

Minkowski (1896), Voronoi (1896), Jacobi-Perron (1907), Furtwangler (1928), Brun (1957), Szekeres (1970), Cusick (1980), Brentjes (1981), Lenstra-Lenstra-Lovasz (1982), Lagarias (1994), Rossner-Schnorr (1996), Vaughan-Clarkson (1996), Arnold (1998), Briggs (2000), etc.

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In any deeper investigation of the problem of simultaneous approximation we are greatly handicapped by the absence of a full analogue of the continued fraction process. There are several analogues, but they all suffer from one of two defects: either they give much poorer approximations than we known to exist, or they involve a series of operations that can be carried out on given $\xi_1, \xi_2, \dots, \xi_n$ but cannot be used to define $\xi_1, \xi_2, \dots, \xi_n$, because we do not know the limitations to which any such sequence is subject. (Davenport, 1956)

Let $\xi = 0.6180\dots$ be a root of $f(x) = x^2 + x - 1$. Then

-1	-1		1
0	x		0.61803
1	$x - 1$	[1, 1, 1]	0.54018
2	$2x - 1$	[1, 1, 1]	0.47214
3	$3x - 2$	[1, 1, 1]	0.43769
4	$5x - 3$	[1, 1, 1]	0.45085
5	$8x - 5$	[1, 1, 1]	0.44582
6	$13x - 8$	[1, 1, 1]	0.44774
7	$21x - 13$	[1, 1, 1]	0.44701
8	$34x - 21$	[1, 1, 1]	0.44729
9	$55x - 34$	[1, 1, 1]	0.44718
10	$89x - 55$	[1, 1, 1]	0.44722
11	$144x - 89$	[1, 1, 1]	0.44721
12	$233x - 144$	[1, 1, 1]	0.44722
13	$377x - 233$	[1, 1, 1]	0.44721
14	$610x - 377$	[1, 1, 1]	0.44721
15	$987x - 610$	[1, 1, 1]	0.44721
16	$1597x - 987$	[1, 1, 1]	0.44721
17	$2584x - 1597$	[1, 1, 1]	0.44721
18	$4181x - 2584$	[1, 1, 1]	0.44721
19	$6765x - 4181$	[1, 1, 1]	0.44721
20	$10946x - 6765$	[1, 1, 1]	0.44721

Let $\xi = 0.4568\dots$ be a root of $f(x) = 11x^2 + 30x - 16$. Then

-1	-1		1
0	x		0.45682
1	$2x - 1$	$[1, 2, 1]$	0.19312
2	$11x - 5$	$[1, 5, 1]$	0.27483
3	$35x - 16$	$[1, 3, 1]$	0.39947
4	$81x - 37$	$[1, 2, 1]$	0.17478
5	$440x - 201$	$[1, 5, 1]$	0.27466
6	$1401x - 640$	$[1, 3, 1]$	0.39950
7	$3242x - 1481$	$[1, 2, 1]$	0.17478
8	$17611x - 8045$	$[1, 5, 1]$	0.27466
9	$56075x - 25616$	$[1, 3, 1]$	0.39950
10	$129761x - 59277$	$[1, 2, 1]$	0.17478
11	$704880x - 322001$	$[1, 5, 1]$	0.27466
12	$2244401x - 1025280$	$[1, 3, 1]$	0.39950
13	$5193682x - 2372561$	$[1, 2, 1]$	0.17478
14	$28212811x - 12888085$	$[1, 5, 1]$	0.27466
15	$89832115x - 41036816$	$[1, 3, 1]$	0.39950
16	$207877041x - 94961717$	$[1, 2, 1]$	0.17478
17	$1129217320x - 515845401$	$[1, 5, 1]$	0.27466
18	$3595529001x - 1642497920$	$[1, 3, 1]$	0.39950
19	$8320275322x - 3800841241$	$[1, 2, 1]$	0.17478
20	$45196905611x - 20646704125$	$[1, 5, 1]$	0.27466

Let $\xi = 1.2599\dots$ be a root of $f(x) = x^3 - 2$. Then

1	$-x + 1$	$[1, -1, 0]$	0.51984
2	$-x^2 + 2x - 1$	$[1, -1, -1]$	0.40535
3	$-3x^2 + 3x + 1$	$[1, 5, 4]$	0.33364
4	$-6x^2 + 2x + 7$	$[1, -3, 3]$	0.40621
5	$-13x^2 + 14x + 3$	$[1, 4, 0]$	1.0027
6	$-8x^2 - 5x + 19$	$[1, 3, -1]$	0.53385
7	$-27x^2 + 11x + 29$	$[1, 1, 1]$	1.1784
8	$3x^2 + 24x - 35$	$[1, -2, 0]$	0.55812
9	$-21x^2 + 59x - 41$	$[0, 1, 2]$	0.44907
10	$-60x^2 + 201x - 158$	$[5, 13, 4]$	4.6835
11	$81x^2 - 260x + 199$	$[5, 14, -6]$	1.3921
12	$465x^2 - 1501x + 1153$	$[5, 4, 10]$	25.553
13	$-546x^2 + 1761x - 1352$	$[5, -28, 2]$	28.800
14	$1011x^2 - 3262x + 2505$	$[5, 6, 4]$	21.834
15	$4590x^2 - 14809x + 11372$	$[5, -8, 7]$	236.92

Let $\xi = 1.2599\dots$ be a root of $f(x) = x^3 - 2$. Then

1	$-x + 1$	$[1, -1, 0]$	0.51984
2	$-x^2 + 2x - 1$	$[1, -1, -1]$	0.40535
3	$-3x^2 + 3x + 1$	$[1, 5, 4]$	0.33364
4	$-6x^2 + 2x + 7$	$[1, -3, 3]$	0.40621
5	$-13x^2 + 14x + 3$	$[1, 4, 0]$	1.0027
6	$-8x^2 - 5x + 19$	$[1, 3, -1]$	0.53385
7	$-27x^2 + 11x + 29$	$[1, 1, 1]$	1.1784
8	$3x^2 + 24x - 35$	$[1, -2, 0]$	0.55812
9	$-21x^2 + 59x - 41$	$[0, 1, 2]$	0.44907
10	$-80x^2 + 100x + 1$	$[1, 3, 3]$	0.34167
11	$180x^2 - 99x - 161$	$[1, 3, -3]$	0.36886
12	$-341x^2 + 459x - 37$	$[1, 4, 0]$	1.0443
13	$-279x^2 - 62x + 521$	$[1, -3, -1]$	0.49699
14	$800x^2 - 496x - 645$	$[1, -1, -1]$	1.0764
15	$217x^2 + 583x - 1079$	$[1, -2, 0]$	0.56745
16	$-366x^2 + 1662x - 1513$	$[0, -1, 2]$	0.49303
17	$-2028x^2 + 3175x - 781$	$[1, -3, 3]$	0.36585
18	$5203x^2 - 3956x - 3275$	$[1, 3, -3]$	0.34332
19	$9159x^2 - 681x - 13681$	$[1, 3, 3]$	0.45334
20	$-22840x^2 + 18999x + 12319$	$[1, -4, 0]$	1.0145

Let $\xi = 0.5436\dots$ be a root of $f(x) = x^3 + x^2 + x - 1$. Then

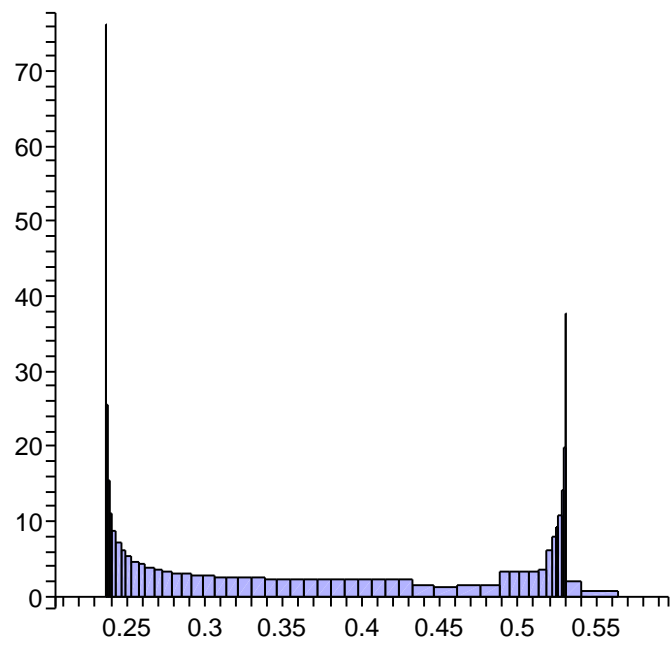
1	$-x^2 - x + 1$	$[1, -1, -1]$	0.48214
2	$-2x^2 + x$	$[1, -2, 0]$	0.23753
3	$3x^2 + 2x - 2$	$[1, -2, 0]$	0.43909
4	$-4x^2 + 4x - 1$	$[1, 3, 1]$	0.25195
5	$-8x^2 - 3x + 4$	$[1, -2, 0]$	0.36944
6	$-5x^2 - 12x + 8$	$[1, -2, 2]$	0.52585
7	$-7x^2 + 13x - 5$	$[1, 1, -1]$	0.29817
8	$-20x^2 - 2x + 7$	$[1, 1, 1]$	0.30221
9	$-18x^2 - 27x + 20$	$[1, -1, 1]$	0.52701
10	$-9x^2 + 38x - 18$	$[1, 1, -1]$	0.36462
11	$-47x^2 + 9x + 9$	$[1, 1, 1]$	0.25421
12	$-56x^2 - 56x + 47$	$[1, -1, 1]$	0.49437
13	$-103x^2 + 56x$	$[1, 2, 0]$	0.23684
14	$-159x^2 - 103x + 103$	$[1, 2, 0]$	0.43562
15	$-206x^2 + 215x - 56$	$[1, 3, -1]$	0.25421
16	$-421x^2 - 150x + 206$	$[1, 2, 0]$	0.36462
17	$271x^2 + 627x - 421$	$[1, 2, -2]$	0.52701
18	$-356x^2 + 692x - 271$	$[1, 1, 1]$	0.30221
19	$-1048x^2 - 85x + 356$	$[1, -1, 1]$	0.29817
20	$963x^2 + 1404x - 1048$	$[1, 1, -1]$	0.52582

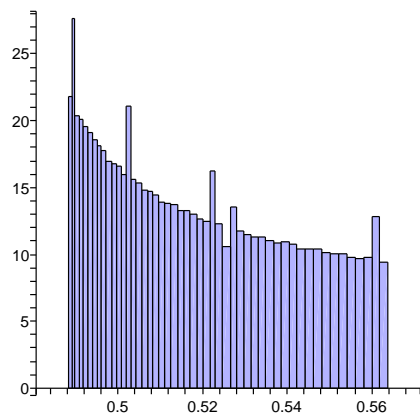
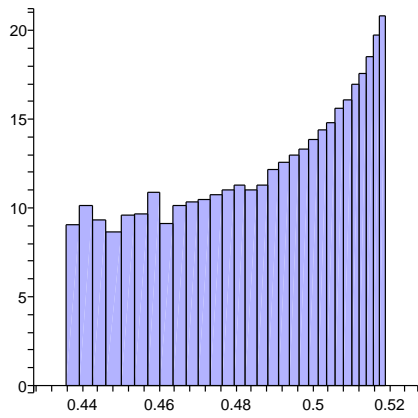
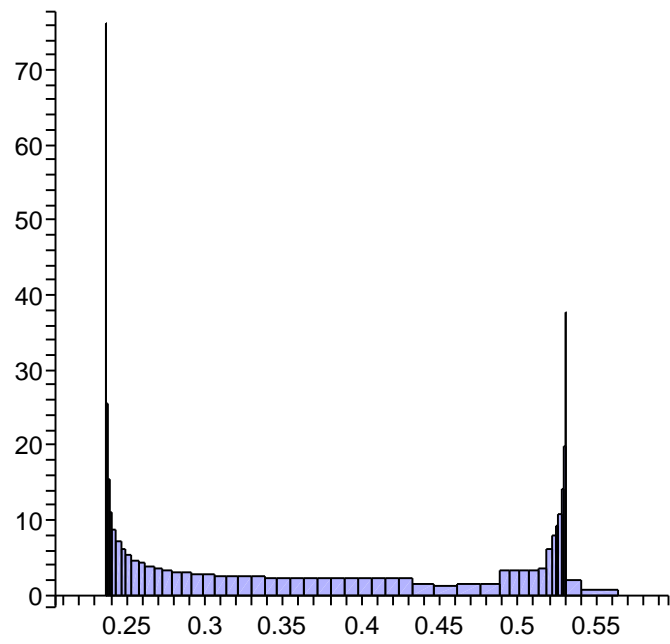
Let $\xi = 0.6180\dots$ be a root of $f(x) = x^2 + x - 1$. Then

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1	$x - 1$	$[1, 1, 1]$	0.54018
2	$2x - 1$	$[1, 1, 1]$	0.47214
3	$3x - 2$	$[1, 1, 1]$	0.43769
4	$5x - 3$	$[1, 1, 1]$	0.45085
5	$8x - 5$	$[1, 1, 1]$	0.44582
6	$13x - 8$	$[1, 1, 1]$	0.44774
7	$21x - 13$	$[1, 1, 1]$	0.44701
8	$34x - 21$	$[1, 1, 1]$	0.44729
9	$55x - 34$	$[1, 1, 1]$	0.44718
10	$89x - 55$	$[1, 1, 1]$	0.44722
11	$144x - 89$	$[1, 1, 1]$	0.44721
12	$233x - 144$	$[1, 1, 1]$	0.44722
13	$377x - 233$	$[1, 1, 1]$	0.44721
14	$610x - 377$	$[1, 1, 1]$	0.44721
15	$987x - 610$	$[1, 1, 1]$	0.44721
16	$1597x - 987$	$[1, 1, 1]$	0.44721
17	$2584x - 1597$	$[1, 1, 1]$	0.44721
18	$4181x - 2584$	$[1, 1, 1]$	0.44721
19	$6765x - 4181$	$[1, 1, 1]$	0.44721
20	$10946x - 6765$	$[1, 1, 1]$	0.44721

Let $\xi = 0.5436\dots$ be a root of $f(x) = x^3 + x^2 + x - 1$. Then

1	$-x^2 - x + 1$	$[1, -1, -1]$	0.48214
2	$-2x^2 + x$	$[1, -2, 0]$	0.23753
3	$3x^2 + 2x - 2$	$[1, -2, 0]$	0.43909
4	$-4x^2 + 4x - 1$	$[1, 3, 1]$	0.25195
5	$-8x^2 - 3x + 4$	$[1, -2, 0]$	0.36944
6	$-5x^2 - 12x + 8$	$[1, -2, 2]$	0.52585
7	$-7x^2 + 13x - 5$	$[1, 1, -1]$	0.29817
8	$-20x^2 - 2x + 7$	$[1, 1, 1]$	0.30221
9	$-18x^2 - 27x + 20$	$[1, -1, 1]$	0.52701
10	$-9x^2 + 38x - 18$	$[1, 1, -1]$	0.36462
11	$-47x^2 + 9x + 9$	$[1, 1, 1]$	0.25421
12	$-56x^2 - 56x + 47$	$[1, -1, 1]$	0.49437
13	$-103x^2 + 56x$	$[1, 2, 0]$	0.23684
14	$-159x^2 - 103x + 103$	$[1, 2, 0]$	0.43562
15	$-206x^2 + 215x - 56$	$[1, 3, -1]$	0.25421
16	$-421x^2 - 150x + 206$	$[1, 2, 0]$	0.36462
17	$271x^2 + 627x - 421$	$[1, 2, -2]$	0.52701
18	$-356x^2 + 692x - 271$	$[1, 1, 1]$	0.30221
19	$-1048x^2 - 85x + 356$	$[1, -1, 1]$	0.29817
20	$963x^2 + 1404x - 1048$	$[1, 1, -1]$	0.52582





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-1	-1		-1
0	x	$-x$	0.61803
1	$x - 1$	$-x$ [1, 1, 1]	0.54018
2	$2x - 1$	$-x$ [1, 1, 1]	0.47214
3	$3x - 2$	$-x$ [1, 1, 1]	0.43769
4	$5x - 3$	$-x$ [1, 1, 1]	0.45085
5	$8x - 5$	$-x$ [1, 1, 1]	0.44582
6	$13x - 8$	$-x$ [1, 1, 1]	0.44774
7	$21x - 13$	$-x$ [1, 1, 1]	0.44701
8	$34x - 21$	$-x$ [1, 1, 1]	0.44729
9	$55x - 34$	$-x$ [1, 1, 1]	0.44718
10	$89x - 55$	$-x$ [1, 1, 1]	0.44722
11	$144x - 89$	$-x$ [1, 1, 1]	0.44721
12	$233x - 144$	$-x$ [1, 1, 1]	0.44722
13	$377x - 233$	$-x$ [1, 1, 1]	0.44721
14	$610x - 377$	$-x$ [1, 1, 1]	0.44721
15	$987x - 610$	$-x$ [1, 1, 1]	0.44721
16	$1597x - 987$	$-x$ [1, 1, 1]	0.44721
17	$2584x - 1597$	$-x$ [1, 1, 1]	0.44721
18	$4181x - 2584$	$-x$ [1, 1, 1]	0.44721
19	$6765x - 4181$	$-x$ [1, 1, 1]	0.44721
20	$10946x - 6765$	$-x$ [1, 1, 1]	0.44721

Let $\xi = 0.4142\dots$ be a root of $f(x) = x^2 + 2x - 1$. Then

-1	-1		-1
0	x	$-x$	0.41421
1	$2x - 1$	$-x$	$[1, 2, 1]$ 0.38364
2	$5x - 2$	$-x$	$[1, 2, 1]$ 0.35534
3	$12x - 5$	$-x$	$[1, 2, 1]$ 0.35325
4	$29x - 12$	$-x$	$[1, 2, 1]$ 0.35361
5	$70x - 29$	$-x$	$[1, 2, 1]$ 0.35354
6	$169x - 70$	$-x$	$[1, 2, 1]$ 0.35355
7	$408x - 169$	$-x$	$[1, 2, 1]$ 0.35355
8	$985x - 408$	$-x$	$[1, 2, 1]$ 0.35355
9	$2378x - 985$	$-x$	$[1, 2, 1]$ 0.35355
10	$5741x - 2378$	$-x$	$[1, 2, 1]$ 0.35355
11	$13860x - 5741$	$-x$	$[1, 2, 1]$ 0.35355
12	$33461x - 13860$	$-x$	$[1, 2, 1]$ 0.35355
13	$80782x - 33461$	$-x$	$[1, 2, 1]$ 0.35355
14	$195025x - 80782$	$-x$	$[1, 2, 1]$ 0.35355
15	$470832x - 195025$	$-x$	$[1, 2, 1]$ 0.35355
16	$1136689x - 470832$	$-x$	$[1, 2, 1]$ 0.35355
17	$2744210x - 1136689$	$-x$	$[1, 2, 1]$ 0.35355
18	$6625109x - 2744210$	$-x$	$[1, 2, 1]$ 0.35355
19	$15994428x - 6625109$	$-x$	$[1, 2, 1]$ 0.35355
20	$38613965x - 15994428$	$-x$	$[1, 2, 1]$ 0.35355

Let $\xi = 0.1925\dots$ be a root of $f(x) = x^2 + 5x - 1$. Then

-1	-1		-1
0	x	$-x$	0.19258
1	$5x - 1$	$-x$ [1, 5, 1]	0.18911
2	$26x - 5$	$-x$ [1, 5, 1]	0.18570
3	$135x - 26$	$-x$ [1, 5, 1]	0.18569
4	$701x - 135$	$-x$ [1, 5, 1]	0.18570
5	$3640x - 701$	$-x$ [1, 5, 1]	0.18570
6	$18901x - 3640$	$-x$ [1, 5, 1]	0.18570
7	$98145x - 18901$	$-x$ [1, 5, 1]	0.18570
8	$509626x - 98145$	$-x$ [1, 5, 1]	0.18570
9	$2646275x - 509626$	$-x$ [1, 5, 1]	0.18570
10	$13741001x - 2646275$	$-x$ [1, 5, 1]	0.18570
11	$71351280x - 13741001$	$-x$ [1, 5, 1]	0.18570
12	$370497401x - 71351280$	$-x$ [1, 5, 1]	0.18570
13	$1923838285x - 370497401$	$-x$ [1, 5, 1]	0.18570
14	$9989688826x - 1923838285$	$-x$ [1, 5, 1]	0.18570
15	$51872282415x - 9989688826$	$-x$ [1, 5, 1]	0.18570
16	$269351100901x - 51872282415$	$-x$ [1, 5, 1]	0.18570
17	$1398627786920x - 269351100901$	$-x$ [1, 5, 1]	0.18570
18	$7262490035501x - 1398627786920$	$-x$ [1, 5, 1]	0.18570
19	$37711077964425x - 7262490035501$	$-x$ [1, 5, 1]	0.18570
20	$195817879857626x - 37711077964425$	$-x$ [1, 5, 1]	0.18570

Let $\xi = 0.6942\dots$ be a root of $f(x) = 3x^2 + 8x - 7$. Then

-1	-1			-1
0	x	$-x$		0.69425
1	$x - 1$	$-3/7 x - 1/7$	[1, 1, 1]	0.43238
2	$3x - 2$	$-3/4 x + 1/4$	[1, 2, 1]	0.24829
3	$10x - 7$	$-x$	[1, 3, 1]	0.57458
4	$13x - 9$	$-3/7 x - 1/7$	[1, 1, 1]	0.32896
5	$36x - 25$	$-3/4 x + 1/4$	[1, 2, 1]	0.24659
6	$121x - 84$	$-x$	[1, 3, 1]	0.57540
7	$157x - 109$	$-3/7 x - 1/7$	[1, 1, 1]	0.32880
8	$435x - 302$	$-3/4 x + 1/4$	[1, 2, 1]	0.24660
9	$1462x - 1015$	$-x$	[1, 3, 1]	0.57540
10	$1897x - 1317$	$-3/7 x - 1/7$	[1, 1, 1]	0.32880
11	$5256x - 3649$	$-3/4 x + 1/4$	[1, 2, 1]	0.24660
12	$17665x - 12264$	$-x$	[1, 3, 1]	0.57540
13	$22921x - 15913$	$-3/7 x - 1/7$	[1, 1, 1]	0.32880
14	$63507x - 44090$	$-3/4 x + 1/4$	[1, 2, 1]	0.24660
15	$213442x - 148183$	$-x$	[1, 3, 1]	0.57540
16	$276949x - 192273$	$-3/7 x - 1/7$	[1, 1, 1]	0.32880
17	$767340x - 532729$	$-3/4 x + 1/4$	[1, 2, 1]	0.24660
18	$2578969x - 1790460$	$-x$	[1, 3, 1]	0.57540
19	$3346309x - 2323189$	$-3/7 x - 1/7$	[1, 1, 1]	0.32880
20	$9271587x - 6436838$	$-3/4 x + 1/4$	[1, 2, 1]	0.24660

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2	$-2x^2 + x$	$[1, -2, 0]$	0.23753
3	$3x^2 + 2x - 2$	$[1, -2, 0]$	0.43909
4	$-4x^2 + 4x - 1$	$[1, 3, 1]$	0.25195
5	$-8x^2 - 3x + 4$	$[1, -2, 0]$	0.36944
6	$-5x^2 - 12x + 8$	$[1, -2, 2]$	0.52585
7	$-7x^2 + 13x - 5$	$[1, 1, -1]$	0.29817
8	$-20x^2 - 2x + 7$	$[1, 1, 1]$	0.30221
9	$-18x^2 - 27x + 20$	$[1, -1, 1]$	0.52701
10	$-9x^2 + 38x - 18$	$[1, 1, -1]$	0.36462
11	$-47x^2 + 9x + 9$	$[1, 1, 1]$	0.25421
12	$-56x^2 - 56x + 47$	$[1, -1, 1]$	0.49437
13	$-103x^2 + 56x$	$[1, 2, 0]$	0.23684
14	$-159x^2 - 103x + 103$	$[1, 2, 0]$	0.43562
15	$-206x^2 + 215x - 56$	$[1, 3, -1]$	0.25421
16	$-421x^2 - 150x + 206$	$[1, 2, 0]$	0.36462
17	$271x^2 + 627x - 421$	$[1, 2, -2]$	0.52701
18	$-356x^2 + 692x - 271$	$[1, 1, 1]$	0.30221
19	$-1048x^2 - 85x + 356$	$[1, -1, 1]$	0.29817
20	$963x^2 + 1404x - 1048$	$[1, 1, -1]$	0.52582

Let $\xi = 0.5436\dots$ be a root of $f(x) = x^3 + x^2 + x - 1$. Then

1	$-x^2 - x + 1$	$[1, -1, -1]$	x	0.48214
2	$-2x^2 + x$	$[1, -2, 0]$	$-x^2$	0.23753
3	$3x^2 + 2x - 2$	$[1, -2, 0]$	x	0.43909
4	$-4x^2 + 4x - 1$	$[1, 3, 1]$	x^2	0.25195
5	$-8x^2 - 3x + 4$	$[1, -2, 0]$	$-x$	0.36944
6	$-5x^2 - 12x + 8$	$[1, -2, 2]$	$-x$	0.52585
7	$-7x^2 + 13x - 5$	$[1, 1, -1]$	x	0.29817
8	$-20x^2 - 2x + 7$	$[1, 1, 1]$	$-x$	0.30221
9	$-18x^2 - 27x + 20$	$[1, -1, 1]$	$-x$	0.52701
10	$-9x^2 + 38x - 18$	$[1, 1, -1]$	x	0.36462
11	$-47x^2 + 9x + 9$	$[1, 1, 1]$	$-x$	0.25421
12	$-56x^2 - 56x + 47$	$[1, -1, 1]$	$-x$	0.49437
13	$-103x^2 + 56x$	$[1, 2, 0]$	$-x^2$	0.23684
14	$-159x^2 - 103x + 103$	$[1, 2, 0]$	$-x$	0.43562
15	$-206x^2 + 215x - 56$	$[1, 3, -1]$	$-x^2$	0.25421
16	$-421x^2 - 150x + 206$	$[1, 2, 0]$	$-x$	0.36462
17	$271x^2 + 627x - 421$	$[1, 2, -2]$	x	0.52701
18	$-356x^2 + 692x - 271$	$[1, 1, 1]$	$-x$	0.30221
19	$-1048x^2 - 85x + 356$	$[1, -1, 1]$	$-x$	0.29817
20	$963x^2 + 1404x - 1048$	$[1, 1, -1]$	x	0.52582

Let $\xi = 0.5436\dots$ be a root of $f(x) = x^3 + x^2 + x - 1$. Then

21	$-441x^2 + 2011x - 963$	$[1, 1, 1]$	$-x$	0.36950
22	$-2452x^2 + 522x + 441$	$[1, -1, 1]$	$-x$	0.25197
23	$-2893x^2 + 2533x - 522$	$[0, 1, 1]$	$-x^2 - x$	0.49147
24	$2974x^2 + 2893x - 2452$	$[1, 1, -1]$	$-1/2x^2 - 1/2$	0.49108
25	$-5345x^2 + 3055x - 81$	$[1, 1, 0]$	x^2	0.23692
26	$8400x^2 + 5264x - 5345$	$[1, 2, -1]$	x	0.43099
27	$10609x^2 - 11536x + 3136$	$[1, -3, -1]$	$-x^2$	0.25660
28	$-22145x^2 - 7473x + 10609$	$[1, -2, 0]$	x	0.35977
29	$-14672x^2 - 32754x + 22145$	$[1, 2, 2]$	$-x$	0.52805
30	$18082x^2 - 36817x + 14672$	$[1, -1, 1]$	$-x$	0.30634
31	$-54899x^2 - 3410x + 18082$	$[1, 1, -1]$	x	0.29423
32	$-51489x^2 - 72981x + 54899$	$[1, 1, 1]$	$-x$	0.52447
33	$21492x^2 - 106388x + 51489$	$[1, -1, 1]$	$-x$	0.37440
34	$-127880x^2 + 29997x + 21492$	$[1, 1, -1]$	x	0.24987
35	$-149372x^2 + 136385x - 29997$	$[0, -1, 1]$	$-x^2 - x$	0.49498
36	$-157877x^2 - 149372x + 127880$	$[1, 1, 1]$	$1/2x^2 + 1/2$	0.48766
37	$-277252x^2 + 166382x - 8505$	$[1, 1, 0]$	$-x^2$	0.23717
38	$443634x^2 + 268747x - 277252$	$[1, -2, -1]$	x	0.42632
39	$-545999x^2 + 618521x - 174887$	$[1, 3, 1]$	x^2	0.25913
40	$-1164520x^2 - 371112x + 545999$	$[1, -2, 0]$	$-x$	0.35494

Let $\xi = 0.5436\dots$ be a root of $f(x) = x^3 + x^2 + x - 1$. Then

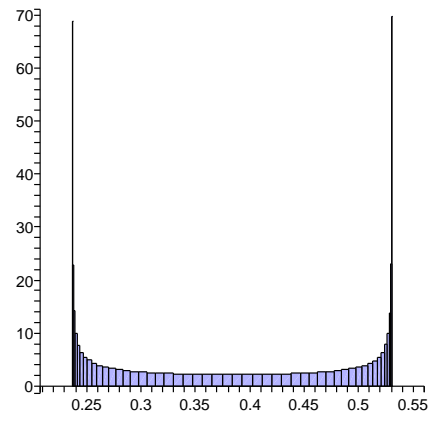
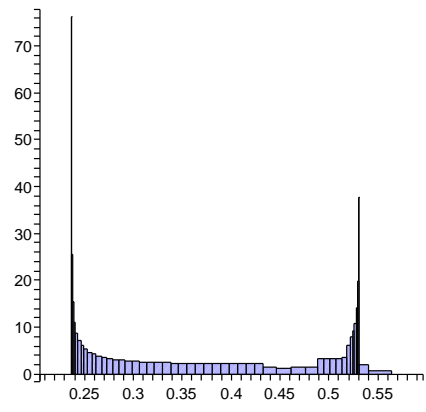
1	$-x^2 - x + 1$	$[1, -1, -1]$	x	0.48214
2	$-2x^2 + x$	$[1, -2, 0]$	$-x^2$	0.23753
3	$3x^2 + 2x - 2$	$[1, -2, 0]$	x	0.43909
4	$-4x^2 + 4x - 1$	$[1, 3, 1]$	x^2	0.25195
5	$-8x^2 - 3x + 4$	$[1, -2, 0]$	$-x$	0.36944
6	$-5x^2 - 12x + 8$	$[1, -2, 2]$	$-x$	0.52585
7	$-7x^2 + 13x - 5$	$[1, 1, -1]$	x	0.29817
8	$-20x^2 - 2x + 7$	$[1, 1, 1]$	$-x$	0.30221
9	$-18x^2 - 27x + 20$	$[1, -1, 1]$	$-x$	0.52701
10	$-9x^2 + 38x - 18$	$[1, 1, -1]$	x	0.36462
11	$-47x^2 + 9x + 9$	$[1, 1, 1]$	$-x$	0.25421
12	$-56x^2 - 56x + 47$	$[1, -1, 1]$	$-x$	0.49437
13	$-103x^2 + 56x$	$[1, 2, 0]$	$-x^2$	0.23684
14	$-159x^2 - 103x + 103$	$[1, 2, 0]$	$-x$	0.43562
15	$-206x^2 + 215x - 56$	$[1, 3, -1]$	$-x^2$	0.25421
16	$-421x^2 - 150x + 206$	$[1, 2, 0]$	$-x$	0.36462
17	$271x^2 + 627x - 421$	$[1, 2, -2]$	x	0.52701
18	$-356x^2 + 692x - 271$	$[1, 1, 1]$	$-x$	0.30221
19	$-1048x^2 - 85x + 356$	$[1, -1, 1]$	$-x$	0.29817
20	$963x^2 + 1404x - 1048$	$[1, 1, -1]$	x	0.52582

Let $\xi = 0.5436\dots$ be a root of $f(x) = x^3 + x^2 + x - 1$. Then

1	x	$[1, -1, -1, 1]$	x	0.54369
2	x^2	$[1, -1, -1, 1]$	x	0.29560
3	$-x^2 - x + 1$	$[1, -1, -1, 1]$	x	0.48214
4	$2x - 1$	$[1, -1, -1, 1]$	x	0.43689
5	$2x^2 - x$	$[1, -1, -1, 1]$	x	0.23753
6	$-3x^2 - 2x + 2$	$[1, -1, -1, 1]$	x	0.43909
7	$x^2 + 5x - 3$	$[1, -1, -1, 1]$	x	0.49150
8	$4x^2 - 4x + 1$	$[1, -1, -1, 1]$	x	0.25195
9	$-8x^2 - 3x + 4$	$[1, -1, -1, 1]$	x	0.36944
10	$5x^2 + 12x - 8$	$[1, -1, -1, 1]$	x	0.52585
11	$7x^2 - 13x + 5$	$[1, -1, -1, 1]$	x	0.29817
12	$-20x^2 - 2x + 7$	$[1, -1, -1, 1]$	x	0.30221
13	$18x^2 + 27x - 20$	$[1, -1, -1, 1]$	x	0.52701
14	$9x^2 - 38x + 18$	$[1, -1, -1, 1]$	x	0.36462
15	$-47x^2 + 9x + 9$	$[1, -1, -1, 1]$	x	0.25421
16	$56x^2 + 56x - 47$	$[1, -1, -1, 1]$	x	0.49437
17	$-103x + 56$	$[1, -1, -1, 1]$	x	0.43562
18	$-103x^2 + 56x$	$[1, -1, -1, 1]$	x	0.23684
19	$159x^2 + 103x - 103$	$[1, -1, -1, 1]$	x	0.43562
20	$-56x^2 - 262x + 159$	$[1, -1, -1, 1]$	x	0.49437

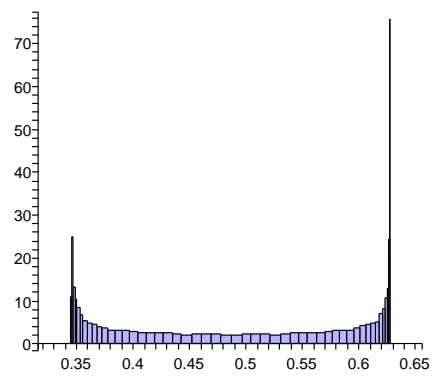
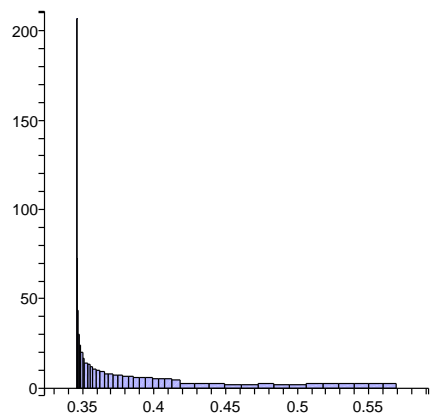
Let $\xi = 0.5436\dots$ be a root of $f(x) = x^3 + x^2 + x - 1$. Then

1	$-x^2 - x + 1$	$-x^2 - x + 1$
2	$-2x^2 + x$	$2x - 1$
3	$3x^2 + 2x - 2$	$2x^2 - x$
4	$-4x^2 + 4x - 1$	$-3x^2 - 2x + 2$
5	$-8x^2 - 3x + 4$	$x^2 + 5x - 3$
6	$-5x^2 - 12x + 8$	$4x^2 - 4x + 1$
7	$-7x^2 + 13x - 5$	$-8x^2 - 3x + 4$
8	$-20x^2 - 2x + 7$	$5x^2 + 12x - 8$
9	$-18x^2 - 27x + 20$	$7x^2 - 13x + 5$
10	$-9x^2 + 38x - 18$	$-20x^2 - 2x + 7$
11	$-47x^2 + 9x + 9$	$18x^2 + 27x - 20$
12	$-56x^2 - 56x + 47$	$9x^2 - 38x + 18$
13	$-103x^2 + 56x$	$-47x^2 + 9x + 9$
14	$-159x^2 - 103x + 103$	$56x^2 + 56x - 47$
15	$-206x^2 + 215x - 56$	$-103x + 56$
16	$-421x^2 - 150x + 206$	$-103x^2 + 56x$
17	$271x^2 + 627x - 421$	$159x^2 + 103x - 103$
18	$-356x^2 + 692x - 271$	$-56x^2 - 262x + 159$
19	$-1048x^2 - 85x + 356$	$-206x^2 + 215x - 56$
20	$963x^2 + 1404x - 1048$	$421x^2 + 150x - 206$



Let $\xi = 0.6823\dots$ be a root of $x^3 + x - 1$. Then

1	$x^2 + x - 1$	$x - 1$
2	$2x^2 - 1$	$x^2 - x$
3	$2x^2 + 3x - 3$	$x^2 + x - 1$
4	$5x^2 + x - 3$	$x^2 - 2x + 1$
5	$8x^2 - 4x - 1$	$2x^2 - 1$
6	$4x^2 + 9x - 8$	$3x - 2$
7	$12x^2 + 5x - 9$	$3x^2 - 2x$
8	$21x^2 - 7x - 5$	$2x^2 + 3x - 3$
9	$7x^2 + 26x - 21$	$3x^2 - 5x + 2$
10	$28x^2 + 19x - 26$	$5x^2 + x - 3$
11	$54x^2 - 9x - 19$	$x^2 - 8x + 5$
12	$63x^2 + 64x - 73$	$8x^2 - 4x - 1$
13	$136x^2 + x - 64$	$4x^2 + 9x - 8$
14	$200x^2 - 135x - 1$	$9x^2 - 12x + 4$
15	$135x^2 + 201x - 200$	$12x^2 + 5x - 9$
16	$335x^2 + 66x - 201$	$5x^2 - 21x + 12$
17	$536x^2 - 269x - 66$	$21x^2 - 7x - 5$
18	$269x^2 + 602x - 536$	$7x^2 + 26x - 21$
19	$805x^2 + 333x - 602$	$26x^2 - 28x + 7$
20	$1407x^2 - 472x - 333$	$28x^2 + 19x - 26$
21	$472x^2 + 1740x - 1407$	$19x^2 - 54x + 28$
22	$1879x^2 + 1268x - 1740$	$54x^2 - 9x - 19$
23	$3619x^2 - 611x - 1268$	$9x^2 + 73x - 54$
24	$4230x^2 + 4276x - 4887$	$73x^2 - 63x + 9$
25	$9117x^2 + 46x - 4276$	$63x^2 + 64x - 73$



Consider a polynomial $f(x) = x^3 + a_2x^2 + a_1x - 1$ in $\mathbb{Z}[x]$ irreducible over \mathbb{Q} . Let ξ, ξ_1 and ξ_2 be its roots. Assume that ξ is real and ξ_1, ξ_2 are complex. We also assume that $|\xi| < 1$.

We introduce a sequence of polynomials $P_n(x) = a_n + b_nx + c_nx^2$ in $\mathbb{Z}[x]$, defined by $P_n(x) = x^n \pmod{f}$. Put

$$K_n = |P_n(\xi)|(a_n^2 + b_n^2 + c_n^2)$$

$$\text{and } I = \inf_n K_n, \quad S = \sup_n K_n, \quad \tilde{K}_n = (K_n - I)/S.$$

THEOREM (HENSLEY-T.): A frequency curve for $\{\tilde{K}_n\}$ is $f(x) = \frac{1}{\pi \sqrt{x(1-x)}}$.

Moreover, $I = \inf_n K_n$ and $S = \sup_n K_n$ are roots of $p(x) = c_6x^6 + \dots + c_2x^2 + c_1x + c_0$ with

$$c_6 = d^3$$

$$c_5 = 4(a_2^4 - a_2^3 - 4a_2^2a_1 + a_2^2 + 4a_2a_1 + a_1^2 - 6a_2 - 3a_1 + 9)d^2$$

$$c_4 = -4(a_2^4 - 4a_2^2a_1 + a_2^2 + 2a_1^2 - 8a_2 - 2a_1 + 3)d^2$$

$$c_3 = -16(a_2^3a_1^2 + 6a_2^4 + 6a_2^2a_1^2 - 4a_2a_1^3 + 2a_1^4 + 4a_2^3 - 18a_2^2a_1 - 4a_1^3 + 10a_2^2 - 10a_2a_1 + 10a_1^2 - 27a_2)d$$

$$c_2 = 16(2a_2^4 + 5a_2^2a_1^2 + a_1^4 + 2a_2^3 - 2a_1^3 + 8a_2^2 + a_1^2 - 6a_2)d$$

$$c_1 = 64a_2^2a_1^4 + 64a_1^6 + 320a_2^3a_1^2 + 64a_2^2a_1^3 + 384a_2a_1^4 - 64a_1^5 + 320a_2^4 + 256a_2^3a_1 + 704a_2^2a_1^2 - 384a_2a_1^3 + 128a_1^4 + 576a_2^3 - 384a_2^2a_1 + 576a_2a_1^2 + 192a_1^3 + 1152a_2^2$$

$$c_0 = -64(a_2^2 + a_2a_1 + a_1^2 + 2a_2 - 2a_1 + 4)(a_2^2 - a_2a_1 + a_1^2),$$

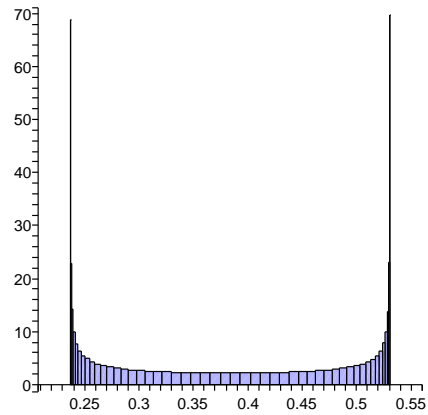
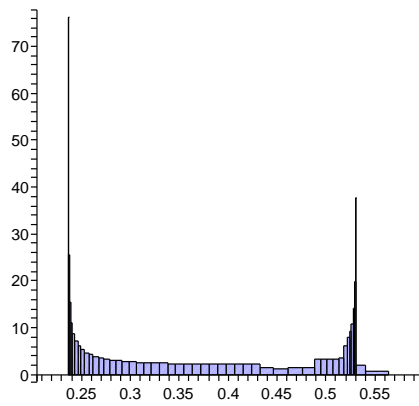
where $d = -4a_1^3 + 4a_2^3 + a_1^2a_2^2 - 18a_1a_2 - 27$ is the discriminant of f .

If $f(x) = x^3 + x^2 + x - 1$, then

$$I = \inf_n K_n = 0.2368.. \quad \text{and} \quad S = \sup_n K_n = 0.5308..$$

are roots of

$$p(x) = (11x^3 + 11x^2 + x - 1)(121x^3 - 143x^2 + 55x - 7)$$

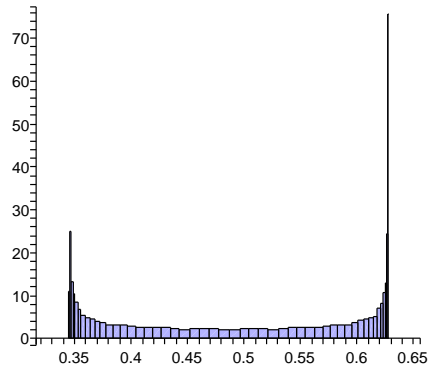
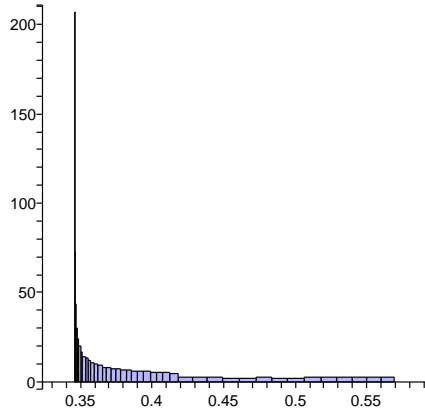


If $f(x) = x^3 + x - 1$, then

$$I = \inf_n K_n = 0.3458.. \quad \text{and} \quad S = \sup_n K_n = 0.6276..$$

are roots of

$$p(x) = 29791 x^6 - 26908 x^5 + 11532 x^4 - 3968 x^3 - 320 x + 192$$



Denote by $c(\xi)$ infimum of those $C > 0$ such that the inequality

$$|(x\xi - y)x| < C$$

has infinitely many solutions in integers x, y with $y \neq 0$.

THEOREM (HURWITZ): $\sup_{\xi} c(\xi) = \frac{1}{\sqrt{5}}$.

Denote by $c^{\infty}(\xi_1, \xi_2)$ infimum of those $C > 0$ such that the inequality

$$|x\xi_1 + y\xi_2 - z| \max(x^2, y^2) < C$$

has infinitely many solutions in integers x, y, z with $z \neq 0$.

CONJECTURE: $\sup_{(\xi_1, \xi_2)} c^{\infty}(\xi_1, \xi_2) = \frac{2}{7}$.

THEOREM (CUSICK, 1972): Let $\xi = \cos(2\pi/7)$. Then

$$c^{\infty}(\xi, \xi^2) = \frac{4}{49}(\xi^2 + 3\xi - 3) = .1874\dots$$

Consider a polynomial $f(x) = x^3 + a_2x^2 + a_1x - 1$ in $\mathbb{Z}[x]$ irreducible over \mathbb{Q} . Let ξ, ξ_1 and ξ_2 be its roots. Assume that ξ is real and ξ_1, ξ_2 are complex. We also assume that $|\xi| < 1$.

We introduce a sequence of polynomials $P_n(x) = a_n + b_nx + c_nx^2$ in $\mathbb{Z}[x]$, defined by $P_n(x) = x^n \pmod{f}$. Put

$$K_n = |P_n(\xi)|(a_n^2 + b_n^2 + c_n^2)$$

$$\text{and } I = \inf_n K_n, \quad S = \sup_n K_n, \quad \tilde{K}_n = (K_n - I)/S.$$

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Moreover, $I = \inf_n K_n$ and $S = \sup_n K_n$ are roots of $p(x) = c_6x^6 + \dots c_2x^2 + c_1x + c_0$ with

$$c_6 = d^3$$

$$c_5 = 4 (a_2^4 - a_2^3 - 4 a_2^2 a_1 + a_2^2 + 4 a_2 a_1 + a_1^2 - 6 a_2 - 3 a_1 + 9) d^2$$

$$c_4 = -4 (a_2^4 - 4 a_2^2 a_1 + a_2^2 + 2 a_1^2 - 8 a_2 - 2 a_1 + 3) d^2$$

$$c_3 = -16 (a_2^3 a_1^2 + 6 a_2^4 + 6 a_2^2 a_1^2 - 4 a_2 a_1^3 + 2 a_1^4 + 4 a_2^3 - 18 a_2^2 a_1 - 4 a_1^3 + 10 a_2^2 - 10 a_2 a_1 + 10 a_1^2 - 27 a_2) d$$

$$c_2 = 16 (2 a_2^4 + 5 a_2^2 a_1^2 + a_1^4 + 2 a_2^3 - 2 a_1^3 + 8 a_2^2 + a_1^2 - 6 a_2) d$$

$$c_1 = 64 a_2^2 a_1^4 + 64 a_1^6 + 320 a_2^3 a_1^2 + 64 a_2^2 a_1^3 + 384 a_2 a_1^4 - 64 a_1^5 + 320 a_2^4 + 256 a_2^3 a_1 + 704 a_2^2 a_1^2 - 384 a_2 a_1^3 + 128 a_1^4 + 576 a_2^3 - 384 a_2^2 a_1 + 576 a_2 a_1^2 + 192 a_1^3 + 1152 a_2^2$$

$$c_0 = -64 (a_2^2 + a_2 a_1 + a_1^2 + 2 a_2 - 2 a_1 + 4) (a_2^2 - a_2 a_1 + a_1^2),$$

where $d = -4a_1^3 + 4a_2^3 + a_1^2 a_2^2 - 18a_1 a_2 - 27$ is the discriminant of f .

Consider a polynomial $f(x) = x^3 + a_2x^2 + a_1x - 1$ in $\mathbb{Z}[x]$ irreducible over \mathbb{Q} . Let ξ, ξ_1 and ξ_2 be its roots. Assume that ξ is real and ξ_1, ξ_2 are complex. We also assume that $|\xi| < 1$.

We introduce a sequence of polynomials $P_n(x) = a_n + b_nx + c_nx^2$ in $\mathbb{Z}[x]$, defined by $P_n(x) = x^n \pmod{f}$. Since $f(\xi) = f(\xi_1) = f(\xi_2) = 0$, we have

$$\begin{aligned} P_n(\xi) &= \xi^n = a_n + b_n\xi + c_n\xi^2, \\ P_n(\xi_1) &= \xi_1^n = a_n + b_n\xi_1 + c_n\xi_1^2, \\ P_n(\xi_2) &= \xi_2^n = a_n + b_n\xi_2 + c_n\xi_2^2. \end{aligned} \tag{1}$$

In other words,

$$\begin{pmatrix} \xi^n \\ \xi_1^n \\ \xi_2^n \end{pmatrix} = V \begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix}, \quad \text{where } V = \begin{pmatrix} 1 & \xi & \xi^2 \\ 1 & \xi_1 & \xi_1^2 \\ 1 & \xi_2 & \xi_2^2 \end{pmatrix} \tag{2}$$

is the Vandermonde matrix. From (2) it follows that

$$\begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix} = V^{-1} \begin{pmatrix} \xi^n \\ \xi_1^n \\ \xi_2^n \end{pmatrix}. \tag{3}$$

Since ξ, ξ_1 and ξ_2 are roots of $f(x)$ and the last coefficient of f is -1 , it follows that $|\xi\xi_1\xi_2| = 1$. Therefore

$$|\xi|^{-1/2} = |\xi_1| = |\xi_2|, \tag{4}$$

because ξ_1 and ξ_2 are complex conjugates. Applying (4) to (3), we get

$$\begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix} = V^{-1} \xi^{-n/2} \begin{pmatrix} \xi^{3n/2} \\ \cos(n\beta) + i \sin(n\beta) \\ \cos(n\beta) - i \sin(n\beta) \end{pmatrix}, \tag{5}$$

where $\beta = \arg(\xi_1)$. Observe that for large n , $\xi^{3n/2}$ is much smaller than $|\cos(n\beta) \pm i \sin(n\beta)|$, therefore

$$\begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix} \approx V^{-1} \xi^{-n/2} \begin{pmatrix} 0 \\ \cos(n\beta) + i \sin(n\beta) \\ \cos(n\beta) - i \sin(n\beta) \end{pmatrix},$$

which can be rewritten as

$$\xi^{n/2} \begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix} \approx \frac{2i}{D} \begin{pmatrix} \xi^{3/2} \sin((n-1)\beta) - \sin((n-2)\beta) \\ \xi^{-1} \sin((n-2)\beta) - \xi^2 \sin(n\beta) \\ \xi \sin(n\beta) - \xi^{-1/2} \sin((n-1)\beta) \end{pmatrix}, \tag{6}$$

where D is the determinant of V .

Put

$$g_1(n) = \xi^{3/2} \sin((n-1)\beta) - \sin((n-2)\beta),$$

$$g_2(n) = \xi^{-1} \sin((n-2)\beta) - \xi^2 \sin(n\beta),$$

$$g_3(n) = \xi \sin(n\beta) - \xi^{-1/2} \sin((n-1)\beta).$$

From (1) and (6) it follows that

$$(P_n(\xi))^{1/2} a_n \approx \frac{2i}{D} g_1(n),$$

$$(P_n(\xi))^{1/2} b_n \approx \frac{2i}{D} g_2(n),$$

$$(P_n(\xi))^{1/2} c_n \approx \frac{2i}{D} g_3(n).$$

(7)

We have

$$(P_n(\xi))^{1/2} a_n \approx A_1 \sin(n\beta) + B_1 \cos(n\beta)$$

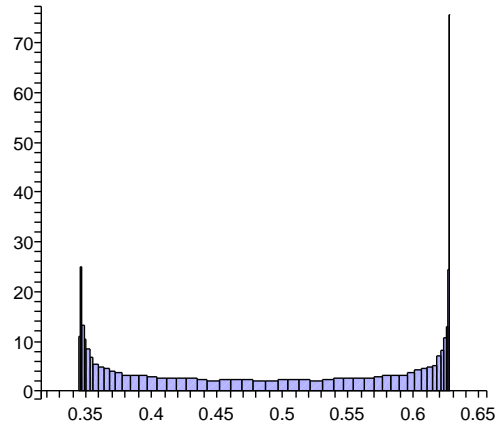
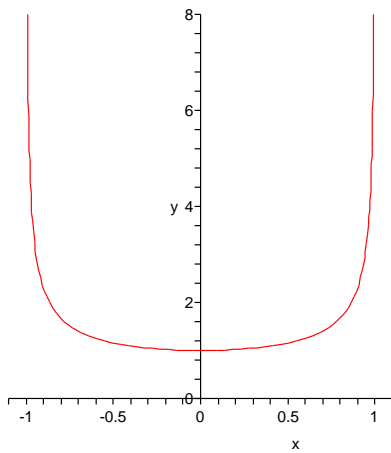
$$(P_n(\xi))^{1/2} b_n \approx A_2 \sin(n\beta) + B_2 \cos(n\beta)$$

$$(P_n(\xi))^{1/2} c_n \approx A_3 \sin(n\beta) + B_3 \cos(n\beta)$$

The density function is

$$\varphi(x) = \frac{1}{\sqrt{A_i^2 + B_i^2 - x^2}}, \quad i = 1, 2, 3.$$

If $A_i^2 + B_i^2 = 1$,



Put

$$\begin{aligned} g_1(n) &= \xi^{3/2} \sin((n-1)\beta) - \sin((n-2)\beta), \\ g_2(n) &= \xi^{-1} \sin((n-2)\beta) - \xi^2 \sin(n\beta), \\ g_3(n) &= \xi \sin(n\beta) - \xi^{-1/2} \sin((n-1)\beta). \end{aligned}$$

From (1) and (6) it follows that

$$\begin{aligned} (P_n(\xi))^{1/2} a_n &\approx \frac{2i}{D} g_1(n), \\ (P_n(\xi))^{1/2} b_n &\approx \frac{2i}{D} g_2(n), \\ (P_n(\xi))^{1/2} c_n &\approx \frac{2i}{D} g_3(n). \end{aligned} \tag{7}$$

To show that the points $(g_1(n), g_2(n), g_3(n))$ trace an ellipse in 3-dimensional space, we note that

$$g_1(n) + \xi g_2(n) + \xi^2 g_3(n) = 0$$

and

$$-\frac{g_1^2(n)}{a^2} + \frac{g_2^2(n)}{b^2} + \frac{g_3^2(n)}{c^2} = 1,$$

where

$$\begin{aligned} a^2 &= \frac{\xi^{3/2} \sin^2 \beta (\xi^3 + 1 - 2\xi^{3/2} \cos \beta)}{\xi^{3/2} + \cos \beta}, \\ b^2 &= -\frac{\sin^2 \beta (\xi^3 + 1 - 2\xi^{3/2} \cos \beta)}{\xi^{1/2} \cos \beta}, \\ c^2 &= \frac{\sin^2 \beta (\xi^3 + 1 - 2\xi^{3/2} \cos \beta)}{\xi(1 + \xi^{3/2} \cos \beta)}. \end{aligned}$$

We now find

$$\inf_n |P_n(\xi)|(a_n^2 + b_n^2 + c_n^2) \quad \text{and} \quad \sup_n |P_n(\xi)|(a_n^2 + b_n^2 + c_n^2).$$

By (7) we have

$$\inf_n |P_n(\xi)|(a_n^2 + b_n^2 + c_n^2) = -\inf_n 4(g_1^2(n) + g_2^2(n) + g_3^2(n))/D^2$$

and

$$\sup_n |P_n(\xi)|(a_n^2 + b_n^2 + c_n^2) = -\sup_n 4(g_1^2(n) + g_2^2(n) + g_3^2(n))/D^2$$

One can see that

$$(g_1(n))^2 + (g_2(n))^2 + (g_3(n))^2 = A \sin(n\beta) \cos(n\beta) + B \cos^2(n\beta) + C,$$

where $A = A(\xi, \beta)$, $B = B(\xi, \beta)$, $C = C(\xi, \beta)$. One can show that the function

$$f(x) = Ax\sqrt{1-x^2} + Bx^2 + C$$

attains its minimum(maximum) values at

$$x_{min} = -\frac{1}{\sqrt{2}} \sqrt{1 - \frac{B}{\sqrt{A^2 + B^2}}} \quad \text{and} \quad x_{max} = \frac{1}{\sqrt{2}} \sqrt{1 + \frac{B}{\sqrt{A^2 + B^2}}}$$

and these values are

$$f_{min} = \frac{B - \sqrt{A^2 + B^2}}{2} + C \quad \text{and} \quad f_{max} = \frac{B + \sqrt{A^2 + B^2}}{2} + C. \tag{8}$$

Using explicit expressions for $A(\xi, \beta)$, $B(\xi, \beta)$ and $C(\xi, \beta)$ in (8) [**here the algebra becomes really ugly and I don't know how to make it more elegant yet**], we deduce that $-\inf_n 4(g_1^2(n)+g_2^2(n)+g_3^2(n))/D^2$ and $-\sup_n 4(g_1^2(n)+g_2^2(n)+g_3^2(n))/D^2$ are roots of the following polynomial:

$$pl = c_6 z^6 + \dots c_2 z^2 + c_1 z + c_0,$$

with

$$c_6 = d^3$$

$$c_5 = 4 (a_2^4 - a_2^3 - 4 a_2^2 a_1 + a_2^2 + 4 a_2 a_1 + a_1^2 - 6 a_2 - 3 a_1 + 9) d^2$$

$$c_4 = -4 (a_2^4 - 4 a_2^2 a_1 + a_2^2 + 2 a_1^2 - 8 a_2 - 2 a_1 + 3) d^2$$

$$c_3 = -16 (a_2^3 a_1^2 + 6 a_2^4 + 6 a_2^2 a_1^2 - 4 a_2 a_1^3 + 2 a_1^4 + 4 a_2^3 - 18 a_2^2 a_1 - 4 a_1^3 + 10 a_2^2 - 10 a_2 a_1 + 10 a_1^2 - 27 a_2) d$$

$$c_2 = 16 (2 a_2^4 + 5 a_2^2 a_1^2 + a_1^4 + 2 a_2^3 - 2 a_1^3 + 8 a_2^2 + a_1^2 - 6 a_2) d$$

$$c_1 = 64 a_2^2 a_1^4 + 64 a_1^6 + 320 a_2^3 a_1^2 + 64 a_2^2 a_1^3 + 384 a_2 a_1^4 - 64 a_1^5 + 320 a_2^4 + 256 a_2^3 a_1 + 704 a_2^2 a_1^2 - 384 a_2 a_1^3 + 128 a_1^4 + 576 a_2^3 - 384 a_2^2 a_1 + 576 a_2 a_1^2 + 192 a_1^3 + 1152 a_2^2$$

$$c_0 = -64 (a_2^2 + a_2 a_1 + a_1^2 + 2 a_2 - 2 a_1 + 4) (a_2^2 - a_2 a_1 + a_1^2),$$

where a_1 and a_2 are coefficients of $f(x) = x^3 + a_2 x^2 + a_1 x - 1$ and

$$d = -4a_1^3 + 4a_2^3 + a_1^2 a_2^2 - 18a_1 a_2 - 27$$

is the discriminant of f . For example,

$$\text{if } f(x) = x^3 + x^2 + x - 1, \text{ then } pl = 64 (11 z^3 + 11 z^2 + z - 1) (121 z^3 - 143 z^2 + 55 z - 7)$$

$$\text{if } f(x) = x^3 + x^2 - 1, \text{ then } pl = 12167 z^6 - 8464 z^5 - 6348 z^4 + 2576 z^3 + 2208 z^2 - 2048 z + 448$$

$$\text{if } f(x) = x^3 + x - 1, \text{ then } pl = 29791 z^6 - 26908 z^5 + 11532 z^4 - 3968 z^3 - 320 z + 192$$

$$\text{if } f(x) = x^3 - x^2 + 2x - 1, \text{ then } pl = 12167 z^6 + 19044 z^4 - 42320 z^3 + 13984 z^2 - 2944 z + 448$$

Consider a polynomial $f(x) = x^3 + a_2x^2 + a_1x - 1$ in $\mathbb{Z}[x]$ irreducible over \mathbb{Q} . Let ξ, ξ_1 and ξ_2 be its roots. Assume that ξ is real and ξ_1, ξ_2 are complex. We also assume that $|\xi| < 1$.

We introduce a sequence of polynomials $P_n(x) = a_n + b_nx + c_nx^2$ in $\mathbb{Z}[x]$, defined by $P_n(x) = x^n \pmod{f}$. Since $f(\xi) = f(\xi_1) = f(\xi_2) = 0$, we have

$$\begin{aligned} P_n(\xi) &= \xi^n = a_n + b_n\xi + c_n\xi^2, \\ P_n(\xi_1) &= \xi_1^n = a_n + b_n\xi_1 + c_n\xi_1^2, \\ P_n(\xi_2) &= \xi_2^n = a_n + b_n\xi_2 + c_n\xi_2^2. \end{aligned} \tag{1}$$

In other words,

$$\begin{pmatrix} \xi^n \\ \xi_1^n \\ \xi_2^n \end{pmatrix} = V \begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix}, \quad \text{where } V = \begin{pmatrix} 1 & \xi & \xi^2 \\ 1 & \xi_1 & \xi_1^2 \\ 1 & \xi_2 & \xi_2^2 \end{pmatrix} \tag{2}$$

is the Vandermonde matrix. From (2) it follows that

$$\begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix} = V^{-1} \begin{pmatrix} \xi^n \\ \xi_1^n \\ \xi_2^n \end{pmatrix}. \tag{3}$$

Since ξ, ξ_1 and ξ_2 are roots of $f(x)$ and the last coefficient of f is -1 , it follows that $|\xi\xi_1\xi_2| = 1$. Therefore

$$|\xi|^{-1/2} = |\xi_1| = |\xi_2|, \tag{4}$$

because ξ_1 and ξ_2 are complex conjugates. Applying (4) to (3), we get

$$\begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix} = V^{-1} \xi^{-n/2} \begin{pmatrix} \xi^{3n/2} \\ \cos(n\beta) + i \sin(n\beta) \\ \cos(n\beta) - i \sin(n\beta) \end{pmatrix}, \tag{5}$$

where $\beta = \arg(\xi_1)$. Observe that for large n , $\xi^{3n/2}$ is much smaller than $|\cos(n\beta) \pm i \sin(n\beta)|$, therefore

$$\begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix} \approx V^{-1} \xi^{-n/2} \begin{pmatrix} 0 \\ \cos(n\beta) + i \sin(n\beta) \\ \cos(n\beta) - i \sin(n\beta) \end{pmatrix},$$

which can be rewritten as

$$\xi^{n/2} \begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix} \approx \frac{2i}{D} \begin{pmatrix} \xi^{3/2} \sin((n-1)\beta) - \sin((n-2)\beta) \\ \xi^{-1} \sin((n-2)\beta) - \xi^2 \sin(n\beta) \\ \xi \sin(n\beta) - \xi^{-1/2} \sin((n-1)\beta) \end{pmatrix}, \tag{6}$$

where D is the determinant of V .

Consider a polynomial $f(x) = x^3 + a_2x^2 + a_1x + 1$ in $\mathbb{Z}[x]$ irreducible over \mathbb{Q} such that $a_1 + a_2 = -3$. Let ξ, ξ_1 and ξ_2 be its roots. Assume that all three roots are real and that the fundamental units are x and $x - 1$. We also assume that $0 < \xi < 1, \xi_1 < 0, \xi_2 > 1$. Then $g(x) = x^3 + (a_2 + 3)x^2 + (a_1 + 2a_2 + 3)x - 1$ is a minimal polynomial of $\xi - 1$. Let γ be such a number that

$$\frac{|\xi_1|}{\xi_2} = \left(\frac{\xi_2 - 1}{1 - \xi_1} \right)^\gamma. \quad (1)$$

Put

$$m = \lfloor n\gamma \rfloor, \quad n \in \mathbb{Z}^+. \quad (2)$$

We introduce a sequence of polynomials $P_n(x) = a_n + b_nx + c_nx^2$ in $\mathbb{Z}[x]$, defined by $P_n(x) = x^n(x-1)^m \pmod{f}$. Since $f(\xi) = f(\xi_1) = f(\xi_2) = 0$, we have

$$\begin{aligned} P_n(\xi) &= \xi^n(\xi - 1)^m = a_n + b_n\xi + c_n\xi^2, \\ P_n(\xi_1) &= \xi_1^n(\xi_1 - 1)^m = a_n + b_n\xi_1 + c_n\xi_1^2, \\ P_n(\xi_2) &= \xi_2^n(\xi_2 - 1)^m = a_n + b_n\xi_2 + c_n\xi_2^2. \end{aligned} \quad (3)$$

In other words,

$$\begin{pmatrix} \xi^n(\xi - 1)^m \\ \xi_1^n(\xi_1 - 1)^m \\ \xi_2^n(\xi_2 - 1)^m \end{pmatrix} = V \begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix}, \quad \text{where } V = \begin{pmatrix} 1 & \xi & \xi^2 \\ 1 & \xi_1 & \xi_1^2 \\ 1 & \xi_2 & \xi_2^2 \end{pmatrix} \quad (4)$$

is the Vandermonde matrix. From (4) it follows that

$$\begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix} = V^{-1} \begin{pmatrix} \xi^n(\xi - 1)^m \\ \xi_1^n(\xi_1 - 1)^m \\ \xi_2^n(\xi_2 - 1)^m \end{pmatrix},$$

so

$$\xi^{n/2}(1 - \xi)^{m/2} \begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix} = V^{-1} \begin{pmatrix} (-1)^m \xi^{3n/2} (1 - \xi)^{3m/2} \\ (-1)^m (\xi^{1/2} \xi_1)^n ((1 - \xi)^{1/2} (1 - \xi_1))^m \\ (\xi^{1/2} \xi_2)^n ((1 - \xi)^{1/2} (\xi_2 - 1))^m \end{pmatrix}. \quad (5)$$

Since ξ, ξ_1 and ξ_2 are roots of $f(x)$ and the last coefficient of f is 1, it follows that $|\xi\xi_1\xi_2| = 1$. Similarly, since $\xi - 1, \xi_1 - 1$ and $\xi_2 - 1$ are roots of $g(x)$ and the last coefficient of g is -1 , it follows that $|(\xi - 1)(\xi_1 - 1)(\xi_2 - 1)| = 1$. Also, for a large m the first entry is much smaller than the last two. From this, (1) and (2) it follows that

$$\xi^{n/2}(1 - \xi)^{m/2} \begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix} \approx (-1)^m V^{-1} \begin{pmatrix} 0 \\ \nu \\ \nu^{-1} \end{pmatrix}, \quad \text{where } \nu = \left(\frac{\xi_2 - 1}{1 - \xi_1} \right)^{\frac{n\gamma - m}{2}}.$$

Since

$$V^{-1} = \frac{1}{D} \begin{pmatrix} \xi_1\xi_2(\xi_2 - \xi_1) & \xi\xi_2(\xi - \xi_2) & \xi\xi_1(\xi_1 - \xi) \\ \xi_1^2 - \xi_2^2 & \xi_2^2 - \xi^2 & \xi^2 - \xi_1^2 \\ \xi_2 - \xi_1 & \xi - \xi_2 & \xi_1 - \xi \end{pmatrix},$$

where D is the determinant of V , we have

$$\xi^{n/2}(1-\xi)^{m/2} \begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix} \approx \frac{(-1)^m}{D} \begin{pmatrix} \xi\xi_2(\xi-\xi_2)\nu - \xi\xi_1(\xi-\xi_1)\nu^{-1} \\ (\xi_2^2 - \xi^2)\nu + (\xi^2 - \xi_1^2)\nu^{-1} \\ (\xi - \xi_2)\nu + (\xi_1 - \xi)\nu^{-1} \end{pmatrix}. \quad (6)$$

Put

$$g_1(n) = \xi\xi_2(\xi - \xi_2)\nu - \xi\xi_1(\xi - \xi_1)\nu^{-1},$$

$$g_2(n) = (\xi_2^2 - \xi^2)\nu + (\xi^2 - \xi_1^2)\nu^{-1},$$

$$g_3(n) = (\xi - \xi_2)\nu + (\xi_1 - \xi)\nu^{-1}.$$

From (3) and (6) it follows that

$$(P_n(\xi))^{1/2}a_n \approx \frac{(-1)^m}{D}g_1(n),$$

$$(P_n(\xi))^{1/2}b_n \approx \frac{(-1)^m}{D}g_2(n), \quad (7)$$

$$(P_n(\xi))^{1/2}c_n \approx \frac{(-1)^m}{D}g_3(n).$$

To show that the points $(g_1(n), g_2(n), g_3(n))$ trace a hyperbola in 3-dimensional space, we note that

$$g_1(n) + \xi g_2(n) + \xi^2 g_3(n) = 0$$

and

$$-\frac{g_1^2(n)}{a^2} + \frac{g_2^2(n)}{b^2} + \frac{g_3^2(n)}{c^2} = 1,$$

where

$$a^2 = \frac{-2D\xi^3(\xi_2 - \xi_1)}{2\xi + \xi_1 + \xi_2}, \quad b^2 = \frac{-2D\xi(\xi_2 - \xi_1)}{\xi_1 + \xi_2}, \quad c^2 = \frac{2D(\xi_2 - \xi_1)}{\xi\xi_1 + \xi\xi_2 + 2\xi_1\xi_2}.$$

Consider a polynomial $f(x) = 2x^3 - 2x^2 - 2x + 1$. It is irreducible over \mathbb{Q} and all its roots $\xi = 0.4030\dots, \xi_1 = -0.8546\dots$ and $\xi_2 = 1.4516\dots$ are real.

We introduce a sequence of polynomials $P_n(x) = a_n + b_n x + c_n x^2$ in $\mathbb{Z}[x]$, defined by

$$P_n(x) = 2^k(x^2 + 12x - 5)^n \pmod{f},$$

were

$$k = \begin{cases} 2\lfloor n/3 \rfloor & \text{if } n \equiv 0 \text{ or } 1 \pmod{3} \\ 2\lfloor n/3 \rfloor + 1 & \text{otherwise.} \end{cases}$$

One can show that P_n have integer coefficients. Since $f(\xi) = f(\xi_1) = f(\xi_2) = 0$, we have

$$P_n(\xi) = 2^k(\xi^2 + 12\xi - 5)^n = a_n + b_n\xi + c_n\xi^2,$$

$$P_n(\xi_1) = 2^k(\xi_1^2 + 12\xi_1 - 5)^n = a_n + b_n\xi_1 + c_n\xi_1^2,$$

$$P_n(\xi_2) = 2^k(\xi_2^2 + 12\xi_2 - 5)^n = a_n + b_n\xi_2 + c_n\xi_2^2.$$

In other words,

$$2^k \begin{pmatrix} (\xi^2 + 12\xi - 5)^n \\ (\xi_1^2 + 12\xi_1 - 5)^n \\ (\xi_2^2 + 12\xi_2 - 5)^n \end{pmatrix} = V \begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix}, \quad \text{where } V = \begin{pmatrix} 1 & \xi & \xi^2 \\ 1 & \xi_1 & \xi_1^2 \\ 1 & \xi_2 & \xi_2^2 \end{pmatrix}$$

is the Vandermonde matrix. From this it follows that

$$\begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix} = 2^k V^{-1} \begin{pmatrix} (\xi^2 + 12\xi - 5)^n \\ (\xi_1^2 + 12\xi_1 - 5)^n \\ (\xi_2^2 + 12\xi_2 - 5)^n \end{pmatrix},$$

so

$$2^{k/2}(\xi^2 + 12\xi - 5)^{n/2} \begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix} = 2^{3k/2} V^{-1} \begin{pmatrix} (\xi^2 + 12\xi - 5)^{3n/2} \\ (\xi_1^2 + 12\xi_1 - 5)^n (\xi^2 + 12\xi - 5)^{n/2} \\ (\xi_2^2 + 12\xi_2 - 5)^n (\xi^2 + 12\xi - 5)^{n/2} \end{pmatrix}$$

$$\approx V^{-1} \begin{pmatrix} 0 \\ [2(\xi_1^2 + 12\xi_1 - 5)(\xi^2 + 12\xi - 5)^{1/2}]^n \\ [2(\xi_2^2 + 12\xi_2 - 5)(\xi^2 + 12\xi - 5)^{1/2}]^n \end{pmatrix}.$$

A straightforward calculation shows that $2(\xi_1^2 + 12\xi_1 - 5)(\xi^2 + 12\xi - 5)^{1/2} \approx -0.9999592172$ and $2(\xi_2^2 + 12\xi_2 - 5)(\xi^2 + 12\xi - 5)^{1/2} \approx 1.000040785$. Note that $1.000040785^{10000} \approx 1.5$. In other words, the drift of the scaled errors will be almost invisible for at least 10000 iterations.