On approximation of real numbers by algebraic numbers of bounded degree

by

K. I. Tsishchanka

<u>Theorem 1</u> **(Dirichlet, 1842): For any real irrational number ξ there exist infinitely many rational numbers p/q such that**

$$\left|\xi - \frac{p}{q}\right| < \frac{1}{q^2}$$

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<u>THEOREM 1′</u>: For any real irrational number ξ there exist infinitely many polynomials $P(x) = ax + b$ with integer coefficients such that

$$|P(\xi)| < c(\xi)\lceil P \rceil^{-1}, \quad \lceil P \rceil = \max\{|a|,\ |b|\}.$$

THEOREM 2: **For any real number ξ which is not rational or quadratic irrational, there exist infinitely many polynomials $P(x) = ax^2 + bx + c$ with integer coefficients such that**

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<u>DEFINITION</u>: **If α is a root of the polynomial equation**

$$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = 0,$$

where a_i's are integers and α satisfies no similar equation of degree $< n$, then α is an algebraic number of degree n.

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<u>EXAMPLE</u>:

number	polynomial	degree
$\frac{p}{q}$	$qx - p$	1
$\frac{1+\sqrt{3}}{2}$	$2x^2 - 2x - 1$	2
$\sqrt{2} + \sqrt{3}$	$x^4 - 10x + 1$	4

<u>Theorem 2</u>: For any real number ξ which is not rational or quadratic irrational, there exist infinitely many polynomials $P(x) = ax^2 + bx + c$ with integer coefficients such that

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<u>Notation</u>: Denote by A_n be the set of algebraic numbers of degree $\leq n$.

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Example: **A_1 is the set of all rational numbers.**

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Notation: **Denote by A_n be the set of algebraic numbers of degree $\leq n$.**

Example: **A_1 is the set of all rational numbers. A_2 is the set of all rational numbers and quadratic irrationals, etc.**

Theorem 1′: **For any real irrational number ξ there exist infinitely many polynomials $P(x) = ax + b$ with integer coefficients such that**

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<u>THEOREM 1″</u>: For any real number $\xi \notin A_1$ there exist infinitely many polynomials $P(x) = ax + b$ with integer coefficients such that

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<u>THEOREM 2</u>: **For any real number $\xi \notin A_2$ there exist infinitely many polynomials $P(x) = ax^2 + bx + c$ with integer coefficients such that**

$$|P(\xi)| < c(\xi, n)\lceil P\rceil^{-2}, \quad \lceil P\rceil = \max\{|a|,\ |b|, |c|\}.$$

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$$|P(\xi)| < c(\xi, n)\overline{|P|}^{-n}, \quad \overline{|P|} = \max\{|a_n|, \ldots, |a_1|, |a_0|\}.$$

<u>Proof</u>:

PROOF: **Let $H > 1$ be some integer.**

<u>Proof</u>: Let $H > 1$ be some integer. Consider all polynomials $P(x) \in Z[x]$ of degree $\leq n$ such that

$$\overline{|P|} = \max\{|a_n|, \dots, |a_1|, |a_0|\} \leq H. \qquad (*)$$

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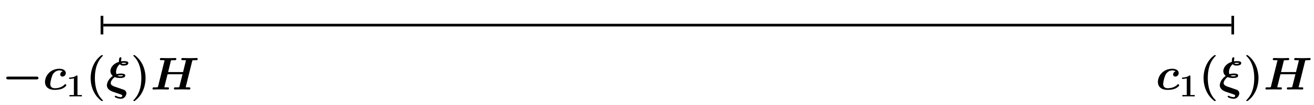

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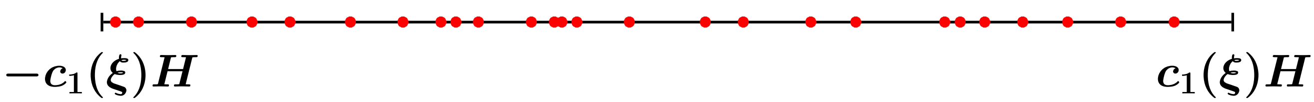

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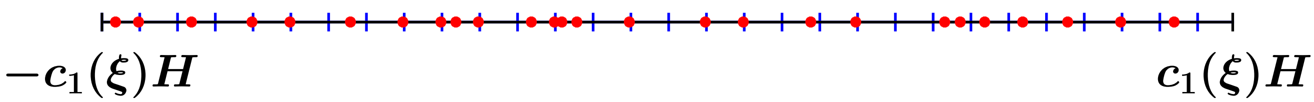

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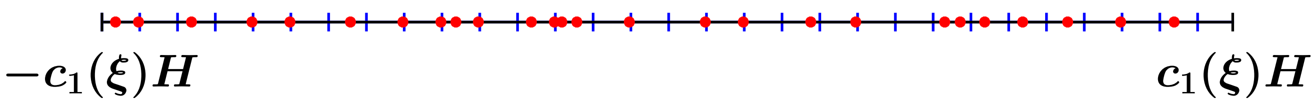

$$\textit{Number of subintervals} = (2H+1)^{n+1} - 1$$

Proof: Let $H > 1$ be some integer. Consider all polynomials $P(x) \in Z[x]$ of degree $\leq n$ such that

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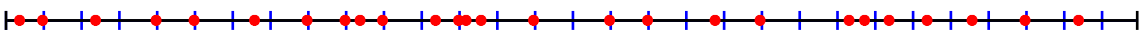

$-c_1(\xi)H$ $\qquad$ $c_1(\xi)H$

$$Number\ of\ subintervals = (2H+1)^{n+1} - 1$$

$$Length = \frac{2c_1(\xi)H}{(2H+1)^{n+1} - 1} < \frac{2c_1(\xi)H}{H^{n+1}} = c_2(\xi)H^{-n}$$

<u>Proof</u>: **Let $H > 1$ be some integer. Consider all polynomials $P(x) \in Z[x]$ of degree $\leq n$ such that**

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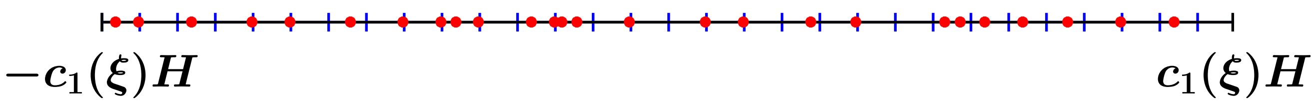

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By the Pigeonhole Principle there exist P_1 and P_2 with

$$|\underbrace{P_1(\xi) - P_2(\xi)}_{P(\xi)}| \leq c_2(\xi)H^{-n} \leq c_2(\xi)\lceil P \rceil^{-n} \ \blacksquare$$

THEOREM 3: **For any real number $\xi \notin A_n$ there exist infinitely many polynomials $P(x) \in Z[x]$ of degree $\leq n$ such that**

$$|P(\xi)| < c(\xi, n)\lceil P \rceil^{-n}.$$

THEOREM 3: **For any real number $\boldsymbol{\xi} \notin \boldsymbol{A_n}$ there exist infinitely many polynomials $\boldsymbol{P(x) \in Z[x]}$ of degree $\boldsymbol{\leq n}$ such that**

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THEOREM 4 (SPRINDŽUK, 1964): **Let $\boldsymbol{\omega}$ be some number with $\boldsymbol{\omega > n}$. Then for almost all real numbers $\boldsymbol{\xi}$ there are only finitely many polynomials $\boldsymbol{P(x) \in Z[x]}$ of degree $\boldsymbol{\leq n}$ such that**

$$|P(\xi)| < \lceil P \rceil^{-\omega}.$$

Theorem 3: **For any real number $\xi \notin A_n$ there exist infinitely many polynomials $P(x) \in Z[x]$ of degree $\leq n$ such that**

$$|P(\xi)| \ll \lceil P \rceil^{-n}.$$

Theorem 4 (Sprindžuk, 1964): **Let ω be some number with $\omega > n$. Then for almost all real numbers ξ there are only finitely many polynomials $P(x) \in Z[x]$ of degree $\leq n$ such that**

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THEOREM 3: **For any real number $\xi \notin A_n$ there exist infinitely many polynomials $P(x) \in Z[x]$ of degree $\leq n$ such that**

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The implicit constant in $\ll$ depends on ξ and n only.

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$$\left|\boldsymbol{\xi} - \frac{\boldsymbol{p}}{\boldsymbol{q}}\right| < \boldsymbol{q}^{-2}$$

$$\left|\xi - \frac{p}{q}\right| < q^{-2} \longrightarrow |q\xi - p| < q^{-1}$$

$$\left|\boldsymbol{\xi} - \frac{\boldsymbol{p}}{\boldsymbol{q}}\right| < \boldsymbol{q}^{-2} \quad \longrightarrow \quad |\boldsymbol{q}\boldsymbol{\xi} - \boldsymbol{p}| < \boldsymbol{q}^{-1}$$

$$\downarrow$$

$$|\boldsymbol{P}(\boldsymbol{\xi})| \ll \lceil \boldsymbol{P} \rceil^{-n}$$

$$
\begin{array}{ccc}
\left|\xi - \dfrac{p}{q}\right| < q^{-2} & \longrightarrow & |q\xi - p| < q^{-1} \\
 & & \downarrow \\
|\xi - \alpha| & \longleftarrow & |P(\xi)| \ll \lceil P \rceil^{-n}
\end{array}
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$$|a_n(\xi - \alpha_1)\ldots(\xi - \alpha_n)| \ll \overline{|P|}^{-n}$$

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$$|x^2 - 1000x + 1000| \ll 1000^{-2} \quad \text{at} \quad \xi = 1.001002003\ldots$$

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$$|\xi - \alpha|^n \ll \overline{|P|}^{-n}$$

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|\xi - \alpha| \ll \overline{|P|}^{-n-1} & \longleftarrow & |P(\xi)| \ll \overline{|P|}^{-n}
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\begin{array}{ccc}
\left|\xi - \dfrac{p}{q}\right| < q^{-2} & \longrightarrow & |q\xi - p| < q^{-1} \\
 & & \big\downarrow \\
|\xi - \alpha| \ll H(\alpha)^{-n-1} & \longleftarrow & |P(\xi)| \ll \lceil P \rceil^{-n}
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$$n = 1 \;\Rightarrow\; |\xi - \alpha| \ll H(\alpha)^{-2} \;\; \text{(Dirichlet,1842)}$$

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$n = 1 \;\Rightarrow\; |\xi - \alpha| \ll H(\alpha)^{-2}$ (Dirichlet,1842)

$n = 2 \;\Rightarrow\; |\xi - \alpha| \ll H(\alpha)^{-3}$ (Davenport – Schmidt, 1967)

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$$n = 1 \;\Rightarrow\; |\xi - \alpha| \ll H(\alpha)^{-2} \text{ (Dirichlet,1842)}$$

$$n = 2 \;\Rightarrow\; |\xi - \alpha| \ll H(\alpha)^{-3} \text{ (Davenport – Schmidt, 1967)}$$

$$n > 2 \;\Rightarrow\; |\xi - \alpha| \ll H(\alpha)^{-\frac{n}{2}-\frac{3}{2}} \text{ (Wirsing, 1961)}$$

Theorem 5 **(Wirsing, 1961): For any real number $\xi \notin A_n$ there exist infinitely many algebraic numbers $\alpha \in A_n$ with**

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Let $n = 2$, then

$$\frac{|P'(x)|}{|P(x)|} = \frac{|[a_2(x-\alpha_1)(x-\alpha_2)]'|}{|a_2(x-\alpha_1)(x-\alpha_2)|} = \frac{|(x-\alpha_1)+(x-\alpha_2)|}{|(x-\alpha_1)(x-\alpha_2)|}$$

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Proof of the Theorem: **Assume to the contrary that there exists a real number $\xi \notin A_n$ such that**

$$|\xi - \alpha| \gg H(\alpha)^{-\frac{n}{2}-\frac{3}{2}} \tag{2}$$

for any algebraic number $\alpha \in A_n$.

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PROOF OF THE THEOREM: **Assume to the contrary that there exists a real number $\xi \notin A_n$ such that**

$$|\xi - \alpha| \gg H(\alpha)^{-\omega}, \quad \omega = \frac{n}{2} + \frac{3}{2} \tag{2}$$

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$$H(\alpha)^{-\omega} \ll \frac{|P(\xi)|}{|P'(\xi)|} \tag{3}$$

<u>THEOREM 5</u> (Wirsing, 1961): For any real number $\boldsymbol{\xi} \notin \boldsymbol{A_n}$ there exist infinitely many algebraic numbers $\boldsymbol{\alpha} \in \boldsymbol{A_n}$ with

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<u>LEMMA 1</u>: We have

$$|\boldsymbol{\xi} - \boldsymbol{\alpha}| \ll \frac{|\boldsymbol{P}(\boldsymbol{\xi})|}{|\boldsymbol{P}'(\boldsymbol{\xi})|} \tag{1}$$

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$$|P'(\xi)| \ll |P(\xi)| \boxed{P}^{\omega} \tag{3}$$

<u>Lemma 2</u>: **There are infinitely many polynomials P, $Q \in Z[x]$ of degree $\leq n$, such that**

$$|P(\xi)| \ll \lceil P \rceil^{-n}$$

$$|Q(\xi)| \ll \lceil P \rceil^{-n}$$

$$\lceil Q \rceil \ll \lceil P \rceil$$

and

P, Q have no common root

LEMMA 3: **Let $P, Q \in Z[x]$ be polynomials of degree d with $1 < d \leq n$. Suppose that P and Q have no common root. Then at least one of the following estimates is true:**

$$1 \ll \max\{|P(\xi)|, |Q(\xi)|\}^2 \max\{\lceil P \rceil, \lceil Q \rceil\}^{2n-2}$$

$$1 \ll \max\left\{|P(\xi)||P'(\xi)||Q'(\xi)|, |Q(\xi)||P'(\xi)|^2\right\} \lceil P \rceil^{n-2} \lceil Q \rceil^{n-1}$$

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<u>Lemma 3</u>: **Let $P, Q \in Z[x]$ be polynomials of degree d with $1 < d \leq n$. Suppose that P and Q have no common root. Then at least one of the following estimates is true:**

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$$\begin{aligned} 1 &\ll \max\{|P(\xi)|, |Q(\xi)|\}^2 \max\{\lceil P \rceil, \lceil Q \rceil\}^{2n-2} \\ &\ll \lceil P \rceil^{-2n} \lceil P \rceil^{2n-2} \end{aligned}$$

LEMMA 3: **Let $P, Q \in Z[x]$ be polynomials of degree d with $1 < d \leq n$. Suppose that P and Q have no common root. Then at least one of the following estimates is true:**

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$$\begin{aligned} 1 &\ll \max\{|P(\xi)|, |Q(\xi)|\}^2 \max\{\lceil P \rceil, \lceil Q \rceil\}^{2n-2} \\ &\ll \lceil P \rceil^{-2n} \lceil P \rceil^{2n-2} \\ &\ll \lceil P \rceil^{-2} \end{aligned}$$

<u>LEMMA 3</u>: **Let $P, Q \in Z[x]$ be polynomials of degree d with $1 < d \leq n$. Suppose that P and Q have no common root. Then at least one of the following estimates is true:**

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$$\begin{aligned} 1 &\ll |P(\xi)||P'(\xi)||Q'(\xi)| \lceil P \rceil^{n-2} \lceil Q \rceil^{n-1} \\ &\ll \lceil P \rceil^{-n} \lceil P \rceil^{\omega-n} \lceil P \rceil^{\omega-n} \lceil P \rceil^{n-2} \lceil P \rceil^{n-1} \\ &\ll \lceil P \rceil^{2\omega-n-3} \end{aligned}$$

<u>LEMMA 3</u>: **Let $P, Q \in Z[x]$ be polynomials of degree d with $1 < d \leq n$. Suppose that P and Q have no common root. Then at least one of the following estimates is true:**

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$$\begin{aligned} 1 &\ll |P(\xi)||P'(\xi)||Q'(\xi)|\lceil P \rceil^{n-2} \lceil Q \rceil^{n-1} \\ &\ll \lceil P \rceil^{-n} \lceil P \rceil^{\omega-n} \lceil P \rceil^{\omega-n} \lceil P \rceil^{n-2} \lceil P \rceil^{n-1} \\ &\ll \lceil P \rceil^{2\omega-n-3} \;\Rightarrow\; 2\omega - n - 3 > 0 \end{aligned}$$

Lemma 3: Let $P, Q \in Z[x]$ be polynomials of degree d with $1 < d \leq n$. Suppose that P and Q have no common root. Then at least one of the following estimates is true:

$$1 \ll \max\left\{|P(\xi)|, |Q(\xi)|\right\}^2 \max\left\{\lceil P \rceil, \lceil Q \rceil\right\}^{2n-2}$$

$$1 \ll \max\left\{|P(\xi)||P'(\xi)||Q'(\xi)|, |Q(\xi)||P'(\xi)|^2\right\} \lceil P \rceil^{n-2} \lceil Q \rceil^{n-1}$$

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LEMMA 3: Let $P, Q \in Z[x]$ be polynomials of degree d with $1 < d \leq n$. Suppose that P and Q have no common root. Then at least one of the following estimates is true:

$$1 \ll \max\{|P(\xi)|, |Q(\xi)|\}^2 \max\left\{\lceil P \rceil, \lceil Q \rceil\right\}^{2n-2}$$

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PROOF: **Let**

$$\boldsymbol{P(x) = a_\ell x^\ell + \ldots + a_1 x + a_0 = a_\ell(x-\alpha_1)\ldots(x-\alpha_\ell)}$$
$$\boldsymbol{Q(x) = b_m x^m + \ldots + b_1 x + b_0 = b_m(x-\beta_1)\ldots(x-\beta_m)}$$

Consider

$$\boldsymbol{a_\ell^m b_m^\ell \prod_{\substack{1 \le i \le \ell \\ 1 \le j \le m}} (\alpha_i - \beta_j)}$$

PROOF: **Let**

$$P(x) = a_\ell x^\ell + \ldots + a_1 x + a_0 = a_\ell (x - \alpha_1) \ldots (x - \alpha_\ell)$$
$$Q(x) = b_m x^m + \ldots + b_1 x + b_0 = b_m (x - \beta_1) \ldots (x - \beta_m)$$

Consider

$$R(P, Q) = a_\ell^m b_m^\ell \prod_{\substack{1 \le i \le \ell \\ 1 \le j \le m}} (\alpha_i - \beta_j)$$

PROOF: **Let**

$$P(x) = a_\ell x^\ell + \ldots + a_1 x + a_0 = a_\ell (x - \alpha_1) \ldots (x - \alpha_\ell)$$
$$Q(x) = b_m x^m + \ldots + b_1 x + b_0 = b_m (x - \beta_1) \ldots (x - \beta_m)$$

Consider

$$R(P, Q) = a_\ell^m b_m^\ell \prod_{\substack{1 \le i \le \ell \\ 1 \le j \le m}} (\alpha_i - \beta_j)$$

LEMMA 3: **Let $P, Q \in Z[x]$ be polynomials of degree d with $1 < d \le n$. Suppose that P and Q have no common root...**

PROOF: **Let**

$$P(x) = a_\ell x^\ell + \ldots + a_1 x + a_0 = a_\ell(x - \alpha_1)\ldots(x - \alpha_\ell)$$
$$Q(x) = b_m x^m + \ldots + b_1 x + b_0 = b_m(x - \beta_1)\ldots(x - \beta_m)$$

Consider

$$R(P, Q) = a_\ell^m b_m^\ell \prod_{\substack{1 \le i \le \ell \\ 1 \le j \le m}} (\alpha_i - \beta_j)$$

<u>Proof</u>: **Let**

$$\begin{aligned} P(x) &= a_\ell x^\ell + \ldots + a_1 x + a_0 = a_\ell (x - \alpha_1) \ldots (x - \alpha_\ell) \\ Q(x) &= b_m x^m + \ldots + b_1 x + b_0 = b_m (x - \beta_1) \ldots (x - \beta_m) \end{aligned}$$

We have

$$R(P, Q) = a_\ell^m b_m^\ell \prod_{\substack{1 \le i \le \ell \\ 1 \le j \le m}} (\alpha_i - \beta_j) \ne 0$$

PROOF: **Let**

$$P(x) = a_\ell x^\ell + \ldots + a_1 x + a_0 = a_\ell (x-\alpha_1)\ldots(x-\alpha_\ell)$$
$$Q(x) = b_m x^m + \ldots + b_1 x + b_0 = b_m (x-\beta_1)\ldots(x-\beta_m)$$

We have

$$|R(P,Q)| = |a_\ell^m b_m^\ell \prod_{\substack{1\le i\le \ell \\ 1\le j\le m}} (\alpha_i - \beta_j)| > 0$$

<u>PROOF</u>: **Let**

$$\begin{aligned}
P(x) &= a_\ell x^\ell + \ldots + a_1 x + a_0 = a_\ell (x-\alpha_1)\ldots(x-\alpha_\ell) \\
Q(x) &= b_m x^m + \ldots + b_1 x + b_0 = b_m (x-\beta_1)\ldots(x-\beta_m)
\end{aligned}$$

We have

$$|R(P,Q)| = |a_\ell^m b_m^\ell \prod_{\substack{1 \le i \le \ell \\ 1 \le j \le m}} (\alpha_i - \beta_j)| > 0$$

$$R(P,Q) = \begin{vmatrix}
a_\ell & \ldots & a_1 & a_0 & \\
\ddots & & \ddots & \ddots & \\
& a_\ell & \ldots & a_1 & a_0 \\
b_m & \ldots & b_1 & b_0 & \\
\ddots & & \ddots & \ddots & \\
& b_m & \ldots & b_1 & b_0
\end{vmatrix}
\begin{matrix} \left.\vphantom{\begin{matrix}a\\a\\a\end{matrix}}\right\} m \\ \left.\vphantom{\begin{matrix}a\\a\\a\end{matrix}}\right\} \ell \end{matrix}$$

Proof: **Let**

$$\boldsymbol{P(x) = a_\ell x^\ell + \ldots + a_1 x + a_0 = a_\ell(x-\alpha_1)\ldots(x-\alpha_\ell)}$$
$$\boldsymbol{Q(x) = b_m x^m + \ldots + b_1 x + b_0 = b_m(x-\beta_1)\ldots(x-\beta_m)}$$

We have

$$\boldsymbol{|R(P,Q)| = |a_\ell^m b_m^\ell \prod_{\substack{1 \le i \le \ell \\ 1 \le j \le m}} (\alpha_i - \beta_j)| \ge 1}$$

<u>PROOF</u>: **Let**

$$\begin{aligned} P(x) &= a_\ell x^\ell + \ldots + a_1 x + a_0 = a_\ell (x-\alpha_1)\ldots(x-\alpha_\ell) \\ Q(x) &= b_m x^m + \ldots + b_1 x + b_0 = b_m (x-\beta_1)\ldots(x-\beta_m) \end{aligned}$$

We have

$$|R(P,Q)| = |a_\ell^m b_m^\ell \prod_{\substack{1 \le i \le \ell \\ 1 \le j \le m}} (\alpha_i - \beta_j)| \geq 1$$

$$1 \ll \max\{|P(\xi)|, |Q(\xi)|\}^2 \max\{\lceil P \rceil, \lceil Q \rceil\}^{2n-2}$$

$$1 \ll \max\left\{|P(\xi)||P'(\xi)||Q'(\xi)|, |Q(\xi)||P'(\xi)|^2\right\} \lceil P \rceil^{n-2} \lceil Q \rceil^{n-1}$$

$$1 \ll \max\left\{|Q(\xi)||P'(\xi)||Q'(\xi)|, |P(\xi)||Q'(\xi)|^2\right\} \lceil P \rceil^{n-1} \lceil Q \rceil^{n-2}$$

PROOF: **Let**

$$\begin{aligned} P(x) &= a_\ell x^\ell + \ldots + a_1 x + a_0 = a_\ell (x-\alpha_1)\ldots(x-\alpha_\ell) \\ Q(x) &= b_m x^m + \ldots + b_1 x + b_0 = b_m (x-\beta_1)\ldots(x-\beta_m) \end{aligned}$$

We have

$$|R(P,Q)| = |a_\ell^m b_m^\ell \prod_{\substack{1\le i\le \ell \\ 1\le j\le m}} (\alpha_i - \beta_j)| \geq 1$$

$$\begin{vmatrix} a_\ell & \ldots & a_1 & a_0 & \\ \ddots & & \ddots & \ddots & \\ & a_\ell & \ldots & a_1 & a_0 \\ b_m & \ldots & b_1 & b_0 & \\ \ddots & & \ddots & \ddots & \\ & b_m & \ldots & b_1 & b_0 \end{vmatrix}$$

PROOF: **Let**

$$P(x) = a_\ell x^\ell + \ldots + a_1 x + a_0 = a_\ell (x-\alpha_1)\ldots(x-\alpha_\ell)$$
$$Q(x) = b_m x^m + \ldots + b_1 x + b_0 = b_m (x-\beta_1)\ldots(x-\beta_m)$$

We have

$$|R(P,Q)| = |a_\ell^m b_m^\ell \prod_{\substack{1\le i\le \ell \\ 1\le j\le m}} (\alpha_i - \beta_j)| \ge 1$$

$$\begin{vmatrix} a_\ell & \ldots & a_1 & a_0 & \\ \ddots & & \ddots & \ddots & \\ & a_\ell & \ldots & a_1 & a_0 \\ b_m & \ldots & b_1 & b_0 & \\ \ddots & & \ddots & \ddots & \\ & b_m & \ldots & b_1 & b_0 \end{vmatrix} \equiv \begin{vmatrix} \frac{P^{(\ell)}(0)}{\ell!} & \ldots & P'(0) & P(0) & \\ \ddots & & \ddots & \ddots & \\ & \frac{P^{(\ell)}(0)}{\ell!} & \ldots & P'(0) & P(0) \\ \frac{Q^{(m)}(0)}{m!} & \ldots & Q'(0) & Q(0) & \\ \ddots & & \ddots & \ddots & \\ & \frac{Q^{(m)}(0)}{m!} & \ldots & Q'(0) & Q(0) \end{vmatrix}$$

<u>Proof</u>: **Let**

$$P(x) = a_\ell x^\ell + \ldots + a_1 x + a_0 = a_\ell (x-\alpha_1)\ldots(x-\alpha_\ell)$$
$$Q(x) = b_m x^m + \ldots + b_1 x + b_0 = b_m (x-\beta_1)\ldots(x-\beta_m)$$

We have

$$|R(P,Q)| = |a_\ell^m b_m^\ell \prod_{\substack{1\le i\le \ell \\ 1\le j\le m}} (\alpha_i - \beta_j)| \ge 1$$

$$\begin{vmatrix} a_\ell & \ldots & a_1 & a_0 & \\ \ddots & & \ddots & \ddots & \\ & a_\ell & \ldots & a_1 & a_0 \\ b_m & \ldots & b_1 & b_0 & \\ \ddots & & \ddots & \ddots & \\ & b_m & \ldots & b_1 & b_0 \end{vmatrix} \equiv \begin{vmatrix} \frac{P^{(\ell)}(\xi)}{\ell!} & \ldots & P'(\xi) & P(\xi) & \\ \ddots & & \ddots & \ddots & \\ & \frac{P^{(\ell)}(\xi)}{\ell!} & \ldots & P'(\xi) & P(\xi) \\ \frac{Q^{(m)}(\xi)}{m!} & \ldots & Q'(\xi) & Q(\xi) & \\ \ddots & & \ddots & \ddots & \\ & \frac{Q^{(m)}(\xi)}{m!} & \ldots & Q'(\xi) & Q(\xi) \end{vmatrix}$$

PROOF: **Let**

$$P(x) = a_\ell x^\ell + \ldots + a_1 x + a_0 = a_\ell (x-\alpha_1)\ldots(x-\alpha_\ell)$$
$$Q(x) = b_m x^m + \ldots + b_1 x + b_0 = b_m (x-\beta_1)\ldots(x-\beta_m)$$

We have

$$|R(P,Q)| = |a_\ell^m b_m^\ell \prod_{\substack{1\le i\le \ell \\ 1\le j\le m}} (\alpha_i - \beta_j)| \ge 1$$

$$\left\| \begin{matrix} a_\ell & \ldots & a_1 & a_0 & \\ \ddots & & \ddots & \ddots & \\ & a_\ell & \ldots & a_1 & a_0 \\ b_m & \ldots & b_1 & b_0 & \\ \ddots & & \ddots & \ddots & \\ & b_m & \ldots & b_1 & b_0 \end{matrix} \right\| \equiv \left\| \begin{matrix} \frac{P^{(\ell)}(\xi)}{\ell!} & \ldots & P'(\xi) & P(\xi) & \\ \ddots & & \ddots & \ddots & \\ & \frac{P^{(\ell)}(\xi)}{\ell!} & \ldots & P'(\xi) & P(\xi) \\ \frac{Q^{(m)}(\xi)}{m!} & \ldots & Q'(\xi) & Q(\xi) & \\ \ddots & & \ddots & \ddots & \\ & \frac{Q^{(m)}(\xi)}{m!} & \ldots & Q'(\xi) & Q(\xi) \end{matrix} \right\|$$

$$\ll \max\{|P(\xi)|, |Q(\xi)|\}^2 \max\{\lceil P \rceil, \lceil Q \rceil\}^{2n-2}$$

PROOF: **Let**

$$\begin{aligned} P(x) &= a_\ell x^\ell + \ldots + a_1 x + a_0 = a_\ell (x-\alpha_1)\ldots(x-\alpha_\ell) \\ Q(x) &= b_m x^m + \ldots + b_1 x + b_0 = b_m (x-\beta_1)\ldots(x-\beta_m) \end{aligned}$$

We have

$$|R(P,Q)| = |a_\ell^m b_m^\ell \prod_{\substack{1\le i\le \ell \\ 1\le j\le m}} (\alpha_i - \beta_j)| \ge 1$$

$$\left\| \begin{matrix} a_\ell & \ldots & a_1 & a_0 & \\ \ddots & & \ddots & \ddots & \\ & a_\ell & \ldots & a_1 & a_0 \\ b_m & \ldots & b_1 & b_0 & \\ \ddots & & \ddots & \ddots & \\ & b_m & \ldots & b_1 & b_0 \end{matrix} \right\| \equiv \left\| \begin{matrix} \frac{P^{(\ell)}(\xi)}{\ell!} & \ldots & P'(\xi) & P(\xi) & \\ \ddots & & \ddots & \ddots & \\ & \frac{P^{(\ell)}(\xi)}{\ell!} & \ldots & P'(\xi) & P(\xi) \\ \frac{Q^{(m)}(\xi)}{m!} & \ldots & Q'(\xi) & Q(\xi) & \\ \ddots & & \ddots & \ddots & \\ & \frac{Q^{(m)}(\xi)}{m!} & \ldots & Q'(\xi) & Q(\xi) \end{matrix} \right\|$$

$$\ll \max\left\{|P(\xi)||P'(\xi)||Q'(\xi)|, |Q(\xi)||P'(\xi)|^2\right\} \lceil P \rceil^{n-2} \lceil Q \rceil^{n-1}$$

PROOF: **Let**

$$\begin{aligned} P(x) &= a_\ell x^\ell + \ldots + a_1 x + a_0 = a_\ell (x-\alpha_1)\ldots(x-\alpha_\ell) \\ Q(x) &= b_m x^m + \ldots + b_1 x + b_0 = b_m (x-\beta_1)\ldots(x-\beta_m) \end{aligned}$$

We have

$$|R(P,Q)| = |a_\ell^m b_m^\ell \prod_{\substack{1\le i\le \ell \\ 1\le j\le m}} (\alpha_i - \beta_j)| \geq 1$$

$$\left\| \begin{matrix} a_\ell & \ldots & a_1 & a_0 & \\ \ddots & & \ddots & \ddots & \\ & a_\ell & \ldots & a_1 & a_0 \\ b_m & \ldots & b_1 & b_0 & \\ \ddots & & \ddots & \ddots & \\ & b_m & \ldots & b_1 & b_0 \end{matrix} \right\| \equiv \left\| \begin{matrix} \frac{P^{(\ell)}(\xi)}{\ell!} & \ldots & P'(\xi) & P(\xi) & \\ \ddots & & \ddots & \ddots & \\ & \frac{P^{(\ell)}(\xi)}{\ell!} & \ldots & P'(\xi) & P(\xi) \\ \frac{Q^{(m)}(\xi)}{m!} & \ldots & Q'(\xi) & Q(\xi) & \\ \ddots & & \ddots & \ddots & \\ & \frac{Q^{(m)}(\xi)}{m!} & \ldots & Q'(\xi) & Q(\xi) \end{matrix} \right\|$$

$$\ll \max\left\{|Q(\xi)||P'(\xi)||Q'(\xi)|, |P(\xi)||Q'(\xi)|^2\right\} \lceil P \rceil^{n-1} \lceil Q \rceil^{n-2}$$

$$\begin{vmatrix}
\frac{P^{(5)}(\xi)}{5!} & \frac{P^{(4)}(\xi)}{4!} & \frac{P^{(3)}(\xi)}{3!} & \frac{P''(\xi)}{2!} & P'(\xi) & P(\xi) & 0 & 0 & 0 & 0 \\
0 & \frac{P^{(5)}(\xi)}{5!} & \frac{P^{(4)}(\xi)}{4!} & \frac{P^{(3)}(\xi)}{3!} & \frac{P''(\xi)}{2!} & P'(\xi) & P(\xi) & 0 & 0 & 0 \\
0 & 0 & \frac{P^{(5)}(\xi)}{5!} & \frac{P^{(4)}(\xi)}{4!} & \frac{P^{(3)}(\xi)}{3!} & \frac{P''(\xi)}{2!} & P'(\xi) & P(\xi) & 0 & 0 \\
0 & 0 & 0 & \frac{P^{(5)}(\xi)}{5!} & \frac{P^{(4)}(\xi)}{4!} & \frac{P^{(3)}(\xi)}{3!} & \frac{P''(\xi)}{2!} & P'(\xi) & P(\xi) & 0 \\
0 & 0 & 0 & 0 & \frac{P^{(5)}(\xi)}{5!} & \frac{P^{(4)}(\xi)}{4!} & \frac{P^{(3)}(\xi)}{3!} & \frac{P''(\xi)}{2!} & P'(\xi) & P(\xi) \\
\frac{Q^{(5)}(\xi)}{5!} & \frac{Q^{(4)}(\xi)}{4!} & \frac{Q^{(3)}(\xi)}{3!} & \frac{Q''(\xi)}{2!} & Q'(\xi) & Q(\xi) & 0 & 0 & 0 & 0 \\
0 & \frac{Q^{(5)}(\xi)}{5!} & \frac{Q^{(4)}(\xi)}{4!} & \frac{Q^{(3)}(\xi)}{3!} & \frac{Q''(\xi)}{2!} & Q'(\xi) & Q(\xi) & 0 & 0 & 0 \\
0 & 0 & \frac{Q^{(5)}(\xi)}{5!} & \frac{Q^{(4)}(\xi)}{4!} & \frac{Q^{(3)}(\xi)}{3!} & \frac{Q''(\xi)}{2!} & Q'(\xi) & Q(\xi) & 0 & 0 \\
0 & 0 & 0 & \frac{Q^{(5)}(\xi)}{5!} & \frac{Q^{(4)}(\xi)}{4!} & \frac{Q^{(3)}(\xi)}{3!} & \frac{Q''(\xi)}{2!} & Q'(\xi) & Q(\xi) & 0 \\
0 & 0 & 0 & 0 & \frac{Q^{(5)}(\xi)}{5!} & \frac{Q^{(4)}(\xi)}{4!} & \frac{Q^{(3)}(\xi)}{3!} & \frac{Q''(\xi)}{2!} & Q'(\xi) & Q(\xi)
\end{vmatrix}$$

$$
\left|\begin{array}{ccccccc|ccc}
\frac{P^{(5)}(\xi)}{5!} & \frac{P^{(4)}(\xi)}{4!} & \frac{P^{(3)}(\xi)}{3!} & \frac{P''(\xi)}{2!} & P'(\xi) & P(\xi) & 0 & 0 & 0 & 0 \\
0 & \frac{P^{(5)}(\xi)}{5!} & \frac{P^{(4)}(\xi)}{4!} & \frac{P^{(3)}(\xi)}{3!} & \frac{P''(\xi)}{2!} & P'(\xi) & P(\xi) & 0 & 0 & 0 \\
0 & 0 & \frac{P^{(5)}(\xi)}{5!} & \frac{P^{(4)}(\xi)}{4!} & \frac{P^{(3)}(\xi)}{3!} & \frac{P''(\xi)}{2!} & P'(\xi) & P(\xi) & 0 & 0 \\
0 & 0 & 0 & \frac{P^{(5)}(\xi)}{5!} & \frac{P^{(4)}(\xi)}{4!} & \frac{P^{(3)}(\xi)}{3!} & \frac{P''(\xi)}{2!} & P'(\xi) & P(\xi) & 0 \\
0 & 0 & 0 & 0 & \frac{P^{(5)}(\xi)}{5!} & \frac{P^{(4)}(\xi)}{4!} & \frac{P^{(3)}(\xi)}{3!} & \frac{P''(\xi)}{2!} & P'(\xi) & P(\xi) \\
\frac{Q^{(5)}(\xi)}{5!} & \frac{Q^{(4)}(\xi)}{4!} & \frac{Q^{(3)}(\xi)}{3!} & \frac{Q''(\xi)}{2!} & Q'(\xi) & Q(\xi) & 0 & 0 & 0 & 0 \\
0 & \frac{Q^{(5)}(\xi)}{5!} & \frac{Q^{(4)}(\xi)}{4!} & \frac{Q^{(3)}(\xi)}{3!} & \frac{Q''(\xi)}{2!} & Q'(\xi) & Q(\xi) & 0 & 0 & 0 \\
0 & 0 & \frac{Q^{(5)}(\xi)}{5!} & \frac{Q^{(4)}(\xi)}{4!} & \frac{Q^{(3)}(\xi)}{3!} & \frac{Q''(\xi)}{2!} & Q'(\xi) & Q(\xi) & 0 & 0 \\
0 & 0 & 0 & \frac{Q^{(5)}(\xi)}{5!} & \frac{Q^{(4)}(\xi)}{4!} & \frac{Q^{(3)}(\xi)}{3!} & \frac{Q''(\xi)}{2!} & Q'(\xi) & Q(\xi) & 0 \\
0 & 0 & 0 & 0 & \frac{Q^{(5)}(\xi)}{5!} & \frac{Q^{(4)}(\xi)}{4!} & \frac{Q^{(3)}(\xi)}{3!} & \frac{Q''(\xi)}{2!} & Q'(\xi) & Q(\xi)
\end{array}\right|
$$

$$
\left|\begin{array}{ccccccc|ccc}
\boxed{P} & \boxed{P} & \boxed{P} & \boxed{P} & \boxed{P} & \boxed{P} & 0 & 0 & 0 & 0 \\
0 & \boxed{P} & \boxed{P} & \boxed{P} & \boxed{P} & \boxed{P} & \boxed{P} & 0 & 0 & 0 \\
0 & 0 & \boxed{P} & \boxed{P} & \boxed{P} & \boxed{P} & \boxed{P} & P(\xi) & 0 & 0 \\
0 & 0 & 0 & \boxed{P} & \boxed{P} & \boxed{P} & \boxed{P} & P'(\xi) & P(\xi) & 0 \\
0 & 0 & 0 & 0 & \boxed{P} & \boxed{P} & \boxed{P} & \frac{P''(\xi)}{2!} & P'(\xi) & P(\xi) \\
\boxed{Q} & \boxed{Q} & \boxed{Q} & \boxed{Q} & \boxed{Q} & \boxed{Q} & 0 & 0 & 0 & 0 \\
0 & \boxed{Q} & \boxed{Q} & \boxed{Q} & \boxed{Q} & \boxed{Q} & \boxed{Q} & 0 & 0 & 0 \\
0 & 0 & \boxed{Q} & \boxed{Q} & \boxed{Q} & \boxed{Q} & \boxed{Q} & Q(\xi) & 0 & 0 \\
0 & 0 & 0 & \boxed{Q} & \boxed{Q} & \boxed{Q} & \boxed{Q} & Q'(\xi) & Q(\xi) & 0 \\
0 & 0 & 0 & 0 & \boxed{Q} & \boxed{Q} & \boxed{Q} & \frac{Q''(\xi)}{2!} & Q'(\xi) & Q(\xi)
\end{array}\right|
$$

$$
\left|\begin{array}{ccccccc|ccc}
\boxed{P} & \boxed{P} & \boxed{P} & \boxed{P} & \boxed{P} & \boxed{P} & 0 & 0 & 0 & 0 \\
0 & \boxed{P} & \boxed{P} & \boxed{P} & \boxed{P} & \boxed{P} & \boxed{P} & 0 & 0 & 0 \\
0 & 0 & \boxed{P} & \boxed{P} & \boxed{P} & \boxed{P} & \boxed{P} & P(\xi) & 0 & 0 \\
0 & 0 & 0 & \boxed{P} & \boxed{P} & \boxed{P} & \boxed{P} & P'(\xi) & P(\xi) & 0 \\
0 & 0 & 0 & 0 & \boxed{P} & \boxed{P} & \boxed{P} & \boxed{P} & P'(\xi) & P(\xi) \\
\boxed{Q} & \boxed{Q} & \boxed{Q} & \boxed{Q} & \boxed{Q} & \boxed{Q} & 0 & 0 & 0 & 0 \\
0 & \boxed{Q} & \boxed{Q} & \boxed{Q} & \boxed{Q} & \boxed{Q} & \boxed{Q} & 0 & 0 & 0 \\
0 & 0 & \boxed{Q} & \boxed{Q} & \boxed{Q} & \boxed{Q} & \boxed{Q} & Q(\xi) & 0 & 0 \\
0 & 0 & 0 & \boxed{Q} & \boxed{Q} & \boxed{Q} & \boxed{Q} & Q'(\xi) & Q(\xi) & 0 \\
0 & 0 & 0 & 0 & \boxed{Q} & \boxed{Q} & \boxed{Q} & \boxed{Q} & Q'(\xi) & Q(\xi)
\end{array}\right|
$$

$$\left|\begin{array}{ccccccc|ccc}
\boxed{P} & \boxed{P} & \boxed{P} & \boxed{P} & \boxed{P} & \boxed{P} & 0 & 0 & 0 & 0 \\
0 & \boxed{P} & \boxed{P} & \boxed{P} & \boxed{P} & \boxed{P} & \boxed{P} & 0 & 0 & 0 \\
0 & 0 & \boxed{P} & \boxed{P} & \boxed{P} & \boxed{P} & \boxed{P} & \boxed{P} & 0 & 0 \\
0 & 0 & 0 & \boxed{P} & \boxed{P} & \boxed{P} & \boxed{P} & P'(\xi) & P(\xi) & 0 \\
0 & 0 & 0 & 0 & \boxed{P} & \boxed{P} & \boxed{P} & \boxed{P} & P'(\xi) & P(\xi) \\
\boxed{Q} & \boxed{Q} & \boxed{Q} & \boxed{Q} & \boxed{Q} & \boxed{Q} & 0 & 0 & 0 & 0 \\
0 & \boxed{Q} & \boxed{Q} & \boxed{Q} & \boxed{Q} & \boxed{Q} & \boxed{Q} & 0 & 0 & 0 \\
0 & 0 & \boxed{Q} & \boxed{Q} & \boxed{Q} & \boxed{Q} & \boxed{Q} & \boxed{Q} & 0 & 0 \\
0 & 0 & 0 & \boxed{Q} & \boxed{Q} & \boxed{Q} & \boxed{Q} & Q'(\xi) & Q(\xi) & 0 \\
0 & 0 & 0 & 0 & \boxed{Q} & \boxed{Q} & \boxed{Q} & \boxed{Q} & Q'(\xi) & Q(\xi)
\end{array}\right|$$

$$
\left\|\begin{array}{ccccccc:ccc}
\overline{|P|} & \overline{|P|} & \overline{|P|} & \overline{|P|} & \overline{|P|} & \overline{|P|} & 0 & 0 & 0 & 0 \\
0 & \overline{|P|} & \overline{|P|} & \overline{|P|} & \overline{|P|} & \overline{|P|} & \overline{|P|} & 0 & 0 & 0 \\
0 & 0 & \overline{|P|} & \overline{|P|} & \overline{|P|} & \overline{|P|} & \overline{|P|} & \overline{|P|} & 0 & 0 \\
0 & 0 & 0 & \overline{|P|} & \overline{|P|} & \overline{|P|} & \overline{|P|} & P'(\xi) & P(\xi) & 0 \\
0 & 0 & 0 & 0 & \overline{|P|} & \overline{|P|} & \overline{|P|} & \overline{|P|} & P'(\xi) & P(\xi) \\
\overline{|Q|} & \overline{|Q|} & \overline{|Q|} & \overline{|Q|} & \overline{|Q|} & \overline{|Q|} & 0 & 0 & 0 & 0 \\
0 & \overline{|Q|} & \overline{|Q|} & \overline{|Q|} & \overline{|Q|} & \overline{|Q|} & \overline{|Q|} & 0 & 0 & 0 \\
0 & 0 & \overline{|Q|} & \overline{|Q|} & \overline{|Q|} & \overline{|Q|} & \overline{|Q|} & \overline{|Q|} & 0 & 0 \\
0 & 0 & 0 & \overline{|Q|} & \overline{|Q|} & \overline{|Q|} & \overline{|Q|} & Q'(\xi) & Q(\xi) & 0 \\
0 & 0 & 0 & 0 & \overline{|Q|} & \overline{|Q|} & \overline{|Q|} & \overline{|Q|} & Q'(\xi) & Q(\xi)
\end{array}\right\|
$$

$$
\ll \max\{|P(\xi)|, |Q(\xi)|\}^2 \max\left\{\overline{|P|}, \overline{|Q|}\right\}^8
$$

$$
\left\| \begin{array}{ccccccc|ccc}
\boxed{P} & \boxed{P} & \boxed{P} & \boxed{P} & \boxed{P} & \boxed{P} & 0 & 0 & 0 & 0 \\
0 & \boxed{P} & \boxed{P} & \boxed{P} & \boxed{P} & \boxed{P} & \boxed{P} & 0 & 0 & 0 \\
0 & 0 & \boxed{P} & \boxed{P} & \boxed{P} & \boxed{P} & \boxed{P} & \boxed{P} & 0 & 0 \\
0 & 0 & 0 & \boxed{P} & \boxed{P} & \boxed{P} & \boxed{P} & P'(\xi) & P(\xi) & 0 \\
0 & 0 & 0 & 0 & \boxed{P} & \boxed{P} & \boxed{P} & \boxed{P} & P'(\xi) & P(\xi) \\
\boxed{Q} & \boxed{Q} & \boxed{Q} & \boxed{Q} & \boxed{Q} & \boxed{Q} & 0 & 0 & 0 & 0 \\
0 & \boxed{Q} & \boxed{Q} & \boxed{Q} & \boxed{Q} & \boxed{Q} & \boxed{Q} & 0 & 0 & 0 \\
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0 & 0 & 0 & \boxed{Q} & \boxed{Q} & \boxed{Q} & \boxed{Q} & Q'(\xi) & Q(\xi) & 0 \\
0 & 0 & 0 & 0 & \boxed{Q} & \boxed{Q} & \boxed{Q} & \boxed{Q} & Q'(\xi) & Q(\xi)
\end{array} \right\|
$$

$$
\ll \max\left\{|P(\xi)||P'(\xi)||Q'(\xi)|, |Q(\xi)||P'(\xi)|^2\right\} \boxed{P}^3 \boxed{Q}^4
$$

$$\left\| \begin{array}{ccccccc|ccc}
\lceil P \rceil & \lceil P \rceil & \lceil P \rceil & \lceil P \rceil & \lceil P \rceil & \lceil P \rceil & 0 & 0 & 0 & 0 \\
0 & \lceil P \rceil & \lceil P \rceil & \lceil P \rceil & \lceil P \rceil & \lceil P \rceil & \lceil P \rceil & 0 & 0 & 0 \\
0 & 0 & \lceil P \rceil & \lceil P \rceil & \lceil P \rceil & \lceil P \rceil & \lceil P \rceil & \lceil P \rceil & 0 & 0 \\
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\lceil Q \rceil & \lceil Q \rceil & \lceil Q \rceil & \lceil Q \rceil & \lceil Q \rceil & \lceil Q \rceil & 0 & 0 & 0 & 0 \\
0 & \lceil Q \rceil & \lceil Q \rceil & \lceil Q \rceil & \lceil Q \rceil & \lceil Q \rceil & \lceil Q \rceil & 0 & 0 & 0 \\
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\end{array} \right\|$$

$$\ll \max\left\{|Q(\xi)||P'(\xi)||Q'(\xi)|, |P(\xi)||Q'(\xi)|^2\right\} \lceil P \rceil^4 \lceil Q \rceil^3$$

Proof: Let

$$P(x) = a_\ell x^\ell + \ldots + a_1 x + a_0 = a_\ell (x - \alpha_1) \ldots (x - \alpha_\ell)$$
$$Q(x) = b_m x^m + \ldots + b_1 x + b_0 = b_m (x - \beta_1) \ldots (x - \beta_m)$$

We have

$$|R(P, Q)| = |a_\ell^m b_m^\ell \prod_{\substack{1 \le i \le \ell \\ 1 \le j \le m}} (\alpha_i - \beta_j)| \ge 1$$

<u>PROOF</u>: Let

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$$|R(P,Q)| \ll \begin{cases} \max\{|P(\xi)|, |Q(\xi)|\}^2 \max\left\{\boxed{P}, \boxed{Q}\right\}^{2n-2} \\ \max\left\{|P(\xi)||P'(\xi)||Q'(\xi)|, |Q(\xi)||P'(\xi)|^2\right\} \boxed{P}^{n-2} \boxed{Q}^{n-1} \\ \max\left\{|Q(\xi)||P'(\xi)||Q'(\xi)|, |P(\xi)||Q'(\xi)|^2\right\} \boxed{P}^{n-1} \boxed{Q}^{n-2} \end{cases}$$

LEMMA 3: Let $P, Q \in Z[x]$ be polynomials of degree d with $1 < d \leq n$. Suppose that P and Q have no common root. Then at least one of the following estimates is true:

$$1 \ll \max\{|P(\xi)|, |Q(\xi)|\}^2 \max\{\lceil P \rceil, \lceil Q \rceil\}^{2n-2}$$

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<u>THEOREM 5</u> **(Wirsing, 1961): For any real number $\xi \notin A_n$ there exist infinitely many algebraic numbers $\alpha \in A_n$ with**

$$|\xi - \alpha| \ll H(\alpha)^{-\frac{n}{2}-\frac{3}{2}}$$

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$$|\xi - \alpha| \ll H(\alpha)^{-\frac{n}{2}-\lambda_n}, \quad \lim_{n\to\infty} \lambda_n = 3$$

n	Th. 5, 1961	Th. 6, 1961	Th. 7, 1993	Conjecture
3	3	3.28	3.5	4
4	3.5	3.82	4.12	5
5	4	4.35	4.71	6
6	4.5	4.87	5.28	7
7	5	5.39	5.84	8
8	5.5	5.9	6.39	9
9	6	6.41	6.93	10
10	6.5	6.92	7.47	11
15	9	9.44	10.09	16
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50	26.5	26.98	27.84	51
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n	Th. 5, 1961	Th. 6, 1961	Th. 7, 1993	Th. 9, 2005	Conjecture
3	3	3.28	3.5	3.73	4
4	3.5	3.82	4.12	4.45	5
5	4	4.35	4.71	5.14	6
6	4.5	4.87	5.28	5.76	7
7	5	5.39	5.84	6.36	8
8	5.5	5.9	6.39	6.93	9
9	6	6.41	6.93	7.50	10
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Consider a sequence of polynomials $P_i \in Z[x]$ of degree $\leq n$ such that

(i) $\frac{1}{2} > |P_1(\xi)| > |P_2(\xi)| > \ldots > |P_i(\xi)| > \ldots$

(ii) $\overline{|P_1|} < \overline{|P_2|} < \ldots < \overline{|P_i|} < \ldots$

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<u>EXAMPLE</u>: **Let** $n = 1$ **and** $\xi = \dfrac{1+\sqrt{5}}{2}$.

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<u>Example</u>: **Let $n = 1$ and $\xi = \dfrac{1+\sqrt{5}}{2}$. Then**

$P_1(x) = x - 2$	$\lvert P_1(\xi)\rvert \approx 0.3819$	$\boxed{P_1} = 2$
$P_2(x) = 2x - 3$	$\lvert P_2(\xi)\rvert \approx 0.2361$	$\boxed{P_2} = 3$
$P_3(x) = 3x - 5$	$\lvert P_3(\xi)\rvert \approx 0.1459$	$\boxed{P_3} = 5$
$P_4(x) = 5x - 8$	$\lvert P_4(\xi)\rvert \approx 0.0902$	$\boxed{P_4} = 8$
$P_5(x) = 8x - 13$	$\lvert P_5(\xi)\rvert \approx 0.0557$	$\boxed{P_5} = 13$
$P_6(x) = 13x - 21$	$\lvert P_6(\xi)\rvert \approx 0.0344$	$\boxed{P_6} = 21$
$P_7(x) = 21x - 34$	$\lvert P_7(\xi)\rvert \approx 0.0213$	$\boxed{P_7} = 34$

Consider a sequence of polynomials $P_i \in Z[x]$ of degree $\leq n$ such that

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LEMMA: **There are infinitely many polynomials P, $Q \in Z[x]$ of degree $\leq n$, such that**

$$|P(\xi)| \ll \lceil P \rceil^{-n}$$

$$|Q(\xi)| \ll \lceil P \rceil^{-n}$$

$$\lceil Q \rceil \ll \lceil P \rceil$$

and

P, Q **have no common root**

Consider a sequence of polynomials $P_i \in Z[x]$ of degree $\leq n$ such that

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<u>Lemma 1</u>: **For any $i \geq 1$ we have**

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<u>Lemma 2</u>: If i is sufficiently large and P_i is irreducible, then

$$1 < |P_i(\xi)|^3 \lceil P_i \rceil^{2\omega + 2n - 3}$$

LEMMA: Let $P, Q \in Z[x]$ be polynomials of degree d with $1 < d \leq n$. Suppose that P and Q have no common root. Then at least one of the following estimates is true:

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PROOF OF THEOREM 5:

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Proof of Theorem 5:

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<u>THEOREM 8</u> **(Minkowski's Linear Forms Theorem): Suppose that β_{ij}, $1 \leq i, j \leq n$, are real numbers with determinant $\pm D$. Suppose that $\lambda_1, \ldots, \lambda_n$ are positive with $\lambda_1 \ldots \lambda_n = D$. Then there exists an integer point x $= (x_1, \ldots, x_n) \neq 0$ such that**

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$$\overline{|G|} \approx |G'(\xi)|$$

THEOREM: For any real number $\xi \notin A_n$ there exist infinitely many algebraic numbers $\alpha \in A_n$ with

$$|\xi - \alpha| \ll H(\alpha)^{-\omega}$$

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$$|\xi - \alpha| \ll \frac{|P(\xi)|}{|P'(\xi)|} \tag{1}$$

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PROOF OF THE THEOREM: **Assume to the contrary that there exists a real number $\xi \notin A_n$ such that**

$$|\xi - \alpha| \gg H(\alpha)^{-\omega} \tag{2}$$

for any algebraic number $\alpha \in A_n$. From (1) and (2) it follows that

$$|P'(\xi)| \ll |P(\xi)| \boxed{P}^{\omega} \tag{3}$$

Consider a sequence of polynomials $P_i \in Z[x]$ of degree $\leq n$ such that

(i) $\frac{1}{2} > |P_1(\xi)| > |P_2(\xi)| > \ldots > |P_i(\xi)| > \ldots$

(ii) $\lceil P_1 \rceil < \lceil P_2 \rceil < \ldots < \lceil P_i \rceil < \ldots$

(iii) for any $P \in Z[x],\ \deg P \leq n,\ P \not\equiv 0,$ with $|P(\xi)| < |P_i(\xi)|$ we have

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Fix any $i \geq 1$. By Minkowski's Linear Forms Theorem there is a polynomial $G(x) = a_n x^n + \ldots + a_1 x + a_0 \in Z[x]$ of degree $\leq n$ having

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<u>Lemma 1</u>: **For any $i \geq 1$ we have**

$$|P_i(\xi)| < \overline{|P_i|}^{-n}$$

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$$1 < |P_i(\xi)|^3 \, \overline{|P_i|}^{2\omega+2n-3}$$

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Proof of Theorem 7:

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where α is the root of P_i closest to ξ.

EXAMPLE: Consider $P(x) = x^2 - 1000x + 1000$. Then

$$|x^2 - 1000x + 1000| \ll 1000^{-2} \quad \text{at} \quad \xi = 1.001002003...$$

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EXAMPLE: For any $\xi \in R$ and any $H > 0$, there exist $a_n, \ldots, a_1, a_0 \in Z$ such that

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\end{aligned}$$

We have

$$\lceil G \rceil \gg |G'(\xi)| \ll |G(\xi)| \lceil G \rceil^{\omega} \ll |P_i(\xi)| \lceil G \rceil^{\omega}$$

therefore

$$\left(|P_i(\xi)|^{-1} \lceil P_{i+1} \rceil^{-n+1} \right)^{1-\omega} \ll |P_i(\xi)|$$

which can be rewritten as

$$|P_i(\xi)| \ll \lceil P_{i+1} \rceil^{-(n-1)\frac{\omega-1}{\omega-2}}$$

Consider a sequence of polynomials $P_i \in Z[x]$ of degree $\leq n$ such that

(i) $\frac{1}{2} > |P_1(\xi)| > |P_2(\xi)| > \ldots > |P_i(\xi)| > \ldots$

(ii) $\overline{|P_1|} < \overline{|P_2|} < \ldots < \overline{|P_i|} < \ldots$

(iii) for any $P \in Z[x],\ \deg P \leq n,\ P \not\equiv 0,$ with $|P(\xi)| < |P_i(\xi)|$ we have

$$\overline{|P|} \geq \overline{|P_{i+1}|}.$$

Fix any $i \geq 1$. By Minkowski's Linear Forms Theorem there is a polynomial $G(x) = a_n x^n + \ldots + a_1 x + a_0 \in Z[x]$ of degree $\leq n$ having

$$\begin{aligned}
&|G(\xi)| < |P_i(\xi)| \\
&|a_n| \leq c^{-1}\overline{|P_{i+1}|} \\
&\ldots \\
&|a_3| \leq c^{-1}\overline{|P_{i+1}|} \\
&|a_2| \leq c^{(n-2)/2}|P_i(\xi)|^{-1/2}\overline{|P_{i+1}|}^{-(n-2)/2} \\
&|a_1| \leq c^{(n-2)/2}|P_i(\xi)|^{-1/2}\overline{|P_{i+1}|}^{-(n-2)/2}
\end{aligned}$$

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(i) $\frac{1}{2} > |P_1(\xi)| > |P_2(\xi)| > \ldots > |P_i(\xi)| > \ldots$

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\end{aligned}$$

Polynomials

$$P_i(x), \quad P_{i+1}(x), \quad G(x)$$

are linearly independent.

$$\boldsymbol{P_i(x) = 100x^5 + 100x^4 + 100x^3 + 100x^2 + 100x + 100}$$

$$\boldsymbol{P_{i+1}(x) = 200x^5 + 200x^4 + 200x^3 + 200x^2 + 200x + 200}$$
$$\boldsymbol{P_i(x) = 100x^5 + 100x^4 + 100x^3 + 100x^2 + 100x + 100}$$

$$\boldsymbol{G_1(x) = 190x^5 + 300x^4 + 300x^3 + 300x^2 + 300x + 300}$$

$$\boldsymbol{P_{i+1}(x) = 200x^5 + 200x^4 + 200x^3 + 200x^2 + 200x + 200}$$

$$\boldsymbol{P_i(x) = 100x^5 + 100x^4 + 100x^3 + 100x^2 + 100x + 100}$$

$$\boldsymbol{G_2(x) = 190x^5 + 290x^4 + 400x^3 + 400x^2 + 400x + 400}$$

$$\boldsymbol{G_1(x) = 190x^5 + 300x^4 + 300x^3 + 300x^2 + 300x + 300}$$

$$\boldsymbol{P_{i+1}(x) = 200x^5 + 200x^4 + 200x^3 + 200x^2 + 200x + 200}$$

$$\boldsymbol{P_i(x) = 100x^5 + 100x^4 + 100x^3 + 100x^2 + 100x + 100}$$

$$\boldsymbol{G_3(x) = 190x^5 + 290x^4 + 390x^3 + 500x^2 + 500x + 500}$$

$$\boldsymbol{G_2(x) = 190x^5 + 290x^4 + 400x^3 + 400x^2 + 400x + 400}$$

$$\boldsymbol{G_1(x) = 190x^5 + 300x^4 + 300x^3 + 300x^2 + 300x + 300}$$

$$\boldsymbol{P_{i+1}(x) = 200x^5 + 200x^4 + 200x^3 + 200x^2 + 200x + 200}$$

$$\boldsymbol{P_i(x) = 100x^5 + 100x^4 + 100x^3 + 100x^2 + 100x + 100}$$

Using

$$P_i(x), \quad P_{i+1}(x), \quad G_1(x), \ldots, G_{n-2}(x)$$

we construct $L_i(x) \in Z[x]$ of degree $\leq n$ such that

$$|L_i(\xi)| < |P_{i-1}(\xi)|^{\frac{1}{\omega-1}} \boxed{P_{i-1}}^{-n+1}$$

$$\boxed{L_i} < |P_{i-1}(\xi)|^{3-\omega-\frac{\omega-2}{\omega-1}} \boxed{P_i}^{-(n-2)(\omega-1)} \boxed{P_{i-1}}^{-n+2}$$

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$$\overline{|L_i|} < |P_{i-1}(\xi)|^{3-\omega-\frac{\omega-2}{\omega-1}} \overline{|P_i|}^{-(n-2)(\omega-1)} \overline{|P_{i-1}|}^{-n+2}$$

$$\overline{|L_i|} \approx |L_i'(\xi)|$$

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$$|L_i(\xi)| < |P_{i-1}(\xi)|^{\frac{1}{\omega-1}} \left|\overline{P_{i-1}}\right|^{-n+1}$$

$$\left|\overline{L_i}\right| < |P_{i-1}(\xi)|^{3-\omega-\frac{\omega-2}{\omega-1}} \left|\overline{P_i}\right|^{-(n-2)(\omega-1)} \left|\overline{P_{i-1}}\right|^{-n+2}$$

$$\left|\overline{L_i}\right| \approx |L_i'(\xi)|$$

$$1 < |P_i(\xi)| \left|\overline{P_{i+1}}\right|^{\frac{2\omega+n-2}{3}(1-\varrho)} \left|\overline{P_i}\right|^{\frac{n-1}{3}(1-\varrho)+\Phi(n)\varrho}$$

Using

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$$\Phi(n) = 2\omega + 2n - 3 - 2(n-1)\frac{\omega-1}{\omega-2}$$

Using

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$$\Phi(n) = 2\omega + 2n - 3 - 2(n-1)\frac{\omega-1}{\omega-2}$$

$$|L_i(\xi)| \gg \overline{|L_i|}^{-\omega+1}$$

<u>THEOREM 9</u>: **For any real number $\xi \notin A_n$ there exist infinitely many algebraic numbers $\alpha \in A_n$ with**

$$|\xi - \alpha| \ll H(\alpha)^{-\omega},$$

where

$$\omega = \frac{n}{2} + \lambda_n, \quad \lim_{n\to\infty} \lambda_n = 4$$

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$$4x^5 - (4n+18)x^4 + (n^2+11n+30)x^3 - (2n^2+10n+22)x^2 \\ + (2n^2+7n+4)x + n^2 - 5n + 2 \quad \text{if} \quad n = 3, 4, 5$$

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$$2x^5 - (n + 12)x^4 + (2n + 30)x^3 + (2n - 41)x^2 - (3n - 29)x + 2n - 10 \quad \text{if} \quad n > 5.$$

n	Th. 5, 1961	Th. 6, 1961	Th. 7, 1993	Th. 9, 2005	Conjecture
3	3	3.28	3.5	3.73	4
4	3.5	3.82	4.12	4.45	5
5	4	4.35	4.71	5.14	6
6	4.5	4.87	5.28	5.76	7
7	5	5.39	5.84	6.36	8
8	5.5	5.9	6.39	6.93	9
9	6	6.41	6.93	7.50	10
10	6.5	6.92	7.47	8.06	11
15	9	9.44	10.09	10.77	16
20	11.5	11.95	12.67	13.40	21
50	26.5	26.98	27.84	28.70	51
100	51.5	51.99	52.92	53.84	101

<u>THEOREM 10</u> **(Davenport - Schmidt, 1968): Let $n \geq 3$. Let ξ be real, but not algebraic of degree ≤ 2. Then there are infinitely many algebraic integers α of degree ≤ 3 which satisfy**

$$0 < |\xi - \alpha| \ll H(\alpha)^{-\eta}, \quad \eta = \frac{1}{2}(3 + \sqrt{5}) = 2.618...$$

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<u>Theorem 11</u> (Roy, 2001): There exist real numbers ξ such that for any algebraic integer α of degree ≤ 3, we have

$$|\xi - \alpha| \gg H(\alpha)^{-\eta}$$

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$$\boldsymbol{\xi} = \cfrac{\mathbf{1}}{\mathbf{3} + \cfrac{\mathbf{1}}{\mathbf{2}+\cfrac{\mathbf{1}}{\mathbf{4}+\cfrac{\mathbf{1}}{\mathbf{2}+\cfrac{\mathbf{1}}{\mathbf{2}+\cfrac{\mathbf{1}}{\mathbf{4}+\cfrac{\mathbf{1}}{\mathbf{2}+\cfrac{\mathbf{1}}{\mathbf{4}+\cfrac{\mathbf{1}}{\mathbf{2}+\cfrac{\mathbf{1}}{\mathbf{2}+\cfrac{\mathbf{1}}{\mathbf{4}+}}}}}}}}}}}$$

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