

CYCLE INTEGRALS OF THE J-FUNCTION AND MOCK MODULAR FORMS

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ABSTRACT. In this paper we construct certain mock modular forms of weight $1/2$ whose Fourier coefficients are cycle integrals of the modular j -function and whose shadows are weakly holomorphic forms of weight $3/2$. As an application we construct through a Shimura-type lift a holomorphic function that transforms with a rational period function having poles at certain real quadratic integers. This function yields a real quadratic analogue of the Borcherds product.

1. INTRODUCTION

An interesting new class of modular forms has emerged in the last several years. They generalize Ramanujan's mock theta functions and are thus called the mock modular forms. The generalization is based on an observation of Zwegers [51, 52], who established a pairing between mock theta and actual theta functions. A prime example is provided by the mock-theta function

$$f(\tau) = \sum_{n \geq 0} \frac{q^n}{(1+q)^2 \cdots (1+q^n)^2}$$

and the weight $3/2$ unary theta series

$$g(\tau) = \sum_{n \equiv 1 \pmod{6}} n q^{n^2/24}.$$

They are paired by a non-holomorphic modular form $h(\tau)$ of weight $1/2$ whose Fourier expansion has holomorphic part $q^{-1/24} f(\tau)$ and that satisfies the differential equation

$$\frac{\partial}{\partial \bar{\tau}} h(\tau) = c y^{-1/2} \overline{g(\tau)}$$

for a certain non-zero constant c . Here as usual $q = e(\tau) = e^{2\pi i \tau}$ and $\tau = x + iy \in \mathcal{H}$, the upper half-plane. Such pairings provide us with a much better understanding of the position of mock theta functions in the modular universe and have led to a number of new results about them. They also motivate the study of other kinds of mock modular forms f of weight k that are paired with holomorphic modular forms g of weight $2 - k$ in a similar way (see [38] and [50] for surveys on some of these developments).

Apparently unrelated to this is the work of Borcherds and Zagier on weakly holomorphic forms of half-integral weight and traces of singular moduli of the classical j -function. A special value of $j(\tau)$ when τ is an imaginary quadratic irrational is called a singular modulus and is well-known to be an algebraic integer. Here j is defined by

$$j(\tau) = \frac{E_4(\tau)^3}{\Delta(\tau)} = q^{-1} + 744 + 196884q + \cdots,$$

where E_4 and Δ are the usual modular forms for $\Gamma = \text{PSL}(2, \mathbb{Z})$. Let

$$\theta(\tau) = 1 + 2q + 2q^4 + 2q^9 + 2q^{16} + \cdots$$

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be the weight $1/2$ theta series and consider the weight $3/2$ weakly holomorphic form defined by

$$(1.1) \quad g_1(\tau) = \theta\left(\tau + \frac{1}{2}\right) \frac{E_4(4\tau)}{\Delta(4\tau)^{1/4}} = q^{-1} - 2 + 248q^2 - 492q^4 + 4119q^7 - 7256q^8 + \dots$$

Zagier [49] gave the remarkable result that the Fourier coefficients $b(n)$ of g are given in terms of traces of singular moduli of $j_1(\tau) = j(\tau) - 744$:

$$b(n) = - \sum_{Q \in \Gamma \backslash \mathcal{Q}_{-n}} w_Q^{-1} j_1\left(\frac{-b + \sqrt{-n}}{2a}\right).$$

Here for any discriminant d , that is a nonzero integer $d \equiv 0, 1 \pmod{4}$, we let \mathcal{Q}_d be the set of integral binary quadratic forms $Q(x, y) = ax^2 + bxy + cy^2$ of discriminant $d = b^2 - 4ac$ that are positive definite if $d < 0$ and are acted on as usual by Γ , resulting in finitely many classes $\Gamma \backslash \mathcal{Q}_d$. Also $w_Q = 1$ unless $Q \sim a(x^2 + y^2)$, when $w_Q = 2$, or $Q \sim a(x^2 + xy + y^2)$, when $w_Q = 3$. It follows from the main result of this paper (Theorem 2) that there is a mock modular form of weight $1/2$

$$(1.2) \quad f_1(\tau) = \sum_{\substack{n > 0 \\ n \equiv 0, 1 \pmod{4}}} a(n)q^n$$

that is paired with Zagier's function $g_1(\tau)$ and whose non-square Fourier coefficients are cycle integrals of j_1 associated to positive discriminants n :

$$(1.3) \quad a(n) = a(n, 1) = \sum_{Q \in \Gamma \backslash \mathcal{Q}_n} \int_{C_Q} j_1(\tau) \frac{d\tau}{Q(\tau, 1)}.$$

Here C_Q is any smooth curve from any $z \in \mathcal{H}$ to $g_Q z$ where g_Q is a certain distinguished generator of the infinite cyclic group of automorphs of Q defined below in (2.10). The pairing is furnished by a non-holomorphic modular form h of weight $1/2$ whose Fourier expansion has holomorphic part given by $f_1(\tau)$ from (1.2) and that satisfies

$$\frac{\partial}{\partial \bar{\tau}} h(\tau) = cy^{-\frac{1}{2}} \overline{g_1(\tau)},$$

for a certain non-zero constant c .

It follows from Theorem 1 that f_1 is part of a natural basis $\{f_d\}_{d \equiv 0, 1 \pmod{4}}$ for the space $\mathbb{M}_{1/2}$ of all mock modular forms of weight $1/2$ for $\Gamma_0(4)$ whose Fourier coefficients are supported on $n \equiv 0, 1 \pmod{4}$ and that are paired with weakly holomorphic forms of weight $3/2$ whose Fourier coefficients are supported on $n \equiv 0, 3 \pmod{4}$. The subset $\{f_d\}_{d \leq 0}$ coincides with the basis given by Borcherds [2] for the subspace comprising weakly holomorphic modular forms, which are mock modular being paired with 0. For each $d \leq 0$ the function f_d is uniquely determined by having a q -expansion of the form

$$f_d(\tau) = q^d + \sum_{n > 0} a(n, d)q^n.$$

Thus $f_0(\tau) = \theta(\tau)$ and the first few terms of the next function are

$$f_{-3}(\tau) = q^{-3} - 248q + 26752q^4 - 85995q^5 + 1707264q^8 + \dots$$

Zagier showed that g_1 from (1.1) is the first member of a basis $\{g_d\}_{0 < d \equiv 0, 1 \pmod{4}}$ for the space of all weakly holomorphic forms of weight $3/2$ for $\Gamma_0(4)$ whose Fourier coefficients are supported on integers $n \equiv 0, 3 \pmod{4}$. The Fourier expansion of g_d is "dual" to that of f_d , namely

$$(1.4) \quad g_d(\tau) = q^{-d} - \sum_{n \leq 0} a(d, n)q^{|n|}.$$

In order to obtain the desired basis $\{f_d\}_{d \equiv 0,1 \pmod{4}}$ for the space $\mathbb{M}_{1/2}$ of mock modular forms, for each positive $d \equiv 0, 1 \pmod{4}$ we construct a $f_d \in \mathbb{M}_{1/2}$ that is paired with g_d . This f_d has a Fourier expansion of the form

$$f_d(\tau) = \sum_{n>0} a(n, d) q^n$$

with coefficients $a(n, d)$ that satisfy the symmetry relation

$$\sqrt{n} a(n, d) = \sqrt{d} a(d, n)$$

and that are, as in the case $d = 1$, related to cycle integrals of modular functions. It is important to observe that mock modular forms paired with forms having poles at $i\infty$ have largely been ignored until now.

As we will show in Theorem 3, there is also an interesting mock modular form of weight $1/2$ that is *not quite* in $\mathbb{M}_{1/2}$ with Fourier expansion

$$(1.5) \quad f^+(\tau) = \sum_{\substack{n>0 \\ n \equiv 0,1 \pmod{4}}} n^{-\frac{1}{2}} H(n) q^n$$

where, for n a fundamental discriminant,

$$H(n) = 2h(n) \log \epsilon_n.$$

Here $h(n)$ is the narrow class number of $\mathbb{Q}(\sqrt{n})$ and ϵ_n is its smallest unit > 1 of norm 1. Now $f^+(\tau)$ is paired with a harmonic (but not holomorphic!) form of weight $3/2$ whose Fourier expansion has holomorphic part

$$(1.6) \quad f^-(\tau) = \sum_{n \leq 0} H(n) q^{|n|},$$

where $H(n)$ for $n \leq 0$ is the usual Hurwitz class number. This $f^-(\tau)$ is itself mock modular of weight $3/2$, being paired with $\theta(\tau)$, a fact discovered by Zagier in 1975 [47]. This example gives some reason to hope that there might be a connection between the cycle integrals of j and abelian extensions of real quadratic fields. So far this hope has not been realized.

There is also an unexpected connection between mock modular forms of weight $1/2$ and modular integrals having rational period functions. Define for each $d \equiv 0, 1 \pmod{4}$

$$(1.7) \quad F_d(\tau) = -H(d) - \sum_{m \geq 1} \left(\sum_{n|m} n a(n^2, d) \right) q^m.$$

Note that F_d is the derivative of the formal Shimura lift of f_d . When $d < 0$ Borcherds showed that F_d is a meromorphic modular form of weight 2 for Γ having a simple pole with residue w_Q^{-1} at each point

$$\tau_Q = \frac{-b + \sqrt{d}}{2a} \in \mathcal{H}$$

of discriminant d . From this he deduced corresponding properties of the infinite product

$$q^{-H(d)} \prod_{m \geq 1} (1 - q^m)^{a(m^2, d)}.$$

In case $d = 0$ one finds since $H(0) = -1/12$ that this product is $\Delta(\tau)^{1/12}$, and we have that

$$F_0(\tau) = \frac{1}{12} - 2 \sum_{n \geq 1} \sigma(n) q^n = \frac{1}{12} E_2(\tau).$$

This is a holomorphic modular integral of weight 2 with a rational period function:

$$F_0(\tau) - \tau^{-2}F_0(-\frac{1}{\tau}) = -\frac{1}{2\pi i} \tau^{-1}.$$

As we will show in Theorem 4, for each $d > 0$ not a square the function F_d is a modular integral of weight 2 with a quadratic rational period function:

$$(1.8) \quad F_d(\tau) - \tau^{-2}F_d(-\frac{1}{\tau}) = 2\sqrt{d} \sum_{\substack{c < 0 < a \\ b^2 - 4ac = d}} (a\tau^2 + b\tau + c)^{-1}.$$

Note that the period function has simple poles at certain real quadratic integers of discriminant d , in analogy to the behavior of F_d when $d < 0$. Such rational period functions and their integrals have been intensively studied after being introduced by Knopp (see [8] for references). At the end of their paper [8], Choie and Zagier raised the problem of explicit construction of a modular integral with a given rational period function. This question was answered in a special case by Parson in [39] using partial hyperbolic Poincare series. Our results show that F_d provides another construction of such a modular integral and gives a real quadratic analogue of (the logarithmic derivative of) the Borcherds product at the same time.

2. STATEMENT OF RESULTS

First we recall some basic facts and notations about weakly holomorphic modular forms of half-integral weight. Set

$$(2.1) \quad j(\gamma, \tau) = \theta(\gamma\tau)/\theta(\tau) \quad \text{for } \gamma \in \Gamma_0(4).$$

For $k \in 1/2 + \mathbb{Z}$ say that f defined on \mathcal{H} has weight k for $\Gamma_0(4)$ (or simply has weight k) if

$$f(\gamma\tau) = j(\gamma, \tau)^{2k} f(\tau) \quad \text{for all } \gamma \in \Gamma_0(4).$$

Let $M_k^!$ be the “plus”-space of weakly holomorphic modular forms of weight k for $\Gamma_0(4)$, comprising functions holomorphic on \mathcal{H} of weight k whose Fourier coefficients $a(n)$ in the expansion $f(\tau) = \sum_n a(n)q^n$ are supported on integers $n \geq n_\infty$ with $(-1)^{k-1/2}n \equiv 0, 1 \pmod{4}$ for some $n_\infty \in \mathbb{Z}$. An easy extension of arguments given in [31, Prop. 2] and [33, p.190] shows that such an f satisfies

$$(-2i\tau)^{-k} f(-\frac{1}{4\tau}) = \alpha \sum_{\substack{n_\infty \leq n \\ n \text{ even}}} a(n)q^{n/4} \quad \text{and} \quad (-2i\tau)^{-k} f(-\frac{1}{4\tau} + \frac{1}{2}) = \alpha \sum_{\substack{n_\infty \leq n \\ n \text{ odd}}} a(n)q^{n/4},$$

where $\alpha = \alpha_k \neq 0$, so that f has at most poles in the three inequivalent cusps of $\Gamma_0(4)$: ∞ , 0 and $1/2$. Let $M_k^+ \subset M_k^!$ denote the subspace of holomorphic forms (having $n_\infty = 0$) and $S_k^+ \subset M_k^+$ the subspace of cusp forms (having $n_\infty = 1$).

In order to pair mock modular forms with ordinary modular forms we now introduce weakly harmonic forms. Let $\Delta_{1/2}$ be the weight $1/2$ Laplacian on \mathcal{H} given by

$$(2.2) \quad \Delta_{1/2} = -y^2(\partial_x^2 + \partial_y^2) + \frac{iy}{2}(\partial_x + i\partial_y) \quad (\tau = x + iy).$$

Suppose that f is a real analytic function on \mathcal{H} of weight $1/2$ for $\Gamma_0(4)$ that is harmonic on \mathcal{H} in the sense that

$$\Delta_{1/2}f = 0.$$

By separation of variables every such f has a (unique) Fourier expansion in the cusp at ∞ of the form

$$(2.3) \quad f(\tau) = \sum_n b(n)\mathcal{M}_n(y)e(nx) + \sum_n a(n)\mathcal{W}_n(y)e(nx).$$

The functions $\mathcal{W}_n(y)$ and $\mathcal{M}_n(y)$ in the Fourier expansion (2.3) are defined in terms of the usual Whittaker functions $W(y) = \pi^{-\frac{1}{2}}y^{-\frac{1}{4}}W_{-\frac{1}{4}, \frac{1}{4}}(y)$ and $M(y) = \pi^{-\frac{1}{2}}y^{-\frac{1}{4}}M_{\frac{1}{4}, \frac{1}{4}}(y)$ by

$$(2.4) \quad \mathcal{W}_n(y) = \begin{cases} |n|^{-\frac{1}{2}}W(4\pi|n|y) & \text{if } n < 0, \\ -4y^{\frac{1}{2}} & \text{if } n = 0, \\ n^{-\frac{1}{2}}e^{-2\pi ny} & \text{if } n > 0, \end{cases}$$

$$(2.5) \quad \mathcal{M}_n(y) = \begin{cases} e^{2\pi|n|y} - W(4\pi|n|y) & \text{if } n < 0, \\ 1 & \text{if } n = 0, \\ 2M(4\pi|n|y) & \text{if } n > 0. \end{cases}$$

See the appendix for basic information on the Whittaker functions. A good reference for details is [35]. Note that when $n \neq 0$ the function $\mathcal{W}_n(y)$ is exponentially decaying while $\mathcal{M}_n(y)$ is exponentially growing in y (see (A.4)). It is also easily seen that the Fourier expansion (2.3) has a uniquely determined holomorphic part of the form $\sum_n c(n)q^n$.

Let $H_{1/2}^!$ denote the space of all such f whose Fourier coefficients at ∞ are supported on integers n with $n \equiv 0, 1 \pmod{4}$ and that have only finitely many non-zero coefficients $b(n)$. As before this is enough to control bad behavior in the other cusps. We will call such an $f \in H_{1/2}^!$ weakly harmonic.¹ Our first result gives a natural basis for $H_{1/2}^!$.

Theorem 1. *For each $d \equiv 0, 1 \pmod{4}$ there is a unique $h_d \in H_{1/2}^!$ with Fourier expansion of the form*

$$(2.6) \quad h_d(\tau) = \mathcal{M}_d(y)e(dx) + \sum_{n \equiv 0, 1 \pmod{4}} a_d(n)\mathcal{W}_n(y)e(nx).$$

The set $\{h_d\}_{d \equiv 0, 1 \pmod{4}}$ forms a basis for $H_{1/2}^!$. The coefficients $a_d(n)$ satisfy the symmetry relation

$$(2.7) \quad a_d(n) = a_n(d)$$

for all integers $n, d \equiv 0, 1 \pmod{4}$. When $d > 0$ we have

$$(2.8) \quad \frac{\partial}{\partial \bar{\tau}} h_d(\tau) = id^{\frac{1}{2}}y^{-\frac{1}{2}}\overline{g_d(\tau)},$$

where $g_d \in M_{3/2}^!$ has Fourier expansion given in (1.4).

Recall that the members of Borcherds basis for $M_{1/2}^!$ are given for $d \leq 0$ with $d \equiv 0, 1 \pmod{4}$ by

$$(2.9) \quad f_d(\tau) = q^d + \sum_{n > 0} a(n, d)q^n.$$

In fact, $M_{1/2}^!$ is a subspace of $H_{1/2}^!$ and Theorem 1 provides an extension of the Borcherds basis to all of $H_{1/2}^!$. We will see that for $d \leq 0$ we have that $h_d = f_d$ from (2.9) and $a(n, d) = n^{-1/2}a_d(n)$ unless $n = d < 0$, in which case $a_d(d) = |d|^{1/2}$. If $d > 0$ let $f_d(\tau) = \sum_{n > 0} a(n, d)q^n$ be the holomorphic part of the Fourier expansion of $2\pi h_d$. Then $a(n, d) = 2\pi n^{-1/2}a_d(n)$. It follows from Theorem 1 that $\{f_d\}_{d \equiv 0, 1 \pmod{4}}$ is a basis for the space $\mathbb{M}_{1/2}$ of all mock modular forms of weight $1/2$ for $\Gamma_0(4)$ that are paired with forms in $M_{3/2}^!$ and whose Fourier coefficients are supported on $n \equiv 0, 1 \pmod{4}$.

In order to state our main result recall that \mathcal{Q}_d is the set of integral binary quadratic forms $Q(x, y) = ax^2 + bxy + cy^2 = [a, b, c]$ of discriminant $d = b^2 - 4ac$ that are positive definite if

¹The definition of harmonic weak Maass forms given in [7] and elsewhere does not apply to the non-holomorphic $f \in H_{1/2}^!$, so we use the terminology *weakly harmonic* to avoid confusion.

$d < 0$. Let $\mathcal{Q}_d^+ \subset \mathcal{Q}_d$ be those forms with $a > 0$, so that $\mathcal{Q}_d = \mathcal{Q}_d^+$ when $d < 0$. Let $Q \mapsto gQ$ be the usual action of Γ that is compatible with linear fractional action on the roots of $Q(\tau, 1) = 0$. Explicitly,

$$(gQ)(x, y) = Q(\delta x - \beta y, -\gamma x + \alpha y), \quad \text{where } g = \pm \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma.$$

As is well known, the resulting set of classes $\Gamma \backslash \mathcal{Q}_d$ is finite and those classes consisting of primitive forms make up an abelian group under composition. Let $\Gamma_Q = \{g \in \Gamma; gQ = Q\}$ be the group of automorphs of Q . If $d < 0$ then $w_Q = \#\Gamma_Q = 1$ unless $Q \sim a(x^2 + y^2)$ or $Q \sim a(x^2 + xy + y^2)$, in which case $w_Q = 2$ or 3 , respectively. If $d > 0$ is not a square then Γ_Q is infinite cyclic with a distinguished generator denoted by g_Q , which for primitive Q is given by

$$(2.10) \quad g_Q = \pm \begin{pmatrix} \frac{t+bu}{2} & cu \\ -au & \frac{t-bu}{2} \end{pmatrix}$$

where t, u are the smallest positive integral solutions of $t^2 - du^2 = 4$. If $\delta = \gcd(a, b, c)$ then $g_Q = g_{Q/\delta}$.

Suppose that D is a fundamental discriminant and that d is a discriminant. For $Q = [a, b, c]$ with discriminant dD let

$$(2.11) \quad \chi(Q) = \chi_D(Q) = \begin{cases} \left(\frac{D}{r}\right) & \text{if } (a, b, c, D) = 1 \text{ where } Q \text{ represents } r \text{ and } (r, D) = 1, \\ 0, & \text{if } (a, b, c, D) > 1. \end{cases}$$

It is well-known that χ is well-defined on classes $\Gamma \backslash \mathcal{Q}_{dD}$, that χ restricts to a real character (a genus character) on the group of primitive classes and that all such characters arise this way. We have the identity

$$(2.12) \quad \chi_D(-Q) = (\text{sgn } D)\chi_D(Q).$$

If d is also fundamental we have that $\chi_D = \chi_d$ on $\Gamma \backslash \mathcal{Q}_{dD}$. A good reference for the basic theory of these characters is [19, p. 508].

It is also well-known and easily shown that the space of all weakly holomorphic modular functions of weight 0 for Γ , namely $\mathbb{C}[j]$, has a unique basis $\{j_m\}_{m \geq 0}$ of the form

$$(2.13) \quad j_m(\tau) = q^{-m} + \sum_{n \geq 1} c_m(n)q^n.$$

For instance, j_m can be obtained from j_1 by applying the m -th Hecke operator. Write

$$d\tau_Q = \frac{\sqrt{d} d\tau}{Q(\tau, 1)}$$

and define the general twisted trace for dD not a square by

$$\text{Tr}_{d,D}(j_m) = \begin{cases} \sum \chi(Q)w_Q^{-1}j_m(\tau_Q), & \text{if } dD < 0; \\ \frac{1}{2\pi} \sum \chi(Q) \int_{C_Q} j_m(\tau) d\tau_Q & \text{if } dD > 0, \end{cases}$$

each sum being over $Q \in \Gamma \backslash \mathcal{Q}_{dD}$. Our main result is the following evaluation.

Theorem 2. *Suppose that $m \geq 1$. For $d \equiv 0, 1 \pmod{4}$ and fundamental D with dD not a square we have*

$$(2.14) \quad \text{Tr}_{d,D}(j_m) = \sum_{n|m} \left(\frac{D}{m/n}\right) a_d(n^2D).$$

In particular, for non-square dD this gives

$$(2.15) \quad a_d(D) = a_D(d) = \begin{cases} \sum \chi(Q) w_Q^{-1} j_1(\tau_Q), & \text{if } dD < 0; \\ \frac{1}{2\pi} \sum \chi(Q) \int_{C_Q} j_1(\tau) d\tau_Q, & \text{if } dD > 0, \end{cases}$$

where each sum is over $Q \in \Gamma \backslash \mathcal{Q}_{dD}$.

Theorem 2 generalizes to the case $dD > 0$ known results of Borcherds [2] and Zagier [49]. We remark that Theorem 2 holds trivially when $d, D < 0$ since in this case both sides of (2.14) vanish, as can be seen using (2.12) and the fact that $h_d = f_d$ is weakly holomorphic for $d < 0$. Also, the factor $(2\pi)^{-1}$ in the definition of $\text{Tr}_{d,D}(j_m)$ for $dD > 0$ occurs due to our choice of normalization in the basis elements, which made possible the complete symmetry of the Fourier coefficients $a_d(n)$. This factor can be removed by choosing a different normalization, as we have done in defining the mock modular forms f_d for $d > 0$. We also remark that for fundamental $d \neq D$ we have the identity

$$(2.16) \quad \text{Tr}_{d,D}(j_m) = \sum_{n|m} \left(\frac{d}{m/n}\right) \text{Tr}_{n^2d,D}(j_1).$$

This result is a consequence of a general identity on Hecke operators proved in [48]. It also follows by applying (2.14) to the identity $\text{Tr}_{d,D}(j_m) = \text{Tr}_{D,d}(j_m)$, then using the symmetry relation (2.7) and (2.15).

It is also possible to interpret the traces when $m = 0$ in terms of Fourier coefficients, but for this we must enlarge our space of weight $1/2$ forms further to include solutions v to the inhomogeneous equation

$$(2.17) \quad \Delta_{1/2}(v) = -\frac{1}{4}\theta.$$

Define for $y > 0$ the special functions

$$\alpha(y) = \frac{y^{1/2}}{2\pi^{1/2}} \int_0^\infty t^{-1/2} \log(1+t) e^{-yt} dt \quad \text{and} \quad \beta(y) = \frac{1}{16\pi} \int_1^\infty t^{-3/2} e^{-yt} dt.$$

As was shown in [47] and [20], the function

$$g_0(\tau) = \sum_{n \leq 0} H(n) q^{|n|} + y^{-1/2} \sum_{n \in \mathbb{Z}} \beta(4\pi n^2 y) q^{-n^2}$$

has weight $3/2$ for $\Gamma_0(4)$, where again $H(n)$ is the Hurwitz class number defined for $n \leq 0$. Since

$$(2.18) \quad \frac{\partial}{\partial \bar{\tau}} g_0(\tau) = -\frac{i}{8\pi} y^{-3/2} \overline{\theta(\tau)}$$

we see that g_0 pairs f^- from (1.6) with θ . The next result provides the pairing between the mock modular form f^+ from (1.5) and g_0 .

Theorem 3. *There is a function of weight $1/2$ for $\Gamma_0(4)$ that solves (2.17) of the form*

$$(2.19) \quad v(\tau) = \sum_{0 < n \equiv 0, 1(4)} n^{-\frac{1}{2}} H(n) q^n + 2\pi \sum_{n \leq 0} H(n) \mathcal{W}_n(y) e(nx) + \sum_{n \neq 0} \alpha(4\pi n^2 y) q^{n^2} - \frac{1}{2} \log y$$

where $H(n) = 2\pi \text{Tr}_n(1)$ for positive $n \equiv 0, 1(\text{mod } 4)$ not a square. It also satisfies

$$(2.20) \quad \frac{\partial}{\partial \bar{\tau}} v(\tau) = -2\pi i y^{-1/2} \overline{g_0(\tau)}.$$

The value of $H(n)$ for square $n > 0$ can also be explicitly computed, but it is rather complicated in general. The construction of v is achieved using a limit formula of Kronecker's type. We remark that v occurred implicitly in the paper [11].

It follows from Theorems 2 and 3 that the function F_d defined in (1.7) has for positive non-square d the elegant expansion

$$F_d(\tau) = -2\pi \sum_{m \geq 0} \text{Tr}_d(j_m) q^m,$$

where we write $\text{Tr}_d(j_m) = \text{Tr}_{d,1}(j_m)$. We will see that this series converges in \mathcal{H} , in contrast to when $d < 0$, where F_d has poles at CM points.

Theorem 4. *For each $d > 0$ not a square the function F_d defined in (1.7) is a holomorphic modular integral of weight 2 with rational period function:*

$$(2.21) \quad F_d(\tau) - \tau^{-2} F_d\left(-\frac{1}{\tau}\right) = 2\sqrt{d} \sum_{\substack{c < 0 < a \\ b^2 - 4ac = d}} (a\tau^2 + b\tau + c)^{-1}.$$

Our proof of the functional equation is quite elementary and uses only basic contour shifting techniques. The existence of a holomorphic F satisfying (2.21) with growth conditions was proved by Knopp [28], [29]. He used a certain Poincaré series built out of cocycles, which was used earlier by Eichler [13], to construct it. This Poincaré series has poles but Knopp showed how to get rid of them. However, it appears to be very difficult to compute F explicitly from this construction. Parson [39] gave a more direct construction in weights > 2 using partial versions of certain partial hyperbolic Poincaré series studied by Zagier, but these do not converge when $k = 2$. Moreover, non-trivial complications seem to arise in the application of Hecke's "convergence trick" since the series giving their Fourier coefficients do not converge absolutely. In higher weights it is however clear from their Fourier expansions that they can be used in conjunction with Proposition 7 to get certain generalizations of our results.

It is interesting to examine numerical values of the traces $\text{Tr}_d(j_m)$. For convenience we list some values when $d < 0$ in Table 1 and some approximate values when $d > 0$ in Table 2 below. As a numerical check, the reader can verify that the identity (2.16) holds for $d = 5$, $D = 1$ and $m = 2$. For $d = 20$ there are two classes, represented by $[1, 4, -1]$ and the non-primitive form $[2, 2, -2]$, and (2.16) amounts to the curious identity

$$(2.22) \quad \int_{C_{[1,4,-1]}} j_1(\tau) \frac{2d\tau}{\tau^2 + 4\tau - 1} = \int_{C_{[1,1,-1]}} j_2(\tau) \frac{d\tau}{\tau^2 + \tau - 1}.$$

In general, a comparison shows that the traces for positive discriminants are generally much smaller than those for negative discriminants and appear to exhibit some regular growth behavior in both d and m . For non-square $d > 1$ and $m \in \mathbb{Z}^+$ we will see that we have the conditionally (and slowly) convergent expansion

$$(2.23) \quad \text{Tr}_d(j_m) = -24\sigma_1(m)\text{Tr}_d(1) + \sum_{0 < c \equiv 0(4)} S_m(d; c) \sin\left(\frac{4\pi m\sqrt{d}}{c}\right),$$

where

$$S_m(d; c) = \sum_{b^2 \equiv d \pmod{c}} e\left(\frac{2mb}{c}\right).$$

It is natural to expect that the first term dominates as either m or d gets large, which would in particular imply that as $d \rightarrow \infty$ through non-squares we have the asymptotic formula

$$(2.24) \quad \text{Tr}_d(j_m) \sim -24\sigma_1(m)\text{Tr}_d(1).$$

Such a result is known for negative discriminants (see [9]) except that then there is a large main term that grows like $e^{\pi m \sqrt{|d|}}$. If true, this asymptotic would indicate that there is tremendous

cancellation in the integrals of j over the cycles when d is large, since $\text{Tr}_d(1) \ll_\epsilon d^{\frac{1}{2}+\epsilon}$ and j_m has exponential growth in the cusp.

After seeing our paper, Y. Manin and D. Zagier told us that M. Kaneko also performed calculations of cycle integrals of the j function, being motivated by the “real multiplication programme” of Manin. Kaneko kindly sent us his paper [26], in which he calculated the individual cycle integrals numerically and observed some interesting behaviour in the distribution of their values. This behavior appears to us to be consistent with the conjectured asymptotic given above.

Finally, we remark that there is an obvious similarity between Theorem 2 and the formula of Katok and Sarnak [27] for the traces of a Maass cusp form. The main difference is that there are no Hecke eigenforms in our setting. Nevertheless, the method we use to prove Theorem 2, which involves Poincaré series and identities between Kloosterman sums, can be applied to give another proof of the Katok-Sarnak formula. It would be interesting to approach our results using a regularized theta lift. (See [16], [5] for the use of regularized theta lifts in the negative discriminants case.) Another interesting problem is to understand the nature of the traces on square discriminants.

TABLE 1. Traces for $d < 0$

m	$\text{Tr}_{-3}(j_m)$	$\text{Tr}_{-4}(j_m)$	$\text{Tr}_{-7}(j_m)$	$\text{Tr}_{-8}(j_m)$
0	1/3	1/2	1	1
1	-248	492	-4119	7256
2	53256	287244	16572393	52255768
3	-12288992	153540528	-67515202851	377674781024

TABLE 2. Traces for $d > 0$

m	$\text{Tr}_5(j_m)$	$\text{Tr}_8(j_m)$	$\text{Tr}_{12}(j_m)$	$\text{Tr}_{13}(j_m)$	$\text{Tr}_{17}(j_m)$	$\text{Tr}_{20}(j_m)$
0	0.30634	0.56109	0.83840	0.76060	1.33354	1.48014
1	-11.54175	-19.13749	-28.67973	-23.40950	-43.94521	-37.39837
2	-25.85662	-50.69769	-74.46436	-69.09778	-117.37778	-107.02498
3	-32.00316	-55.08485	-86.06819	-79.31283	-140.89930	-132.79098

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3. EXPONENTIAL SUMS

A crucial ingredient in what follows is an identity connecting the weight $1/2$ Kloosterman sum with a certain exponential sum taken over solutions to a quadratic congruence, a quadratic Weyl sum. In a special case this identity is due to Salié and variants have found many applications in the theory of modular forms. We shall use a general version due essentially to Kohnen [32]. To define the weight $1/2$ Kloosterman sum we need an explicit formula for the theta multiplier in $j(\gamma, \tau)$, which was defined in (2.1). This may be found in [41, p. 447]. As usual, for non-zero $z \in \mathbb{C}$ and $v \in \mathbb{R}$ we define $z^v = |z|^v \exp(iv \arg z)$ with $\arg z \in (-\pi, \pi]$. We have

$$j(\gamma, \tau) = (c\tau + a)^{1/2} \varepsilon_a^{-1} \left(\frac{c}{a} \right) \quad \text{for } \gamma = \pm \begin{pmatrix} * & * \\ c & a \end{pmatrix} \in \Gamma_0(4),$$

where $\left(\frac{c}{a}\right)$ is the extended Kronecker symbol and

$$\varepsilon_a = \begin{cases} 1 & \text{if } a \equiv 1 \pmod{4} \\ i & \text{if } a \equiv 3 \pmod{4}. \end{cases}$$

For $c \in \mathbb{Z}^+$ with $c \equiv 0 \pmod{4}$ and $m, n \in \mathbb{Z}$ let

$$K_{1/2}(m, n; c) = \sum_{a \pmod{c}} \left(\frac{c}{a}\right) \varepsilon_a e\left(\frac{ma+n\bar{a}}{c}\right)$$

be the weight $1/2$ Kloosterman sum. Here $\bar{a} \in \mathbb{Z}$ satisfies

$$a\bar{a} \equiv 1 \pmod{c}.$$

It is convenient to define the modified Kloosterman sum

$$K^+(m, n; c) = (1-i)K_{1/2}(m, n; c) \times \begin{cases} 1 & \text{if } c/4 \text{ is even} \\ 2 & \text{otherwise.} \end{cases}$$

It is easily checked that

$$(3.1) \quad K^+(m, n; c) = K^+(n, m; c) = \overline{K^+(n, m; c)}.$$

The associated exponential sum is defined for $d \equiv 0, 1 \pmod{4}$ and fundamental D by

$$(3.2) \quad S_m(d, D; c) = \sum_{\substack{b \pmod{c} \\ b^2 \equiv Dd \pmod{c}}} \chi\left(\left[\frac{c}{4}, b, \frac{b^2-Dd}{c}\right]\right) e\left(\frac{2mb}{c}\right),$$

where χ was defined in (2.11). Clearly

$$S_{-m}(d, D; c) = \overline{S_m(d, D; c)} = S_m(d, D; c).$$

The following identity is proved by a slight modification of the proof given by Kohlen in [32, Prop. 5, p. 259] (see also [9], [25] and [42]).

Proposition 1. *For positive $c \equiv 0 \pmod{4}$, $d, m \in \mathbb{Z}$ with $d \equiv 0, 1 \pmod{4}$ and D a fundamental discriminant, we have*

$$S_m(d, D; c) = \sum_{n | (m, \frac{c}{4})} \left(\frac{D}{n}\right) \sqrt{\frac{n}{c}} K^+\left(d, \frac{m^2 D}{n^2}; \frac{c}{n}\right).$$

By Möbius inversion in two variables this can be written in the form

$$(3.3) \quad c^{-1/2} K^+(d, m^2 D, c) = \sum_{n | (m, \frac{c}{4})} \mu(n) \left(\frac{D}{n}\right) S_{m/n}(d, D; \frac{c}{n}).$$

Note that this gives an identity for $K^+(d, d', c)$ for any pair $d, d' \equiv 0, 1 \pmod{4}$. An immediate consequence of (3.3) and the obvious upper bound

$$S_m(d, D; c) \ll_{\epsilon} c^{\epsilon}$$

is the upper bound

$$(3.4) \quad K^+(d, d', c) \ll_{\epsilon} c^{1/2+\epsilon},$$

which holds for any $\epsilon > 0$. Furthermore, since for any $m, n \in \mathbb{Z}$ we have

$$K_{1/2}(m, n; c) = \frac{1}{4} K_{1/2}(4m, 4n; 4c),$$

(3.4) implies that for any $m, n \in \mathbb{Z}$

$$K_{1/2}(m, n, c) \ll_{\epsilon} c^{1/2+\epsilon}.$$

This elementary bound correspond to Weil's bound for the ordinary (weight 0) Kloosterman sum

$$K_0(m, n; c) = \sum_{\substack{a \pmod{c} \\ (a, c) = 1}} e\left(\frac{ma+n\bar{a}}{c}\right),$$

which states that (see [44], [21, Lemma 2])

$$(3.5) \quad K_0(m, n; c) \ll_{\epsilon} (m, n, c)^{1/2} c^{1/2+\epsilon}.$$

4. THE BASIS

In this section we prove Theorem 1. The following uniqueness result shows that a weakly harmonic form of weight $1/2$ is determined by the coefficients $b(n)$ in its Fourier expansion (2.3).

Proposition 2. *If $f \in H_{1/2}^1$ has Fourier expansion*

$$(4.1) \quad f(\tau) = \sum_n b(n) \mathcal{M}_n(y) e(nx) + \sum_n a(n) \mathcal{W}_n(y) e(nx),$$

then $f = 0$ if and only if $b(n) = 0$ for all $n \equiv 0, 1 \pmod{4}$.

Proof. Consider the Maass-type differential operator ξ_k defined for any $k \in \mathbb{R}$ through its action on a differentiable function f on \mathcal{H} by

$$\xi_k(f) = 2iy^k \frac{\partial f}{\partial \bar{\tau}}.$$

This operator is studied in some detail in [6]. Here we will record those properties that we require. It is easily checked that

$$\xi_k((\gamma\tau + \delta)^{-k} f(g\tau)) = (\gamma\tau + \delta)^{k-2} (\xi_k f)(g\tau)$$

for any $g \in \text{PSL}(2, \mathbb{R})$. Furthermore,

$$-\xi_{2-k} \circ \xi_k = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + iyk \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right),$$

so in particular $\Delta_{1/2} = -\xi_{3/2} \circ \xi_{1/2}$. A direct calculation using (2.5), (2.4) and the recurrence formulas in [35, p. 301] yields the formulas

$$(4.2) \quad \xi_{1/2}(\mathcal{M}_n(y) e(nx)) = 2|n|^{\frac{1}{2}} q^{-n} \quad \xi_{1/2}(\mathcal{W}_n(y) e(nx)) = \begin{cases} 0 & \text{if } n > 0 \\ -2q^{|n|} & \text{if } n \leq 0. \end{cases}$$

It follows that $\xi_{1/2}$ maps $H_{1/2}^1$ to $M_{3/2}^1$ with kernel $M_{1/2}^1$. Given f in (4.1) with $b(n) = 0$ for all n , we infer that $\xi_{1/2} f \in M_{3/2}^+ = \{0\}$. This implies that $f \in S_{1/2}^+ = \{0\}$, finishing the proof. \square

We now turn to the construction of h_d . It should be observed that the proof of Proposition 2 shows that $M_{1/2}^1$ is a subspace of $H_{1/2}^1$ and that $h_d = f_d$ is forced for all $d \leq 0$. In particular, $h_0 = \theta$. However, we will give a uniform construction using Poincaré series. Due to some delicate convergence issues that arise from this approach, we will define them through analytic continuation. For fixed s with $\text{Re}(s) > 1/2$ and $n \in \mathbb{Z}$ let

$$\mathcal{M}_n(y, s) = \begin{cases} \Gamma(2s)^{-1} (4\pi|n|y)^{-\frac{1}{4}} M_{\frac{1}{4} \text{sgn } n, s - \frac{1}{2}}(4\pi|n|y) & \text{if } n \neq 0, \\ y^{s-\frac{1}{4}} & \text{if } n = 0 \end{cases}$$

$$\mathcal{W}_n(y, s) = \begin{cases} |n|^{-\frac{3}{4}} \Gamma\left(s + \frac{\text{sgn } n}{4}\right)^{-1} (4\pi y)^{-\frac{1}{4}} W_{\frac{1}{4} \text{sgn } n, s - \frac{1}{2}}(4\pi|n|y), & \text{if } n \neq 0 \\ \frac{2^{2s-\frac{1}{2}}}{(2s-1)\Gamma(2s-1/2)} y^{\frac{3}{4}-s}, & \text{if } n = 0, \end{cases}$$

where M and W are the usual Whittaker functions. By (A.6) and (A.7), for $n \neq 0$ we have that $\mathcal{M}_n(y) = \mathcal{M}_n(y, 3/4)$ and $\mathcal{W}_n(y) = \mathcal{W}_n(y, 3/4)$ where $\mathcal{M}_n(y)$ and $\mathcal{W}_n(y)$ were given in (2.5) and (2.4). However, $\mathcal{M}_0(y) = \mathcal{W}_0(y, 3/4)$ and $\mathcal{W}_0(y) = \mathcal{M}_0(y, 3/4)$.² We also need the usual I and J -Bessel functions, defined for fixed ν and $y > 0$ by (see e.g. [34])

$$(4.3) \quad I_\nu(y) = \sum_{k=0}^{\infty} \frac{(y/2)^{\nu+2k}}{k! \Gamma(\nu + k + 1)} \quad \text{and} \quad J_\nu(y) = \sum_{k=0}^{\infty} \frac{(-1)^k (y/2)^{\nu+2k}}{k! \Gamma(\nu + k + 1)}.$$

For $m \in \mathbb{Z}$ let

$$(4.4) \quad \psi_m(\tau, s) = \mathcal{M}_m(y, s)e(mx).$$

It follows from (A.3) and (2.2) that

$$\Delta_{1/2}\psi_m(\tau, s) = (s - \frac{1}{4})(\frac{3}{4} - s)\psi_m(\tau, s).$$

Define the Poincaré series

$$P_m(\tau, s) = \sum_{g \in \Gamma_\infty \setminus \Gamma_0(4)} j(g, \tau)^{-1} \psi_m(g\tau, s),$$

where Γ_∞ is the subgroup of translations. By (A.5) this series converges absolutely and uniformly on compacta for $\text{Re } s > 1$. The function $P_0(\tau, s)$ is the usual weight $1/2$ Eisenstein series. It is clear that for $\text{Re}(s) > 1$ and any m the function $P_m(\tau, s)$ has weight $1/2$ and that P_m satisfies

$$\Delta_{1/2}P_m(\tau, s) = (s - \frac{1}{4})(\frac{3}{4} - s)P_m(\tau, s).$$

As in [32], in order to get forms whose Fourier expansions are supported on $n \equiv 0, 1 \pmod{4}$ we will employ the projection operator $\text{pr}^+ = \frac{2}{3}(U_4 \circ W_4) + \frac{1}{3}$, where

$$(U_4 f)(\tau) = \frac{1}{4} \sum_{\nu \pmod{4}} f\left(\frac{\tau+\nu}{4}\right) \quad \text{and} \quad (W_4 f)(\tau) = \left(\frac{2\tau}{i}\right)^{-1/2} f(-1/4\tau).$$

For each $d \equiv 0, 1 \pmod{4}$ and $\text{Re}(s) > 1$ define

$$(4.5) \quad P_d^+(\tau, s) = \text{pr}^+(P_d(\tau, s)).$$

Proposition 3. *For any $d \equiv 0, 1 \pmod{4}$ and $\text{Re}(s) > 1$ the function $P_d^+(\tau, s)$ has weight $1/2$ and satisfies*

$$\Delta_{1/2}P_d^+(\tau, s) = (s - \frac{1}{4})(\frac{3}{4} - s)P_d^+(\tau, s).$$

Its Fourier expansion is given by

$$(4.6) \quad P_d^+(\tau, s) = \mathcal{M}_d(y, s)e(dx) + \sum_{n \equiv 0, 1(4)} b_d(n, s)\mathcal{W}_n(y, s)e(nx), \quad \text{where}$$

$$(4.7) \quad b_d(n, s) = \sum_{0 < c \equiv 0(4)} K^+(d, n; c) \times \begin{cases} 2^{\frac{1}{2}} \pi |dn|^{\frac{1}{4}} c^{-1} I_{2s-1}\left(\frac{4\pi\sqrt{|dn|}}{c}\right) & \text{if } dn < 0; \\ 2^{\frac{1}{2}} \pi |dn|^{\frac{1}{4}} c^{-1} J_{2s-1}\left(\frac{4\pi\sqrt{|dn|}}{c}\right) & \text{if } dn > 0, \\ \pi^{s+\frac{1}{4}} |d+n|^{s-\frac{1}{4}} c^{-2s} & \text{if } dn = 0, d+n \neq 0, \\ 2^{\frac{1}{2}-2s} \pi^{\frac{1}{2}} \Gamma(2s) c^{-2s} & \text{if } d = n = 0. \end{cases}$$

The sum defining each $b_d(n, s)$ is absolutely convergent.

²This notational switching is inessential but gives a cleaner statement of Theorem 1 and some other results.

Proof. The first statement is clear. So is the last statement using the trivial bound for $K^+(d, n; c)$ and the definitions (4.3).

For the calculation of the Fourier expansion we employ the following lemma, whose proof is standard and follows from an application of Poisson summation using an integral formula found in [15, p. 176]. See [30, Lemma 2, p. 20] or [32] for the prototype result.

Lemma 1. *Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R})$ have $c > 0$ and suppose that $\operatorname{Re}(s) > 1/2$. Then for ψ_m defined in (4.4) with any $m \in \mathbb{Z}$, we have*

$$\sum_{r \in \mathbb{Z}} (c(\tau + r) + d)^{-1/2} \psi_m \left(\frac{a(\tau + r) + b}{c(\tau + r) + d}, s \right) = 2\pi i^{-1/2} \sum_{n \in \mathbb{Z}} e\left(\frac{am+nd}{c}\right) \mathcal{W}_n(y, s) e(nx) \\ \times \begin{cases} c^{-1} |mn|^{\frac{1}{4}} J_{2s-1}(4\pi \sqrt{|mn|} c^{-1}) & \text{if } mn > 0 \\ c^{-1} |mn|^{\frac{1}{4}} I_{2s-1}(4\pi \sqrt{|mn|} c^{-1}) & \text{if } mn < 0 \\ 2^{-\frac{1}{2}} \pi^{s-\frac{3}{4}} c^{-2s} |m+n|^{s-\frac{1}{4}} & \text{if } mn = 0, m+n \neq 0, \\ \pi^{-\frac{1}{2}} (2c)^{-2s} \Gamma(2s) & \text{if } m = n = 0, \end{cases}$$

where both sides of the identity converge uniformly on compact subsets of \mathcal{H} .

With this Lemma, the computation of the Fourier coefficients parallels so closely that given in [30, pp. 18–27] in the holomorphic case that we will omit the details. \square

It is a well-known consequence of the theory of the resolvent kernel that $P_d(\tau, s)$ has an analytic continuation in s to $\operatorname{Re}(s) > 1/2$ except for possibly finitely many simple poles in $(1/2, 1)$. These poles may only occur at points of the discrete spectrum of $\Delta_{1/2}$ on the Hilbert space consisting of weight $1/2$ functions f on \mathcal{H} that satisfy

$$\int_{\Gamma \backslash \mathcal{H}} |f(\tau)|^2 y \, d\mu < \infty \quad (d\mu = y^{-2} dx \, dy),$$

and this space contains the residues.³ It is easily seen from (4.5) that $P_d^+(\tau, s)$ also has an analytic continuation to $\operatorname{Re}(s) > 1/2$ with at most finitely many simple poles in $(1/2, 1)$. Actually, such poles can only occur in $(\frac{1}{2}, \frac{3}{4}]$, since by (3.4) and (4.3) the series in (4.7) giving the Fourier coefficient $b_d(n, s)$ converges absolutely for $\operatorname{Re}(s) > 3/4$. Thus for $\operatorname{Re}(s) > 1/2$ away from these poles the function $P_d^+(\tau, s)$ has weight $1/2$ and satisfies

$$\Delta_{1/2} P_d^+(\tau, s) = (s - \frac{1}{4})(\frac{3}{4} - s) P_d^+(\tau, s).$$

Furthermore a residue at $s = 3/4$ is a weight $1/2$ weakly harmonic form $f \in H_{1/2}^!$. In fact, the Fourier expansion of f can be obtained from that of P_d^+ in (4.6) by taking residues term by term, a process that is easily justified using the integral representations for the Fourier coefficients since the convergence is uniform on compacta. This shows that the Fourier expansion of f is supported on n with $n \equiv 0, 1 \pmod{4}$ and that it can have no exponentially growing terms. Another way to see these facts is to observe that f is the projection of the residue of P_d , which comes from the discrete spectrum. Thus by Proposition 2 applied to $f - b(0)\theta$, we obtain the following result.

Proposition 4. *For each d and each $\tau \in \mathcal{H}$ the function $P_d^+(\tau, s)$ has an analytic continuation around $s = 3/4$ with at most a simple pole there with residue*

$$(4.8) \quad \operatorname{res}_{s=3/4} P_d^+(\tau, s) = \rho_d \theta(\tau),$$

where $\rho_d \in \mathbb{C}$.

³See [15, p.179] and its references, especially [40] and [14]. A very clear treatment when the weight is 0 and the multiplier is trivial is given in [36]. In particular, see Satz 6.8 p.60; the case of weight $1/2$ is similar.

When $d = 0$ this result is well-known. In fact, $b_0(n, s)$ can be computed in terms of Dirichlet L -functions. We have the following formulas (see e.g. [23]).

Lemma 2. *For $m \in \mathbb{Z}^+$ and D a fundamental discriminant we have that*

$$(4.9) \quad \sum_{n|m} \left(\frac{D}{n}\right) b_0\left(\frac{Dm^2}{n^2}, s\right) = 2^{2-4s} \pi^{s+\frac{1}{4}} m^{\frac{3}{2}-2s} |D|^{s-\frac{1}{4}} \sigma_{4s-2}(m) \frac{L_D(2s - \frac{1}{2})}{\zeta(4s - 1)} \quad \text{and}$$

$$(4.10) \quad b_0(0, s) = \pi^{\frac{1}{2}} 2^{\frac{5}{2}-6s} \Gamma(2s) \frac{\zeta(4s - 2)}{\zeta(4s - 1)},$$

where L_D is the Dirichlet L -function.

By Möbius inversion, (4.9) gives for $m \neq 0$ the identity

$$(4.11) \quad b_0(Dm^2, s) = 2^{2-4s} \pi^{s+\frac{1}{4}} |D|^{s-\frac{1}{4}} \frac{L_D(2s - 1/2)}{\zeta(4s - 1)} \sum_{n|m} \mu(m/n) \left(\frac{D}{m/n}\right) n^{\frac{3}{2}-2s} \sigma_{4s-2}(n).$$

This yields a direct proof of Proposition 4 in case $d = 0$. Since $b_0(d, s) = b_d(0, s)$, which is clear from (3.1) and (4.7), a calculation using (4.11) and the (4.10) also gives the constant ρ_d in (4.8):

$$(4.12) \quad \rho_d = \begin{cases} \frac{3}{4\pi} & \text{if } d = 0, \\ \frac{6}{\pi} \sqrt{d} & \text{if } d \text{ is a non-zero square,} \\ 0 & \text{otherwise.} \end{cases}$$

We are finally ready to define the basis functions h_d . For $d \neq 0$ let

$$(4.13) \quad h_d(\tau, s) = P_d^+(\tau, s) - \frac{b_d(0, s)}{b_0(0, s)} P_0^+(\tau, s).$$

It has the Fourier expansion

$$(4.14) \quad h_d(\tau, s) = \mathcal{M}_d(y, s) e(dx) - \frac{b_d(0, s)}{b_0(0, s)} y^{s-\frac{1}{4}} + \sum_{0 \neq n \equiv 0, 1(4)} a_d(n, s) \mathcal{W}_n(y, s) e(nx), \quad \text{where}$$

$$(4.15) \quad a_d(n, s) = b_d(n, s) - \frac{b_d(0, s) b_0(n, s)}{b_0(0, s)}.$$

Proposition 5. *For each nonzero $d \equiv 0, 1 \pmod{4}$ the function $h_d(\tau, s)$ defined in (4.13) has an analytic continuation to $s = 3/4$ and*

$$h_d(\tau, 3/4) = h_d(\tau) \in H_{1/2}^1.$$

The Fourier expansion of each such h_d at ∞ has the form (2.6), where for each nonzero $n \equiv 0, 1 \pmod{4}$ we have

$$a_d(n) = \lim_{s \rightarrow 3/4^+} a_d(n, s).$$

Furthermore, $a_d(0) = 2\sqrt{d}$ if d is a square and $a_d(0) = 0$ otherwise.

Proof. Observe that $h_d(\tau, s)$ defined in (4.13) is holomorphic at $s = 3/4$, since otherwise by Proposition 4 it would have as residue there a nonzero multiple of $\theta(\tau)$, which cannot happen since (4.14) does not yield the constant term in θ . From (4.14) its Fourier expansion is given by

$$h_d(\tau) = \mathcal{M}_d(y) e(dx) + \sum_{n \equiv 0, 1(4)} a_d(n) \mathcal{W}_n(y) e(nx),$$

where $a_d(n) = \lim_{s \rightarrow 3/4^+} a_d(n, s)$ for $n \neq 0$ and, after recalling the definition of $\mathcal{W}_0(y)$ from (2.4), we have that

$$(4.16) \quad a_d(0) = \lim_{s \rightarrow 3/4^+} \frac{b_d(0, s)}{4b_0(0, s)}.$$

Here again we use the integral representations for the Fourier coefficients and the fact that $h_d(\tau, s) \rightarrow h_d(\tau)$ uniformly on compacta as $s \rightarrow 3/4^+$. Thus $h_d \in H_{1/2}^!$ for all $d \neq 0$. The last statement of Proposition 5 can easily be obtained from (4.16), (4.11) and (4.10). \square

Continuing with the proof of Theorem 1, we next show that the symmetry relation (2.7) holds. By (3.1) and (4.7) we have that $b_d(n, s) = b_n(d, s)$, hence by (4.15)

$$(4.17) \quad a_d(n, s) = a_n(d, s).$$

Now (2.7) follows from Proposition 5 and (4.17), where we use that $h_0 = \theta$ in case $nd = 0$. Note that $a_0(0) = 0$. A direct calculation using (4.2) together with (2.7) yields (2.8). This completes the proof of Theorem 1.

5. CYCLE INTEGRALS OF POINCARÉ SERIES

In preparation for the proofs of Theorem 2, in this section we will compute the cycle integrals of certain general Poincaré series, which we will then specialize in order to treat j_m . To begin we need to make some elementary observations about cycle integrals. For $Q \in \mathcal{Q}_d$ with $d > 0$ not a square let S_Q be the oriented semi-circle defined by

$$(5.1) \quad a|\tau|^2 + b \operatorname{Re} \tau + c = 0,$$

directed counterclockwise if $a > 0$ and clockwise if $a < 0$. Clearly

$$(5.2) \quad S_{gQ} = gS_Q,$$

for any $g \in \Gamma$. Given $z \in S_Q$ let C_Q be the directed arc on S_Q from z to $g_Q z$, where g_Q was defined in (2.10). It can easily be checked that C_Q has the same orientation as S_Q . Recall that

$$d\tau_Q = \frac{\sqrt{d} d\tau}{Q(\tau, 1)}.$$

If $\tau' = g\tau$ for some $g \in \Gamma$ we have

$$(5.3) \quad d\tau'_{gQ} = d\tau_Q.$$

For any Γ -invariant function f on \mathcal{H} the integral $\int_{C_Q} f(\tau) d\tau_Q$ is both independent of $z \in S_Q$ and is a class invariant. This is an immediate consequence of the following lemma that expresses this cycle integral as a sum of integrals over arcs in a fixed fundamental domain for Γ . This lemma will also be used in the proof of Theorem 4. Let \mathcal{F} be the standard fundamental domain for Γ

$$\mathcal{F} = \{\tau \in \mathcal{H}; -\frac{1}{2} \leq \operatorname{Re} \tau \leq 0, |\tau| \geq 1\} \cup \{\tau \in \mathcal{H}; 0 < \operatorname{Re} \tau < \frac{1}{2}, |\tau| > 1\}.$$

Lemma 3. *Let $Q \in \mathcal{Q}_d$ be a form with $d > 0$ not a square and $\mathcal{F}' = g\mathcal{F}$ be the image of \mathcal{F} under any fixed $g \in \Gamma$. Suppose that f is Γ -invariant and continuous on S_Q . Then for any $z \in S_Q$ we have*

$$(5.4) \quad \int_{C_Q} f(\tau) d\tau_Q = \sum_{q \in (Q)} \int_{S_q \cap \mathcal{F}'} f(\tau) d\tau_q,$$

where (Q) denotes the class of Q .

Proof. Let $\tilde{f}(\tau) = f(\tau)$ if $\tau \in \mathcal{F}$ and $\tilde{f}(\tau) = 0$ otherwise, so $f(\tau) = \sum_{g \in \Gamma} \tilde{f}(g\tau)$ with only a discrete set of exceptions. Thus

$$\int_{C_Q} f(\tau) d\tau_Q = \int_{C_Q} \sum_{g \in \Gamma} \tilde{f}(g\tau) d\tau_Q = \sum_{g \in \Gamma/\Gamma_Q} \sum_{\sigma \in \Gamma_Q} \int_{C_Q} \tilde{f}(g\sigma\tau) d\tau_Q = \sum_{g \in \Gamma/\Gamma_Q} \int_{S_Q} \tilde{f}(g\tau) d\tau_Q.$$

Take $g\tau$ as a new variable. By (5.2) and (5.3) we get

$$\int_{C_Q} f(\tau) d\tau_Q = \sum_{g \in \Gamma/\Gamma_Q} \int_{S_{gQ}} \tilde{f}(\tau) d\tau_{gQ},$$

which immediately yields (5.4). \square

The general Poincaré series are built from a test function $\phi : \mathbb{R}^+ \rightarrow \mathbb{C}$ assumed to be smooth and to satisfy $\phi(y) = O_\varepsilon(y^{1+\varepsilon})$, for any $\varepsilon > 0$. For any $m \in \mathbb{Z}$ let

$$(5.5) \quad G_m(\tau, \phi) = \sum_{g \in \Gamma_\infty \backslash \Gamma} e(m \operatorname{Re} g\tau) \phi(\operatorname{Im} g\tau).$$

This sum converges uniformly on compacta and defines a smooth Γ -invariant function on \mathcal{H} . We will express its cycle integrals in terms of the sum $S_m(d, D; c)$ from (3.2). Define for $t > 0$ the integral transform.

$$\Phi_m(t) = \int_0^\pi \cos(2\pi m t \cos \theta) \phi(t \sin \theta) \frac{d\theta}{\sin \theta}.$$

For ϕ as above we see that this integral converges absolutely and that $\Phi_m(t) = O_\varepsilon(t^{1+\varepsilon})$. As we have seen, we may assume without loss that $d, D > 0$.

Proposition 6. *Suppose that $d, D > 0$ with dD not a square. Then for all $m \in \mathbb{Z}$*

$$\sum_{Q \in \Gamma \backslash \mathcal{Q}_{dD}} \chi(Q) \int_{C_Q} G_m(\tau, \phi) d\tau_Q = \sum_{0 < c \equiv 0(4)} S_m(d, D; c) \Phi_m\left(\frac{2\sqrt{dD}}{c}\right).$$

Proof. For each Q , interchanging the sum defining P_m and the integral yields

$$(5.6) \quad \int_{C_Q} G_m(\tau, \phi) d\tau_Q = \sum_{g \in \Gamma_\infty \backslash \Gamma} \int_{C_Q} e(m \operatorname{Re} g\tau) \phi(\operatorname{Im} g\tau) d\tau_Q.$$

Now Γ_Q , the group of automorphs of Q , acts freely on $\Gamma_\infty \backslash \Gamma$ so we have that

$$\begin{aligned} \sum_{g \in \Gamma_\infty \backslash \Gamma} \int_{C_Q} e(m \operatorname{Re} g\tau) \phi(\operatorname{Im} g\tau) d\tau_Q &= \\ \sum_{g \in \Gamma_\infty \backslash \Gamma/\Gamma_Q} \sum_{\sigma \in \Gamma_Q} \int_{C_Q} e(m \operatorname{Re} g\sigma\tau) \phi(\operatorname{Im} g\sigma\tau) d\tau_Q &= \\ \sum_{g \in \Gamma_\infty \backslash \Gamma/\Gamma_Q} \int_{S_Q} e(m \operatorname{Re} g\tau) \phi(\operatorname{Im} g\tau) d\tau_Q. \end{aligned}$$

Applying (5.3) and (5.2) in the last expression, we get from (5.6) that

$$\int_{C_Q} G_m(\tau, \phi) d\tau_Q = \sum_{g \in \Gamma_\infty \backslash \Gamma/\Gamma_Q} \int_{S_{gQ}} e(m \operatorname{Re} \tau) \phi(\operatorname{Im} \tau) d\tau_{gQ}$$

and hence that

$$\sum_{Q \in \Gamma \backslash \mathcal{Q}_{dD}} \chi(Q) \int_{C_Q} G_m(\tau, \phi) d\tau_Q = \sum_{Q \in \Gamma_\infty \backslash \mathcal{Q}_{dD}} \chi(Q) \int_{S_Q} e(m \operatorname{Re} \tau) \phi(\operatorname{Im} \tau) d\tau_Q.$$

We now need to parameterize the cycle explicitly. Let

$$(5.7) \quad \tau_Q = \frac{-b}{2a} + \frac{i\sqrt{d}}{2|a|},$$

which is easily seen to be the apex of the circle S_Q . We can parameterize S_Q by $\theta \in (0, \pi)$ via

$$\tau = \begin{cases} \operatorname{Re} \tau_Q + e^{i\theta} \operatorname{Im} \tau_Q & \text{if } a > 0 \\ \operatorname{Re} \tau_Q - e^{-i\theta} \operatorname{Im} \tau_Q & \text{if } a < 0. \end{cases}$$

With this parameterization we find that

$$Q(\tau, 1) = \frac{d}{4a} \cdot \begin{cases} e^{2i\theta} - 1 & \text{if } a > 0 \\ e^{-2i\theta} - 1 & \text{if } a < 0 \end{cases}$$

and hence that $d\tau_Q = d\theta / \sin \theta$. Since $\chi(Q) = \chi(-Q)$ we arrive at the identity

$$\sum_{Q \in \Gamma \backslash \mathcal{Q}_{dD}} \chi(Q) \int_{C_Q} G_m(\tau, \phi) d\tau_Q = 2 \sum_{Q \in \Gamma_\infty \backslash \mathcal{Q}_{dD}^+} \chi(Q) e(m \operatorname{Re} \tau_Q) \Phi_m(\operatorname{Im} \tau_Q).$$

The proof of Proposition 6 is thus reduced to the following lemma. \square

Lemma 4. *Let ϕ be as above and suppose that dD is not a square. Then for all $m \in \mathbb{Z}$ we have the identity*

$$\sum_{\Gamma_\infty \backslash \mathcal{Q}_{dD}^+} \chi(Q) e(m \operatorname{Re} \tau_Q) \phi(\operatorname{Im} \tau_Q) = \frac{1}{2} \sum_{0 < c \equiv 0(4)} S_m(d, D; c) \phi\left(\frac{2\sqrt{|dD|}}{c}\right),$$

where τ_Q is defined in (5.7).

Proof. Under the growth condition on ϕ both series are absolutely convergent, and can be rearranged at will. Consider the left hand side. For $g = \pm \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \in \Gamma_\infty$ and $Q = [a, b, c] \in \mathcal{Q}_{dD}$, $gQ = [a, b - 2ka, *]$ and so the map

$$[a, b, c] \mapsto (a, b \bmod 2a)$$

is Γ_∞ -invariant. Thus

$$\sum_{\Gamma_\infty \backslash \mathcal{Q}_{dD}^+} \chi(Q) e(m \operatorname{Re} \tau_Q) \phi(\operatorname{Im} \tau_Q) = \sum_{a=1}^{\infty} \phi\left(\frac{\sqrt{|dD|}}{2a}\right) \sum_{b(2a)} \chi\left([a, b, \frac{b^2-dD}{4a}]\right) e\left(-\frac{mb}{2a}\right).$$

The sum in b is restricted to those values for which $\frac{b^2-dD}{4a}$ is an integer. This happens exactly when $b^2 \equiv dD \pmod{4a}$. Thus the inner sum is

$$\sum_{\substack{b(2a) \\ b^2 \equiv dD(4a)}} \chi\left([a, b, \frac{b^2-dD}{4a}]\right) e\left(-\frac{mb}{2a}\right) = \frac{1}{2} \sum_{\substack{b(4a) \\ b^2 \equiv dD(4a)}} \chi\left([a, b, \frac{b^2-dD}{4a}]\right) e\left(-\frac{2mb}{4a}\right) = \frac{1}{2} S_m(d, D; 4a).$$

Replace $4a$ by c to finish the proof. \square

We remark that the positive definite version of Proposition 6 is following well-known formula for $dD < 0$:

$$(5.8) \quad \sum_{Q \in \Gamma \backslash \mathcal{Q}_{dD}} w_Q^{-1} \chi(Q) G_m(\tau, \phi) = \frac{1}{2} \sum_{0 < c \equiv 0(4)} S_m(d, D; c) \phi\left(\frac{2\sqrt{|dD|}}{c}\right).$$

This formula is an immediate consequence of Lemma 4.

Now we will specialize the Poincaré series G_m from (5.5) and construct the modular functions j_m . Let $G_m(\tau, s) = G_m(\tau, \phi_{m,s})$, where

$$\phi_{m,s}(y) = \begin{cases} y^s & \text{if } m = 0 \\ 2\pi|m|^{\frac{1}{2}}y^{\frac{1}{2}}I_{s-\frac{1}{2}}(2\pi|m|y) & \text{if } m \neq 0, \end{cases}$$

with $I_{s-1/2}$ the Bessel function as before. The resulting Γ -invariant function satisfies

$$\Delta_0 G_m(\tau, s) = s(1-s)G_m(\tau, s).$$

The function G_0 is the usual Eisenstein series while G_m for $m \neq 0$ was studied by Neunhöffer [36] and Niebur [37], among others. The required analytic properties of $G_m(\tau, s)$ in s are most easily obtained from their Fourier expansions. For the Eisenstein series we have the well known formulas (see e.g. [24])

$$G_0(\tau, s) = y^s + c_0(0, s)y^{1-s} + \sum_{n \neq 0} c_0(n, s)K_{s-\frac{1}{2}}(2\pi|n|y)e(nx),$$

where $K_{s-\frac{1}{2}}$ is the K -Bessel function (see e.g. [34]),

$$c_0(0, s) = \frac{\xi(2s-1)}{\xi(2s)} \quad \text{and for } n \neq 0 \quad c_0(n, s) = \frac{2y^{1/2}}{\xi(2s)}|n|^{s-1/2}\sigma_{1-2s}(|n|),$$

with $\xi(s) = \pi^{-\frac{s}{2}}\Gamma(s/2)\zeta(s)$. For $m \neq 0$ the Fourier expansion of G_m can be found in [15], and is given by

$$G_m(\tau, s) = 2\pi|m|^{\frac{1}{2}}y^{\frac{1}{2}}I_{s-\frac{1}{2}}(2\pi|m|y)e(mx) + c_m(0, s)y^{1-s} \\ + 2|m|^{1/2}y^{1/2} \sum_{n \neq 0} |n|^{1/2}c_m(n, s)K_{s-\frac{1}{2}}(2\pi|n|y)e(nx),$$

where

$$c_m(0, s) = \frac{4\pi|m|^{1-s}\sigma_{2s-1}(|m|)}{(2s-1)\xi(2s)}$$

and

$$c_m(n; s) = \sum_{c>0} c^{-1}K_0(m, n; c) \cdot \begin{cases} I_{2s-1}(4\pi\sqrt{|mn|}c^{-1}) & \text{if } mn < 0 \\ J_{2s-1}(4\pi\sqrt{|mn|}c^{-1}) & \text{if } mn > 0. \end{cases}$$

Define for $m \in \mathbb{Z}^+$ and $\text{Re}(s) > 1$

$$(5.9) \quad j_m(\tau, s) = G_{-m}(\tau, s) - \frac{2m^{1-s}\sigma_{2s-1}(m)}{\pi^{-(s+\frac{1}{2})}\Gamma(s+\frac{1}{2})\zeta(2s-1)}G_0(\tau, s).$$

It follows from its Fourier expansion, Weil's bound (3.5) and (4.3) that $j_m(\tau, s)$ has an analytic continuation to $\text{Re}(s) > 3/4$. Furthermore, since a bounded harmonic function is constant, for $m \in \mathbb{Z}^+$ we have

$$(5.10) \quad j_m(\tau, 1) = j_m(\tau),$$

where j_m was defined in (2.13) (c.f. [37]). Alternatively, we could apply the theory of the resolvent kernel in weight 0 to get the analytic continuation of $j_m(\tau, s)$ up to $\text{Re } s > 1/2$.

In view of (5.10), in order to compute the traces of $j_m(\tau, s)$ it is enough to compute them for $G_m(\tau, s)$. We have the following identities, which are known when $m = 0$ (Dirichlet/Hecke) and when $dD < 0$ (see e.g. [10], [9], [7]). For the convenience of the reader we will give a uniform proof.

Proposition 7. *Let $\operatorname{Re}(s) > 1$ and $m \in \mathbb{Z}$. Suppose that d and D are not both negative and that dD is not a square. Then, when $dD < 0$ we have*

$$\sum_{Q \in \Gamma \backslash Q_{dD}} \frac{\chi(Q)}{w_Q} G_m(\tau_Q, s) = \begin{cases} \sqrt{2\pi} |m|^{\frac{1}{2}} |dD|^{\frac{1}{4}} \sum_{c \equiv 0(4)} \frac{S_m(d, D; c)}{c^{1/2}} I_{s-\frac{1}{2}} \left(\frac{4\pi \sqrt{m^2 |dD|}}{c} \right) & \text{if } m \neq 0 \\ 2^{s-1} |dD|^{\frac{s}{2}} \sum_{c \equiv 0(4)} \frac{S_0(d, D; c)}{c^s} & \text{if } m = 0, \end{cases}$$

while when $dD > 0$ we have

$$\sum_{Q \in \Gamma \backslash Q_{dD}} \frac{\chi(Q)}{B(s)} \int_{C_Q} G_m(\tau, s) d\tau_Q = \begin{cases} \sqrt{2\pi} |m|^{\frac{1}{2}} |dD|^{\frac{1}{4}} \sum_{c \equiv 0(4)} \frac{S_m(d, D; c)}{c^{1/2}} J_{s-\frac{1}{2}} \left(\frac{4\pi \sqrt{m^2 |dD|}}{c} \right) & \text{if } m \neq 0 \\ 2^{s-1} |dD|^{\frac{s}{2}} \sum_{c \equiv 0(4)} \frac{S_0(d, D; c)}{c^s} & \text{if } m = 0, \end{cases}$$

where $B(s) = 2^s \Gamma(\frac{s}{2})^2 / \Gamma(s)$.

Proof. By (5.8) the proof of Proposition 7 reduces to the case $dD > 0$. Applying Proposition 6 when $m = 0$ we use the well-known evaluation

$$\int_0^\pi (\sin \theta)^{s-1} d\theta = 2^{s-1} \frac{\Gamma(\frac{s}{2})^2}{\Gamma(s)}.$$

When $m \neq 0$ we need the following not-so-well-known evaluation to finish the proof. \square

Lemma 5. *For $\operatorname{Re}(s) > 0$ we have*

$$\int_0^\pi \cos(t \cos \theta) I_{s-\frac{1}{2}}(t \sin \theta) \frac{d\theta}{(\sin \theta)^{1/2}} = 2^{s-1} \frac{\Gamma(\frac{s}{2})^2}{\Gamma(s)} J_{s-1/2}(t).$$

Proof. Denote the left hand side by $L_s(t)$. We use the definition of $I_{s-\frac{1}{2}}$ in (4.3) to get

$$L_s(t) = \sum_{k=0}^{\infty} \frac{(t/2)^{s+2k-1/2}}{k! \Gamma(s+k+\frac{1}{2})} \int_0^\pi \cos(t \cos \theta) (\sin \theta)^{s+2k-1} d\theta.$$

Lommel's integral representation [43, p. 47] gives for $\operatorname{Re} v > -1/2$ that

$$J_\nu(y) = \frac{(y/2)^\nu}{\Gamma(\nu + \frac{1}{2}) \Gamma(\frac{1}{2})} \int_0^\pi \cos(y \cos \theta) (\sin \theta)^{2\nu} d\theta.$$

Thus for $\operatorname{Re}(s) > 0$ we have that

$$L_s(t) = \Gamma(\frac{1}{2}) \sum_{k=0}^{\infty} \frac{\Gamma(\frac{s}{2} + k)}{k! \Gamma(s+k+\frac{1}{2})} (t/2)^{s/2+k} J_{(s-1)/2+k}(t).$$

This Neumann series can be evaluated (see [43, p.143,eq.1]) giving for $\operatorname{Re}(s) > 0$

$$L_s(t) = \frac{\Gamma(\frac{1}{2}) \Gamma(\frac{s}{2})}{\Gamma(\frac{s}{2} + \frac{1}{2})} J_{s-1/2}(t).$$

The result follows by the duplication formula for $\Gamma(s)$. \square

6. THE TRACES IN TERMS OF FOURIER COEFFICIENTS

In this section we prove Theorems 2 and 3. We need to express the traces of j_m in terms of the Fourier coefficients of our basis h_d . This is first done for $j_m(\tau, s)$ with $\operatorname{Re}(s) > 1$ by applying Proposition 1 to transform the sum of exponential sums in Proposition 7 into a sum of Kloosterman sums, which is then related to the coefficients of $h_d(\tau, s)$. The method of using Kloosterman sums in this way was first applied by Zagier [46] to base change, then by Kohnen [32] to the Shimura lift and more recently to weakly holomorphic forms in [4], [9], [25] and [7].

Theorem 2 follows from Proposition 5, (5.10) and the next result by taking the limit as $s \rightarrow 1^+$ of both sides of (6.1).

Proposition 8. *Let $m \in \mathbb{Z}^+$ and $\operatorname{Re}(s) > 1$. Suppose that d and D are not both negative and that dD is not a square. Then*

$$(6.1) \quad \sum_{n|m} \left(\frac{D}{n}\right) a_d\left(\frac{m^2 D}{n^2}, \frac{s}{2} + \frac{1}{4}\right) = \begin{cases} \sum_Q \chi(Q) w_Q^{-1} j_m(\tau_Q, s) & \text{if } dD < 0, \\ B(s)^{-1} \sum_Q \chi(Q) \int_{C_Q} j_m(\tau, s) d\tau_Q & \text{if } dD > 0, \end{cases}$$

where each sum on the right hand side is over $Q \in \Gamma \backslash \mathcal{Q}_{dD}$.

Proof. It is convenient to set for any $m \in \mathbb{Z}$

$$T_m(s) = \begin{cases} \sum_Q \chi(Q) w_Q^{-1} G_m(\tau_Q, s) & \text{if } dD < 0, \\ B(s)^{-1} \sum_Q \chi(Q) \int_{C_Q} G_m(\tau, s) d\tau_Q & \text{if } dD > 0, \end{cases}$$

where each sum is over $Q \in \Gamma \backslash \mathcal{Q}_{dD}$. By Propositions 7 and 1 we have for $m \neq 0$ and $\operatorname{Re}(s) > 1$ that

$$T_m(s) = \pi |2m|^{\frac{1}{2}} |dD|^{\frac{1}{4}} \sum_{n|m} \left(\frac{D}{n}\right) n^{-\frac{1}{2}} \sum_{c \equiv 0(4)} c^{-1} K^+(d, \frac{m^2 D}{n^2}; c) \cdot \begin{cases} I_{s-\frac{1}{2}} \left(\frac{4\pi}{c} \sqrt{\frac{m^2}{n^2} |dD|} \right) & \text{if } dD < 0, \\ J_{s-\frac{1}{2}} \left(\frac{4\pi}{c} \sqrt{\frac{m^2}{n^2} |dD|} \right) & \text{if } dD > 0, \end{cases}$$

while when $m = 0$ we have

$$T_0(s) = 2^{s-1} |dD|^{\frac{s}{2}} L_D(s) \sum_{c \equiv 0(4)} c^{-s-1/2} K^+(d, 0; c).$$

Thus by (4.7) we derive that

$$(6.2) \quad T_m(s) = \begin{cases} \sum_{n|m} \left(\frac{D}{n}\right) b_d\left(\frac{m^2 D}{n^2}, \frac{s}{2} + \frac{1}{4}\right), & \text{if } m \neq 0 \\ 2^{s-1} \pi^{-\frac{s+1}{2}} |D|^{\frac{s}{2}} L_D(s) b_d\left(0, \frac{s}{2} + \frac{1}{4}\right) & \text{if } m = 0. \end{cases}$$

In view of (5.9), in order to prove Proposition 8 it is enough to show that

$$(6.3) \quad \sum_{n|m} \left(\frac{D}{n}\right) a_d\left(\frac{m^2 D}{n^2}, \frac{s}{2} + \frac{1}{4}\right) = T_m(s) - \frac{2m^{1-s} \sigma_{2s-1}(m)}{\pi^{-(s+\frac{1}{2})} \Gamma(s + \frac{1}{2}) \zeta(2s-1)} T_0(s).$$

By (4.15) and the first formula of (6.2) the left hand side of (6.3) is

$$T_m(s) - \frac{b_d\left(0, \frac{s}{2} + \frac{1}{4}\right)}{b_0\left(0, \frac{s}{2} + \frac{1}{4}\right)} \sum_{n|m} \left(\frac{D}{n}\right) b_0\left(\frac{m^2 D}{n^2}, \frac{s}{2} + \frac{1}{4}\right).$$

Hence by the second formula of (6.2) we are reduced to showing that

$$b_0(0, \frac{s}{2} + \frac{1}{4})^{-1} \sum_{n|m} \left(\frac{D}{n}\right) b_0\left(\frac{m^2 D}{n^2}, \frac{s}{2} + \frac{1}{4}\right) = \frac{2^s \pi^{s/2} |D|^{s/2} m^{1-s} \sigma_{2s-1}(m) L_D(s)}{\Gamma(s + \frac{1}{2}) \zeta(2s - 1)},$$

which follows by Lemma 2. This finishes the proof of Proposition 8, hence of Theorem 2. \square

We now give a quick proof of Theorem 3. By (4.12) we have

$$\text{Res}_{s=\frac{3}{4}} P_0^+(\tau, s) = \frac{3}{4\pi} \theta(\tau).$$

The function v can now be defined through the limit formula

$$(6.4) \quad v(\tau) = \frac{2\pi}{3} \lim_{s \rightarrow \frac{3}{4}} \left(P_0^+(\tau, s) - \frac{\frac{3}{4\pi} \theta(\tau)}{s - 3/4} \right).$$

It follows from (6.4) that v has weight $1/2$ and satisfies (2.17). Finally, using (6.2) when $m = 0$ and the fact that $G_0(\tau, s)$ has a simple pole at $s = 1$ with residue $3/\pi$, one shows that v has a Fourier expansion of the form (2.19). The formula

$$\Delta_{1/2} = -\xi_{3/2} \circ \xi_{1/2}$$

together with (2.17) and (2.18) easily gives (2.20).

7. RATIONAL PERIOD FUNCTIONS

We now prove Theorem 4. First we give a rough bound for the traces in terms of m when $d > 0$ is not a square that is sufficient to show that F_d is holomorphic in \mathcal{H} .

Proposition 9. *For $d > 0$ not a square and $m \in \mathbb{Z}^+$ we have for all $\epsilon > 0$ that*

$$\text{Tr}_d(j_m) \ll_{d,\epsilon} m^{5/4+\epsilon}.$$

Proof. It follows from [22, Thm 1. p. 110] that for fixed d not a square and $x > 0$, we have for all $\epsilon > 0$ that

$$(7.1) \quad \sum_{0 < n < x} S_m(d, 1; 4n) \ll_{d,\epsilon} (mx)^\epsilon (m^{5/4} + x^{3/4}),$$

after replacing d by $4d$ if necessary. For $1 < s < 2$ we have by (3) and the well-known bound (see e.g. [34, pp. 122])

$$J_\nu(y) \ll_\nu y^{-1/2}$$

that

$$\sum_{0 < n \leq m} S_m(d, 1; 4n) \sqrt{\frac{m}{n}} J_{s-\frac{1}{2}}\left(\pi \sqrt{|d|} \frac{m}{n}\right) \ll_{d,\epsilon} m^{1+\epsilon}.$$

By (4.3) we have for $x > m$

$$\sum_{m < n < x} S_m(d, 1; 4n) \sqrt{\frac{m}{n}} J_{s-\frac{1}{2}}\left(\pi \sqrt{|d|} \frac{m}{n}\right) \ll_{d,\epsilon} m^s \left| \sum_{m < n < x} S_m(d, 1; 4n) n^{-s} \right| + m^{1+\epsilon}.$$

Summation by parts and (7.1) give

$$m^s \sum_{m < n < x} S_m(d, 1; 4n) n^{-s} \ll_{d,\epsilon} m^{5/4+\epsilon}.$$

Now Proposition 9 follows by Proposition 7 and (5.9) by taking $s \rightarrow 1^+$ in the resulting uniform inequality

$$\sum_{Q \in \Gamma \backslash \mathcal{Q}_d} \int_{C_Q} j_m(\tau, s) d\tau_Q \ll_{d, \epsilon} m^{5/4 + \epsilon}$$

and using (5.10). \square

It now follows from Theorems 2 and 3 and Proposition 9 that the function F_d defined in (1.7) for $d > 0$ not a square can be represented by the series

$$(7.2) \quad F_d(\tau) = -2\pi \sum_{m \geq 0} \text{Tr}_d(j_m) q^m,$$

which gives a holomorphic function on \mathcal{H} . The basis $\{j_m\}_{m \geq 0}$ has a generating function that goes back to Faber (see e.g. [1]):

$$(7.3) \quad \sum_{m \geq 0} j_m(z) q^m = \frac{j'(\tau)}{j(z) - j(\tau)}, \quad \text{where } j'(\tau) = \frac{1}{2\pi i} \frac{dj}{d\tau}.$$

Note that this formal series converges when $\text{Im}(\tau) > \text{Im}(z)$ and that for fixed τ not a zero of j' it has a simple pole at $z = \tau$ with residue $(2\pi i)^{-1}$. It follows from (7.3) and (7.2) that for $\text{Im}(\tau)$ sufficiently large we have

$$(7.4) \quad F_d(\tau) = \frac{\sqrt{d}}{2\pi i} \sum_{Q \in \Gamma \backslash \mathcal{Q}_d} \int_{C_Q} \frac{j'(\tau)}{j(\tau) - j(z)} \frac{d\tau}{Q(\tau, 1)},$$

where we take for C_Q an arc on S_Q , the semi-circle defined in (5.1). Let $\mathcal{F}' = -\mathcal{F}^{-1}$ be the image of the standard fundamental domain under inversion $\tau \mapsto -1/\tau$. By (7.4) and Lemma 3 applied to each class of \mathcal{Q}_d and to each fundamental domain \mathcal{F} and \mathcal{F}' , we can write

$$F_d(\tau) = \frac{\sqrt{d}}{4\pi i} \sum_{Q \in \mathcal{Q}_d} \left(\int_{S_Q \cap \mathcal{F}} \frac{j'(\tau)}{j(\tau) - j(z)} \frac{d\tau}{Q(\tau, 1)} + \int_{S_Q \cap \mathcal{F}'} \frac{j'(\tau)}{j(\tau) - j(z)} \frac{d\tau}{Q(\tau, 1)} \right).$$

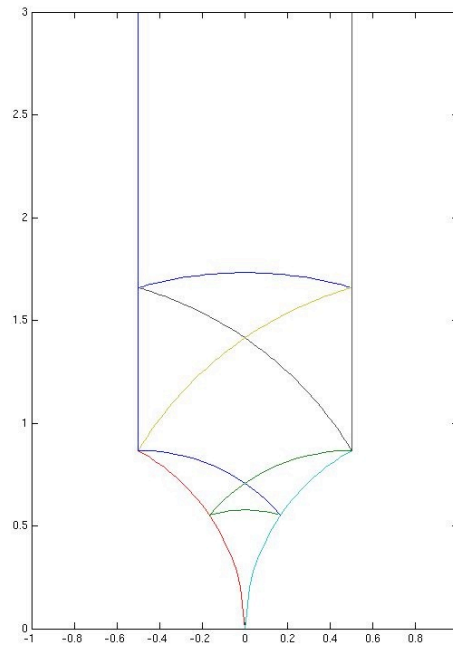
Now it is easily seen that each of these integrals is invariant under $Q \mapsto -Q$, so we may restrict the sum to \mathcal{Q}_d^+ , giving

$$(7.5) \quad F_d(\tau) = \frac{\sqrt{d}}{2\pi i} \sum_{Q \in \mathcal{Q}_d^+} \left(\int_{S_Q \cap \mathcal{F}} \frac{j'(\tau)}{j(\tau) - j(z)} \frac{d\tau}{Q(\tau, 1)} + \int_{S_Q \cap \mathcal{F}'} \frac{j'(\tau)}{j(\tau) - j(z)} \frac{d\tau}{Q(\tau, 1)} \right).$$

Recall from [8] that an indefinite quadratic form $Q = [a, b, c]$ is called *simple* if $c < 0 < a$. It is easily seen that $Q \in \mathcal{Q}_d$ is simple if and only if $Q \in \mathcal{Q}_d^+$ and S_Q intersects $\mathcal{F}'' = \mathcal{F} \cup \mathcal{F}'$. For simple Q let $A_Q = S_Q \cap \mathcal{F}''$ be the arc in \mathcal{F}'' oriented from right to left. Clearly A_Q must connect the two “vertical” sides of \mathcal{F}'' .⁴ Thus from (7.5) we obtain the identity

$$F_d(\tau) = \frac{\sqrt{d}}{2\pi i} \sum_{\substack{Q \text{ simple} \\ b^2 - 4ac = d}} \int_{A_Q} \frac{j'(\tau)}{j(\tau) - j(z)} \frac{dz}{Q(z, 1)}.$$

⁴For example, when $d = 12$ the simple forms are $[1, 0, -3], [1, -2, -2], [1, 2, -2], [3, 0, -1], [2, 2, -1], [2, -2, -1]$. A diagram showing the corresponding arcs A_Q in this case is given in Figure 1.

FIGURE 1. Arcs A_Q when $d = 12$.

Now we deform each arc A_Q in the sum of integrals to B_Q , which is within \mathcal{F}'' and has the same endpoints as A_Q , but travels above τ . By evaluating each resulting residue at τ , we get the formula

$$F_d(\tau) = \frac{\sqrt{d}}{2\pi i} \sum_{\substack{Q \text{ simple} \\ b^2 - 4ac = d}} \int_{B_Q} \frac{j'(\tau)}{j(\tau) - j(z)} \frac{dz}{Q(z, 1)} + \sqrt{d} \sum_{\substack{Q \text{ simple} \\ b^2 - 4ac = d}} Q(\tau, 1)^{-1},$$

which is also valid at $-1/\tau$. A simple calculation now shows that (2.21) holds in a neighborhood of τ , hence for all $\tau \in \mathcal{H}$. Thus Theorem 4 follows.

Finally, for fixed $m \in \mathbb{Z}^+$ the inequality (7.1) can be used to show that the series in Proposition 7 converges when $s = 1$. They yield the formula (2.23) upon using the elementary evaluation

$$J_{1/2}(y) = \sqrt{\frac{2}{\pi y}} \sin y.$$

APPENDIX A. WHITTAKER FUNCTIONS

A standard reference for the theory of Whittaker functions is [45, Chap. 16]. For the convenience of the reader we will record here some of the basic properties of these special functions that we need. Many more properties can be found in [35].

For fixed μ, ν with $\operatorname{Re}(\nu \pm \mu + 1/2) > 0$ the Whittaker functions may be defined for $y > 0$ by [35, pp. 311, 313]

$$(A.1) \quad M_{\mu,\nu}(y) = y^{\nu+\frac{1}{2}} e^{\frac{y}{2}} \frac{\Gamma(1+2\nu)}{\Gamma(\nu+\mu+\frac{1}{2})\Gamma(\nu-\mu+\frac{1}{2})} \int_0^1 t^{\nu+\mu-\frac{1}{2}} (1-t)^{\nu-\mu-\frac{1}{2}} e^{-yt} dt \quad \text{and}$$

$$(A.2) \quad W_{\mu,\nu}(y) = y^{\nu+\frac{1}{2}} e^{\frac{y}{2}} \frac{1}{\Gamma(\nu-\mu+\frac{1}{2})} \int_1^\infty t^{\nu+\mu-\frac{1}{2}} (t-1)^{\nu-\mu-\frac{1}{2}} e^{-yt} dt.$$

Both $M_{\mu,\nu}(y)$ and $W_{\mu,\nu}(y)$ satisfy the second order linear differential equation

$$(A.3) \quad \frac{d^2 w}{dy^2} + \left(-\frac{1}{4} + \mu y^{-1} + \left(\frac{1}{4} - \nu^2\right) y^{-2} \right) w = 0.$$

Their asymptotic behavior as $y \rightarrow \infty$ for fixed μ, ν is easily found from (A.1) and (A.2) by changing variable $t \mapsto t/y$:

$$(A.4) \quad M_{\mu,\nu}(y) \sim \frac{\Gamma(1+2\nu)}{\Gamma(\nu-\mu+\frac{1}{2})} y^{-\mu} e^{y/2} \quad \text{and} \quad W_{\mu,\nu}(y) \sim y^\mu e^{-y/2}.$$

In particular, they are linearly independent. For small y we get directly from (A.1) that

$$(A.5) \quad M_{\mu,\nu}(y) = y^{\nu+\frac{1}{2}} \left(1 + O_{\mu,\nu}(y) \right).$$

It is also apparent from (A.1) and (A.2) that when $\nu - \mu = 1/2$ we have

$$(A.6) \quad M_{\mu,\nu}(y) + (2\mu + 1)W_{\mu,\nu}(y) = \Gamma(2\mu + 2)y^{-\mu} e^{y/2},$$

while when $\nu + \mu = 1/2$ we have from (A.2) that

$$(A.7) \quad W_{\mu,\nu}(y) = y^\mu e^{-y/2}.$$

The I -Bessel and K -Bessel functions are special Whittaker functions [35]:

$$I_\nu(y) = 2^{-2\nu-\frac{1}{2}} \Gamma(\nu + 1)^{-1} y^{-\frac{1}{2}} M_{0,\nu}(2y) \quad \text{and} \quad K_\nu(y) = \sqrt{\frac{\pi}{2y}} W_{0,\nu}(2y).$$

Their asymptotic properties for large y thus follow from (A.4).

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