime to $2N$ but they say the result remains 'doubtless true' with this hypothesis removed. See [Hayashi 95] and [Borcherds 99] for more details.) Particular to weight 2 is the fact that $\mathbb{C}/L$ is defined over $\mathbb{Q}$ and so y_D is a rational point on the elliptic curve $E_f \simeq \mathbb{C}/L$. In contrast, we should stress that for weight $k > 2$, the elliptic curve $E \simeq \mathbb{C}/L$ is not expected to be defined over any number field. For instance, the j-invariants for our examples all appear to be transcendental over $\mathbb{Q}$.

Remark 3.5. For $N = 1$ or a prime, and k odd we can state Conjecture 3.3 in terms of modular forms of half-integer weight. Specifically, let $\phi \in J_{k+1}(N)$ be a Jacobi form with a Fourier expansion as in (3.1), and set

$$g(\tau) = \sum_{M=0}^{\infty} c(M) q^M, \qquad q = e^{2\pi i \tau}$$

where $c(M)$ is defined by,

$$c(M) := \begin{cases} c\left(\frac{M+r^2}{4N}, r\right) & \text{if } M \equiv -r^2 \bmod 4N \text{ for any } r \in \mathbb{Z}; \\ 0, & \text{otherwise.} \end{cases}$$

This function is well-defined because $c(n,r)$ depends only on r^2-4nN when $N=1$ or a prime, and k is odd. Then by [Eichler and Zagier 85, p.69], g is in $M_{k+1/2}(4N)$, the space of modular forms of weight $k+1/2$ and level $4N$. In addition, if $f \in S_{2k}(N)$ is a normalized newform with $\varepsilon = -1$, then the form g defined by ϕ_f is in Shimura correspondence with f.

4. Algorithm

Let $f = \sum_{n\geq 1} a_n q^n \in S_{2k}(N)$ be a normalized newform with rational Fourier coefficients. The sign ε of the functional equation of $L(f,s)$ can be computed with the identity,

$$f\left(\frac{-1}{Nz}\right) = (-1)^k \varepsilon N^k z^{2k} f(z)$$

given by the action of the Fricke involution of level N on f. We will only consider f such that $\varepsilon = -1$ and $L'(f,k) \neq 0$.

The first step is to find a basis of our lattice L, which is the $\mathbb{Z}$-module generated by the periods $\mathcal{P}$ as described above. Suppose p_1, p_2, p_3 are three periods in $\mathcal{P}$. Since L has rank 2, these are linearly dependent over $\mathbb{Z}$, that is

$$a_1p_1 + a_2p_2 + a_3p_3 = 0, \quad \text{for some} \quad a_i \in \mathbb{Z}.$$

We may assume $\gcd(a_1,a_2,a_3) = 1$. Let $d = \gcd(a_1,a_2)$, then there exist integers $x, y \in \mathbb{Z}$ such that $xa_1 + ya_2 = d$. Similarly $\gcd(d,a_3) = 1$ so there exist integers $u, v \in \mathbb{Z}$ such that $ud + va_3 = 1$. Define the matrix M by,

$$M = \begin{pmatrix} a_1 & a_2 & a_3 \\ -y & x & 0 \\ -va_1/d & -va_2/d & u \end{pmatrix}.$$

Observe $M \in GL_3(\mathbb{Z})$ and $M \cdot {}^T(p_1,p_2,p_3) = {}^T(0, -yp_1+xp_2, -va_1p_1/d - va_2p_2/d + up_3)$. Hence $-yp_1+xp_2$ and $-va_1p_1/d - va_2p_2/d + up_3$ are a basis for the $\mathbb{Z}$-module generated by p_1, p_2, p_3.

We would also like our basis elements to have small norm. Given a basis ω_1, ω_2 of a lattice, its norm form is a real bilinear quadratic form defined by the matrix,

$$B = \begin{pmatrix} 2|\omega_1|^2 & 2\operatorname{Re}(\omega_1\bar{\omega}_2) \\ 2\operatorname{Re}(\omega_1\bar{\omega}_2) & 2|\omega_2|^2 \end{pmatrix}.$$

Thus it is equivalent to a reduced form of the same discriminant, that is, there exists $U \in SL_2(\mathbb{Z})$ such that

$${}^TUBU = \begin{pmatrix} 2\alpha & \beta \\ \beta & 2\gamma \end{pmatrix}, \qquad \alpha, \beta, \gamma \in \mathbb{R},$$

with $|\beta| \leq \alpha \leq \gamma$ and $\beta \geq 0$ if either $|\beta| = \alpha$ or $\alpha = \gamma$. Hence $(\omega_1', \omega_2') := (\omega_1, \omega_2)U$ is a 'reduced' basis. For a basis of all of L we simply apply this process iteratively on the elements of $\mathcal{P}$.

In fact it is not hard to see that L is a real lattice, that is, $\bar{L} = L$. Thus given a basis ω_1, ω_2 of L, we may assume $\omega_1 \in i\mathbb{R}$, and therefore $\tau := \omega_2/\omega_1$ has real part equal to either 0 or 1/2. This implies $\mathrm{Re}(L) = \mathrm{Re}(\omega_2)$ which will help simplify our computations.

To actually compute the elements in $\mathcal{P}$ we need to split the path from $(i\infty)$ to $\gamma(i\infty)$ of integration at some point $\tau \in \mathfrak{h}$ which gives,

$$\int_{i\infty}^{\gamma(i\infty)} f(z)z^m dz = \int_{i\infty}^{\gamma(\tau)} f(z)z^m dz - \int_{i\infty}^{\tau} f(z)(az+b)^m(cz+d)^{2k-2-m}dz,$$

for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$. We choose τ to be a point at which f has good convergence. To compute integrals of the form,

$$\int_{i\infty}^{\tau} f(z)z^m dz,$$

we use repeated integration by parts to get the formula

$$\int_{i\infty}^{\tau} f(z)z^m dz = m!\,(-1)^m \sum_{j=-1}^{m-1} \frac{(-1)^{j+1}}{(j+1)!}\tau^{j+1} f_{m-j}(\tau), \tag{4.1}$$

where $f_\ell(\tau)$ is defined to be the ℓ-fold integral of f evaluated at $\tau \in \mathfrak{h}$, that is,

$$f_\ell(\tau) = \frac{1}{(2\pi i)^\ell} \sum_{n\geq 1} \frac{a_n}{n^\ell} q^n, \qquad q = \exp(2\pi i\tau)$$

which is well-defined for any $0 \leq \ell \leq 2k-1$.

The next task is to compute $\alpha(\tau)$ for $\tau \in \mathcal{H}_N$. We could do this using (4.1), but it is computationally faster to use the following identity for α. Recall the modular differential operator,

$$\partial_m := \frac{1}{2\pi i}\frac{d}{dz} - \frac{m}{4\pi y}, \qquad z = x+iy \in \mathfrak{h},$$

for any integer m. Define $\partial_m^\ell(f) := \partial_{m+2(\ell-1)} \circ \cdots \circ \partial_{m+2} \circ \partial_m(f)$ to be the composition of the ℓ operators $\partial_m, \partial_{m+2}, \ldots, \partial_{m+2(\ell-1)}$. Then a straightforward combinatorial argument yields the following identity, whose proof we will omit,

Lemma 4.1. *Let τ be a Heegner point of level N and discriminant D. Then*

$$\alpha(\tau) = \kappa_D \cdot \partial_{-2k+2}^{k-1} \circ f_{2k-1}(\tau),$$

where $\kappa_D = (k-1)!\,(2\pi i)^k(2\pi\sqrt{|D|})^{k-1}$ is a constant depending only on D and $2k$.

A closed formula for ∂_m^ℓ (see [Villegas and Zagier 93] for example) allows us to write α as

$$\alpha(\tau) = \kappa_D(2\pi i)\left(\frac{-y}{\pi}\right)^k \sum_{n\geq 1} p\left(\frac{2k}{2}, \frac{1}{4\pi yn}\right) a_n q^n, \tag{4.2}$$

where $p(m,x)$, is the polynomial,

$$p(m,x) = \sum_{\ell=m}^{2m-1} \binom{m-1}{2m-1-\ell} \frac{(\ell-1)!}{(m-1)!} x^\ell, \qquad m \in \mathbb{Z},\ x \in \mathbb{R}.$$

We compute $\alpha(\tau)$ using (4.2). Also notice that Lemma 4.1 perhaps provides further insight into why the map $\mathcal{H}_N \to \mathbb{C}/L$ inducing α is invariant under $\Gamma_0(N)$. Loosely speaking, this is because integrating f $(2k-1)$-times lowers its weight by $2(2k-1)$ and ∂^{k-1}_{-2k+2} increases its weight by $2(k-1)$ to get something morally of weight 0.

Given a set of Heegner point representatives of level N, discriminant D, and root r, we can use the above to compute $y_{D,r}$. Verifying the first conjecture for each D, r then amounts to choosing a complex number y_f, and an integer t, both non-zero, and showing the linear dependence,

$$\operatorname{Re}(y_{D,r}) - m_{D,r}\operatorname{Re}(y_f) + n_{D,r}\operatorname{Re}(\omega_2)/t = 0 \tag{4.3}$$

for some integers $m_{D,r}, n_{D,r}$. The second conjecture consists of comparing the coefficients $m_{D,r}$ of y_f we get above with the Jacobi form coefficients of the form ϕ_f.

5. Examples

The Fourier coefficients of the forms in these examples were computed using SAGE [Stein and Joyner 05]. The rest of the calculations were done in PARI/GP [PAR 08].

We will always take a set of generators for $\Gamma_0(N)$ which includes the translation matrix $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ but no other matrix whose $(2,1)$ entry is 0. The period integrals for T are always 0 since $i\infty$ is its fixed fixed point, hence we can exclude it from our computations of $\mathcal{P}$. In addition the $(2\pi)^k$ factor in the definitions of y_D and L is left off from the computations, since it is just a scaling factor and requires unnecessary extra precision.

For each example below, we list the number of digits of precision and the number M of terms of f we used. Below that is a set of generators we chose for $\Gamma_0^*(N)$ and the bases, ω_1, ω_2, we got for L from computing $\mathcal{P}$ and applying the lattice reduction algorithm explained in Section 4. We then provide a table listing the m_D which satisfy equation (4.3) for t, y_f of our choosing, and D less than some bound. Without getting into details, the precision we chose depended on the size of the M-th term of f and on the a priori knowledge of the size of the coefficients satisfying (4.3).

Example 5.1. $2k = 10$, $N = 3$. The space of cuspidal newforms of weight 10 and level 3 has dimension 2, but only one form has $\varepsilon = -1$. The first few terms of it are

$$f = q - 36q^2 - 81q^3 + 784q^4 - 1314q^5 + 2916q^6 - 4480q^7 - 9792q^8 + \cdots$$

Precision	60
Number of terms	100
$\Gamma_0^*(3)$	$\left\langle T, \left(\begin{smallmatrix} -1 & 1 \\ -3 & 2 \end{smallmatrix}\right), \omega_3 = \left(\begin{smallmatrix} 0 & -1 \\ 3 & 0 \end{smallmatrix}\right) \right\rangle$
ω_1	$-i \cdot 0.00088850361439085\ldots$
ω_2	$0.00002189032158611\ldots$
y_f	$y_{-8}/2$
t	1

The m_D in Table 1 give, up to scalar multiple, the coefficients of the weight 11/2 level 12 modular form found in [Eichler and Zagier 85, p. 144]. Note we can use

$\lvert D\rvert$	m_D	$\lvert D\rvert$	m_D
8	2	104	380
11	-5	107	-507
20	8	116	-40
23	8	119	-560
35	42	131	235
47	-48	143	-376
56	0	152	-364
59	-155	155	-64
68	160	164	-1440
71	40	167	1528
83	353	179	2635
95	280	191	-400

TABLE 1. $f \in S_{10}(3)$. List of D, m_D such that $y_D - m_D y_f \in L$ for $|D| < 200$.

the theorems of Waldspurger to get information about the values $L(f, D, k)$ from this table. For example, $L(f, -56, 5) = 0$.

Example 5.2. $2k = 18$, $N = 1$.

The weight 18 level 1 eigenform in $S_{18}(1)$ has the closed form

$$f(z) = \frac{-E_6^3(z) + E_4^3(z)E_6(z)}{1728},$$

where $E_{2k}(z)$ is the normalized weight $2k$ Eisenstein series.

Precision	200
Number of terms	100
$\Gamma_0^*(1) = SL_2(\mathbb{Z})$	$\left\langle T, S = \omega_1 = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \right\rangle$
ω_1	$i \cdot 0.0018318767758701917 61\ldots$
ω_2	$0.000000000519923858624\ldots$
y_f	$y_{-3}/3$
t	1

The m_D in Table 2 are identical to the coefficients of the weight 19/2 level 4 half-integer weight form in [Eichler and Zagier 85, p.141], which is in Shimura correspondence with f.

Example 5.3. $2k = 4$, $N = 13$

The dimension of the new cuspidal subspace is 3 in this case, but only one has integer coefficients in its q-expansion.

$$f = q - 5q^2 - 7q^3 + 17q^4 - 7q^5 + 35q^6 - 13q^7 - 45q^8 + 22q^9 + \cdots$$

$\lvert D\rvert$	m_D	$\lvert D\rvert$	m_D
3	1	51	108102
4	-2	52	-93704
7	-16	55	-22000
8	36	56	80784
11	99	59	-281943
15	-240	67	659651
19	-253	68	193392
20	-1800	71	-84816
23	2736	79	-109088
24	-1464	83	-22455
31	-6816	84	-484368
35	27270	87	1050768
39	-6864	88	143176
40	39880	91	195910
43	-66013	95	-370800
47	44064		

TABLE 2. $f \in S_{18}(1)$. List of D, m_D such that $y_D - m_D y_f \in L$ for $|D| < 100$.

Precision	28
Number of terms	250
$\Gamma_0^*(13)$	$\left\langle T, \left(\begin{smallmatrix} 8 & -5 \\ 13 & -8 \end{smallmatrix}\right), \left(\begin{smallmatrix} -3 & 1 \\ -13 & 4 \end{smallmatrix}\right), \left(\begin{smallmatrix} 5 & -2 \\ 13 & -5 \end{smallmatrix}\right), \left(\begin{smallmatrix} -9 & 7 \\ -13 & 10 \end{smallmatrix}\right), \omega_{13} = \left(\begin{smallmatrix} 0 & -1 \\ 13 & 0 \end{smallmatrix}\right) \right\rangle$
ω_1	$i \cdot 0.003124357726009878347400865279\ldots$
ω_2	$-0.04271662498543992056668379773\ldots$ $-i \cdot 0.001562178863004939178984383052\ldots$
y_f	$y_{-3}/3$
t	3

Notice this is the first example of a nonsquare lattice. In fact $\omega_2/\omega_1 = -0.5000\cdots + i \cdot 13.67212999\ldots$ so $\mathrm{Re}(\omega_2/\omega_1) = 1/2$ as explained earlier. This is also the first example where the choice of r matters, since $k = 2$ is not odd. For each D, we chose r in the interval $0 < r < 13$. In addition this is our only example where $t > 1$.

A closed form expression for the weight 3 index 13 Jacobi form $\phi = \phi_f$ corresponding to f was provided to us by Nils Skoruppa,

$$\phi(\tau, z) = \vartheta_1^5 \vartheta_2^3 \vartheta_3 / \eta^3$$

Here η is the usual Dedekind eta-function, $\eta = q^{1/24} \prod_{n \geq 1}(1 - q^n)$ with $q = e^{2\pi i \tau}$, and $\vartheta_a = \sum_{r \in \mathbb{Z}} \left(\frac{-4}{r}\right) q^{\frac{r^2}{8}} \zeta^{\frac{ar}{2}}$ for $a = 1, 2, 3$, $\zeta = e^{2\pi i z}$. (This has a nice product expansion using Jacobi's triple product identity.)

We verify that the (n, r)-th coefficient $c(n, r)$ in the Fourier expansion of ϕ is identically equal to the $m_{D,r}$ in Table 3 for $|D| < 200$.

$\|D\|$	$m_{D,r}$	$\|D\|$	$m_{D,r}$
3	1	107	4
4	-1	116	-8
23	2	120	-13
35	-7	127	14
40	3	131	-3
43	-17	139	29
51	9	152	2
55	-6	155	22
56	1	159	-6
68	-5	168	-21
79	4	179	-17
87	-6	183	-2
88	10	191	-10
95	4	199	4
103	-8		

TABLE 3. $f \in S_4(13)$. List of D, $m_{D,r}$ such that $ty_{D,r} - tm_{D,r}y_f \in L$ with $t = 3$, for $|D| < 200$.

6. More Examples

The coefficients of Jacobi forms are difficult to compute, in particular for the cases when N is composite or when k is even. We chose the previous examples in part because the Fourier coefficients for their Jacobi forms already existed, thanks to the work of Zagier, Eichler, and Skoruppa mentioned above. However, given any weight and level, we can still provide convincing evidence for our conjecture without knowing the exact coefficients of its Jacobi form. This is done using a refinement of Waldspurger [Waldspurger 81] given in [Gross et al. 87, p.527].

Specifically, let $f \in S_{2k}(N)$ be a normalized newform with $\varepsilon = -1$. Let $\phi = \phi_f \in J_{k+1,N}$, with Fourier coefficients denoted by $c(n,r)$, be the Jacobi form corresponding to f as described in Section 3. For a fundamental discriminant D with $\gcd(D,N) = 1$ and square root r modulo $4N$, [Gross et al. 87, Corollary 1] says

$$|D|^{k-1/2}L(f,D,k) \doteq |c(n,r)|^2;$$

here $L(f,D,s)$ is L-series of f twisted by D, and $n \in \mathbb{Z}$ satisfies $D = r^2 - 4Nn$. By $\doteq$ we mean equality up to a nonzero factor depending on $N, 2k, f$, and ϕ, but independent of D. (Gross-Kohnen-Zagier give this constant explicitly in their paper, but for us it is unnecessary.)

Thus given two such discriminants $D_i = r_i^2 - 4Nn_i$, $i = 1,2$, we have

$$\frac{|D_1|^{k-1/2}L(f,D_1,k)}{|D_2|^{k-1/2}L(f,D_2,k)} = \frac{|c(n_1,r_1)|^2}{|c(n_2,r_2)|^2}.$$

Hence by computing central values of twisted L-functions of f, we can test if ratios of squares of our m_{D_i,r_i} are equal to those of $c(n_i,r_i)$.

For the examples below we have the same format as the previous examples along with a fixed choice of discriminant D_1 for which we verified explicitly,

$$\frac{|D_1|^{k-1/2}L(f,D_1,k)}{|D|^{k-1/2}L(f,D,k)} = \frac{m_{D_1,r}^2}{m_{D,r}^2}$$

for all D coprime to N less than a certain bound.

Example 6.1. $2k = 4$, $N = 21$.

The dimension of the new cuspidal subspace of $S_4(21)$ is 4. We chose

$$f = q - 3q^2 - 3q^3 + q^4 - 18q^5 + 9q^6 + 7q^7 + \cdots .$$

Precision	40
Number of terms	500
$\Gamma_0^*(21)$	$\left\langle T, \left(\begin{smallmatrix} -4 & 1 \\ -21 & 5 \end{smallmatrix}\right), \left(\begin{smallmatrix} 11 & -5 \\ 42 & -19 \end{smallmatrix}\right), \left(\begin{smallmatrix} 13 & -9 \\ 42 & -29 \end{smallmatrix}\right), \left(\begin{smallmatrix} 8 & -5 \\ 21 & -13 \end{smallmatrix}\right), \left(\begin{smallmatrix} 26 & -19 \\ 63 & -46 \end{smallmatrix}\right), \left(\begin{smallmatrix} -16 & 13 \\ -21 & 17 \end{smallmatrix}\right) \right\rangle$
ω_1	$i \cdot 0.012130626847574141\ldots$
ω_2	$-0.03257318919429172\ldots$
y_f	y_{-3}
t	1
D_1	-20

For a consistent choice of each r we chose the first positive residue modulo $2N$ which satisfies $D \equiv r^2 \bmod 4N$ for each D.

$\|D\|$	$m_{D,r}$	$\|D\|$	$m_{D,r}$
3	1	111	4
20	-1	119	0
24	-1	131	3
35	0	132	8
47	2	143	2
56	0	152	-7
59	1	159	0
68	-2	164	-2
83	5	167	4
84	0	168	0
87	-4	195	8
104	-3		

TABLE 4. $f \in S_4(21)$. List of D, $m_{D,r}$ such that $y_{D,r} - m_{D,r} y_f \in L$ for $|D| < 200$.

Example 6.2. $2k = 12$, $N = 4$.

The space of new cuspforms in $S_{12}(4)$ is spanned by one normalized newform whose Fourier series begins with,

$$f = q - 516q^3 - 10530q^5 + 49304q^7 + 89109q^9 - 309420q^11 + \cdots .$$

Precision	80
Number of terms	200
$\Gamma_0^*(4)$	$\langle T, \left(\begin{smallmatrix} 1 & -1 \\ 4 & -3 \end{smallmatrix}\right) \rangle$
ω_1	$i \cdot 0.000960627675025996\ldots$
ω_2	$-0.02998129737318938\ldots$
y_f	y_{-7}
t	1
D_1	-7

Similar to the last example, we chose the first positive residue modulo $2N$ which satisfies $D \equiv r^2 \bmod 4N$ for each D.

$\lvert D\rvert$	$m_{D,r}$	$\lvert D\rvert$	$m_{D,r}$
7	1	103	1649
15	5	111	-765
23	-3	119	-90
31	-50	127	2664
39	-35	143	-3729
47	186	151	-505
55	215	159	-2825
71	-315	167	3819
79	-10	183	2539
87	-497	191	1830
95	405	199	-5755

TABLE 5. $f \in S_{12}(4)$. List of D, $m_{D,r}$ such that $y_{D,r} - m_{D,r} y_f \in L$ for $|D| < 200$.

ACKNOWLEDGMENTS

I am deeply grateful to my advisor, Fernando Rodriguez Villegas, for his continuing guidance and support and for sharing his ideas that have enriched this work. I would also like to thank Don Zagier, Winfried Kohnen, and Chad Schoen for their ideas and suggestions. Thanks to Jeffrey Stopple for his careful reading of the manuscript. The coefficients of modular forms were computed with the help of William Stein. This research was partially funded by the Donald D. Harrington Endowment Fellowship.

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