A MATHEMATICAL PROBLEM: EASY OR DIFFUCULT?

Kiryl Tsishchanka

kit@knox.edu

Department of Mathematics

Knox College

DEFINITION:

A prime number is a positive integer greater than 1 that is divisible by no positive integers other than 1 and itself.

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$6 = 3 + 3$	$16 = 3 + 13$
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$4 = 2 + 2$	$14 = 3 + 11$	$24 = 3 + 11$
$6 = 3 + 3$	$16 = 3 + 13$	$26 = 7 + 19$
$8 = 3 + 5$	$18 = 5 + 13$	$28 = 5 + 23$
$10 = 3 + 7$	$20 = 7 + 13$	$30 = 7 + 23$
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$4 = 2 + 2$	$14 = 3 + 11$	$24 = 3 + 11$	$34 = 3 + 31$
$6 = 3 + 3$	$16 = 3 + 13$	$26 = 7 + 19$	$36 = 5 + 31$
$8 = 3 + 5$	$18 = 5 + 13$	$28 = 5 + 23$	$38 = 7 + 31$
$10 = 3 + 7$	$20 = 7 + 13$	$30 = 7 + 23$	$40 = 3 + 37$
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$12=5+7$	$22=5+17$	$32=3+29$	$42=5+37$

Goldbach's Conjecture: Every even positive integer greater that 2 can be written as the sum of two primes.

TRUE OR NOT TRUE?

Numbers 8 and 9 are the only consecutive powers.

Cathalan's Conjecture:

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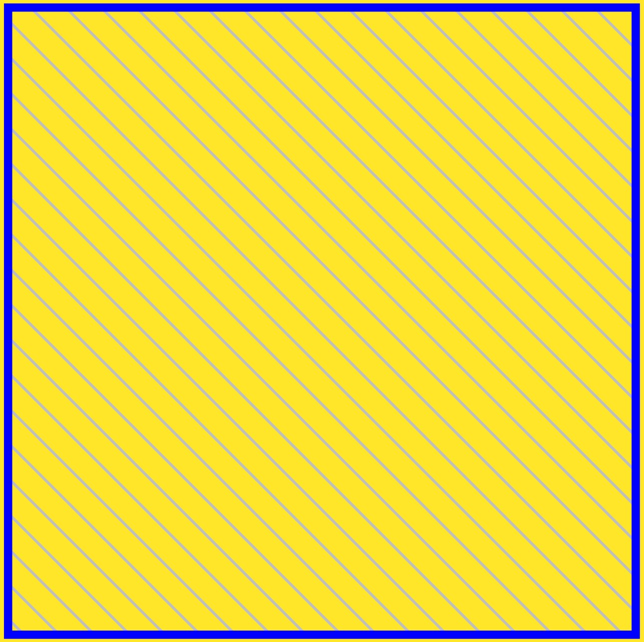
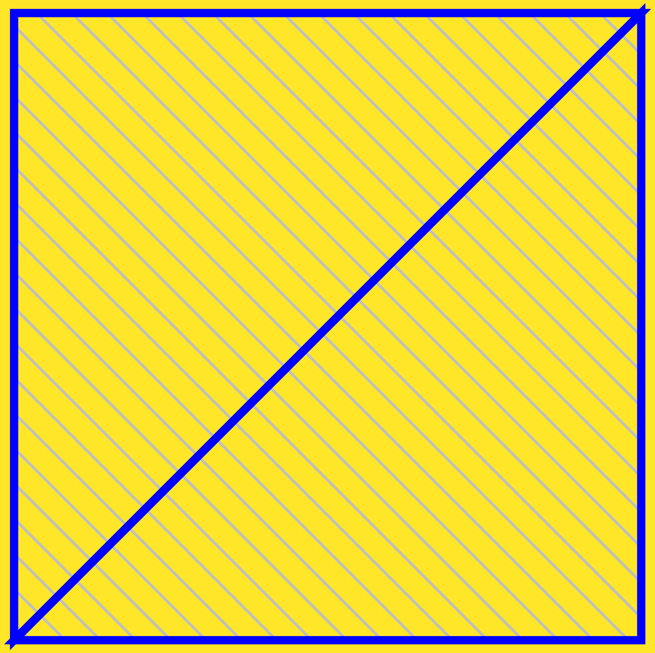
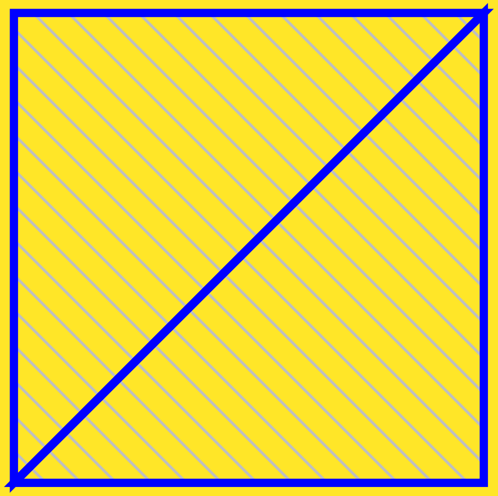
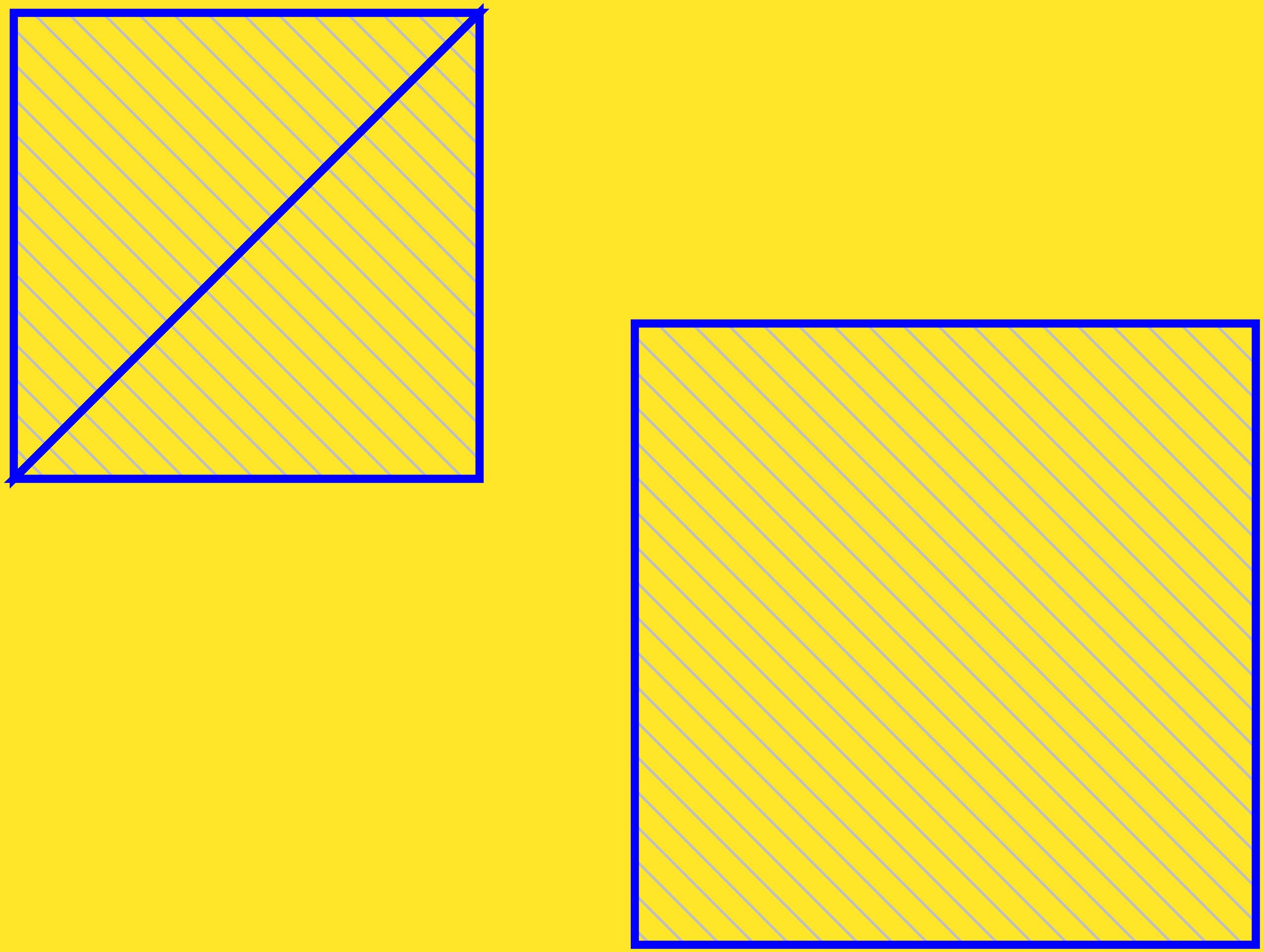
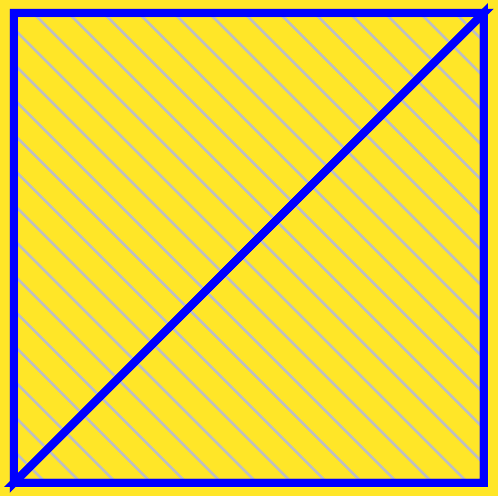
TRUE (Michalesku, 2002).

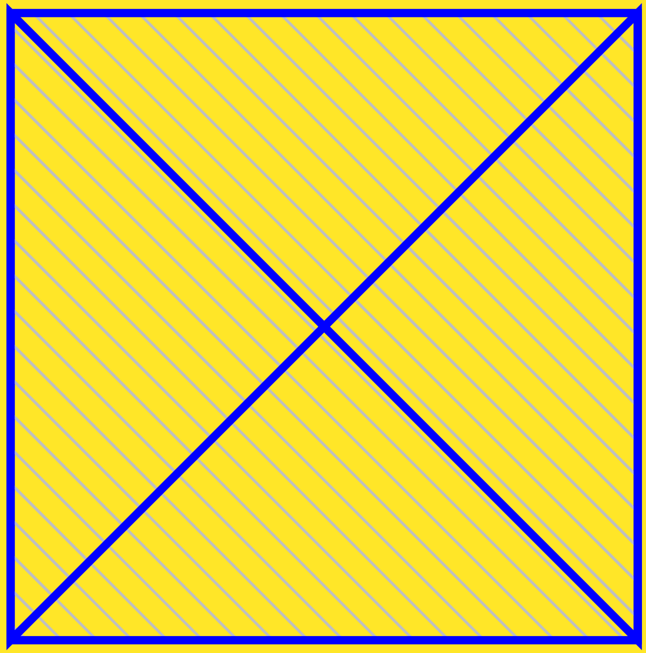


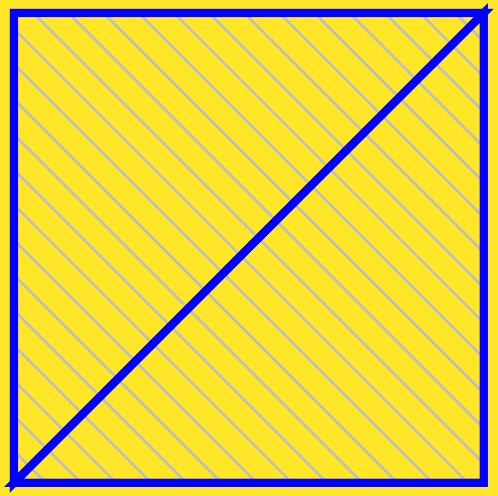


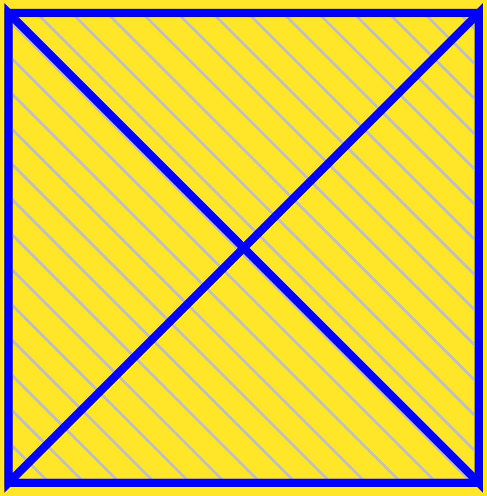


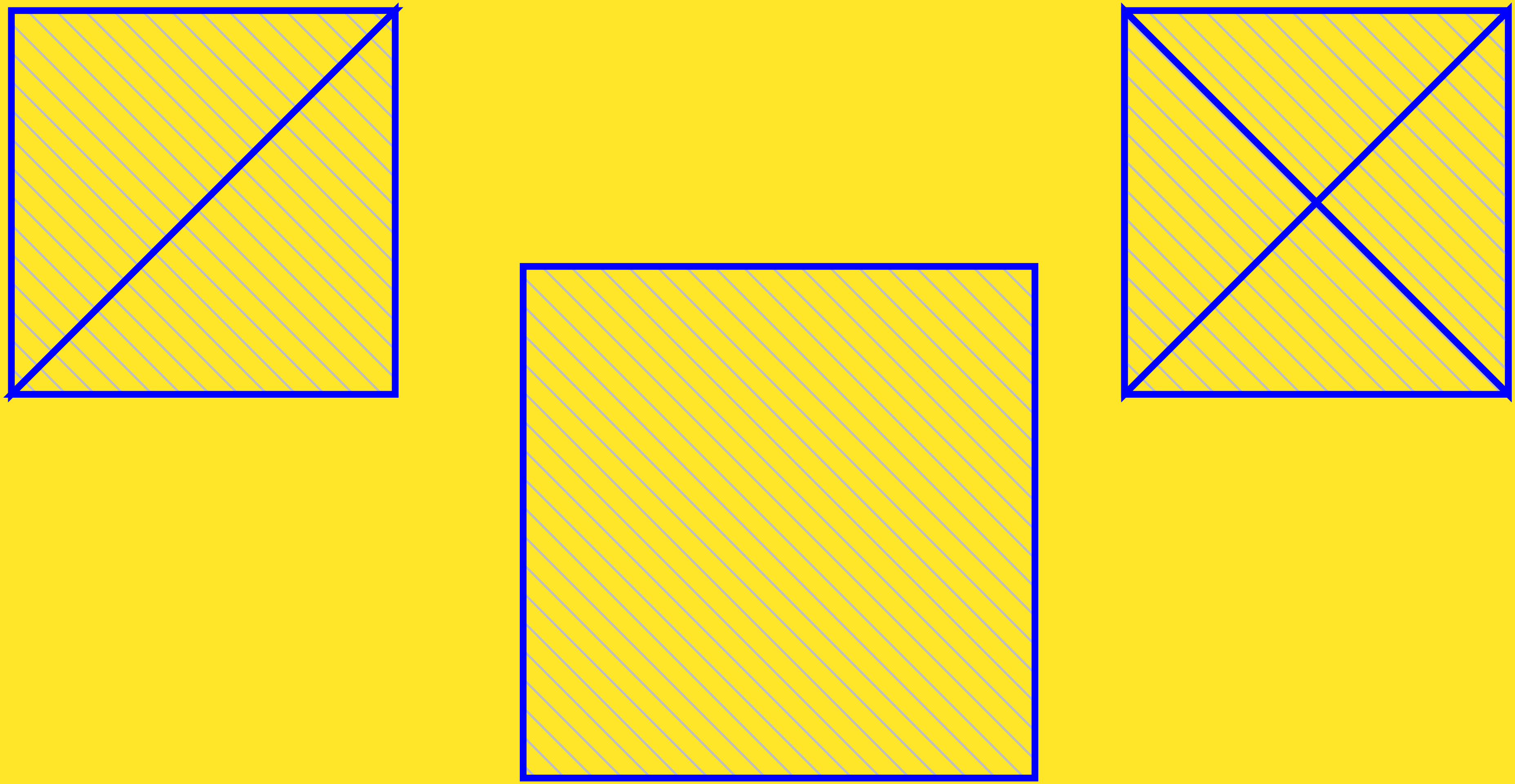


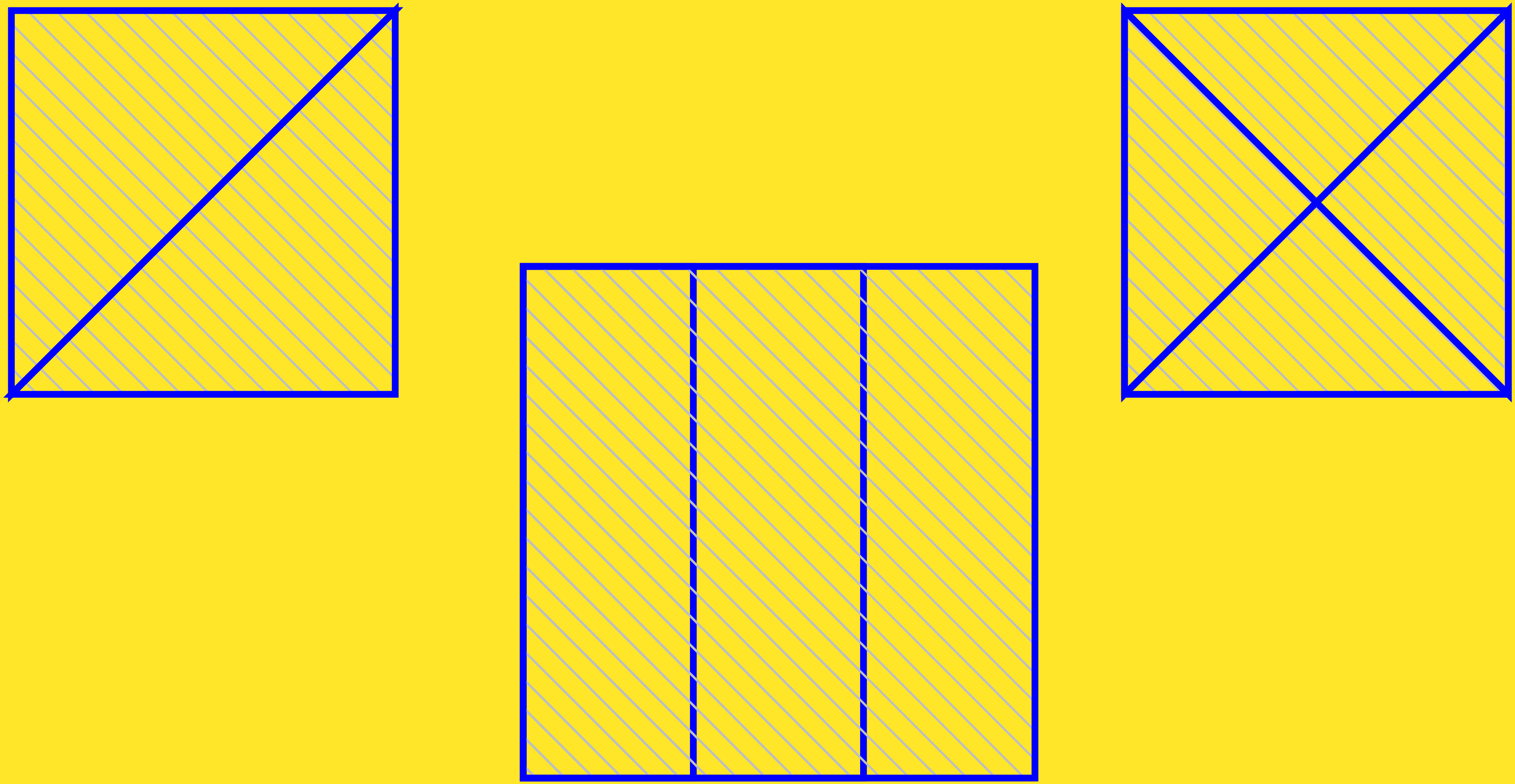


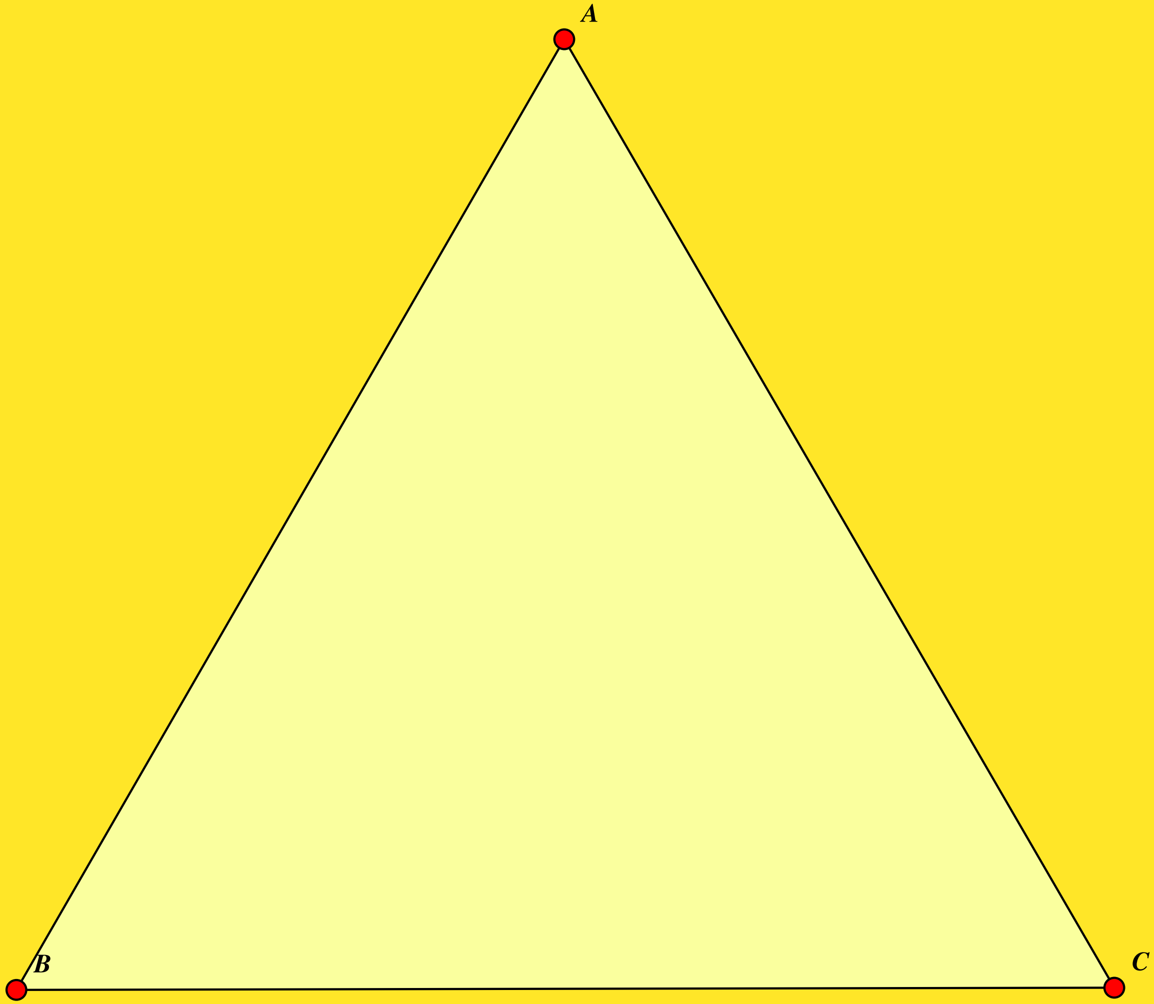


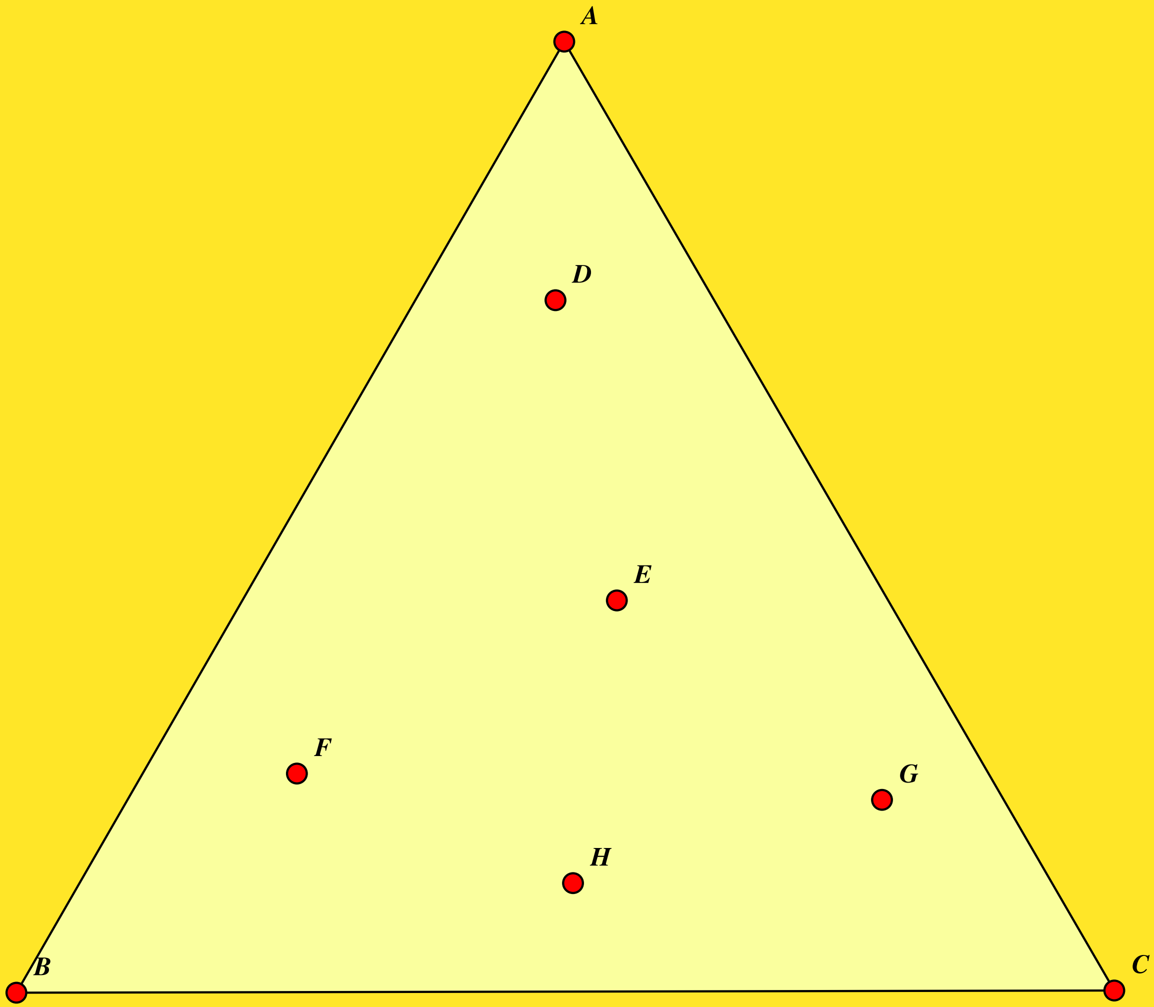









A
B
C


A
D
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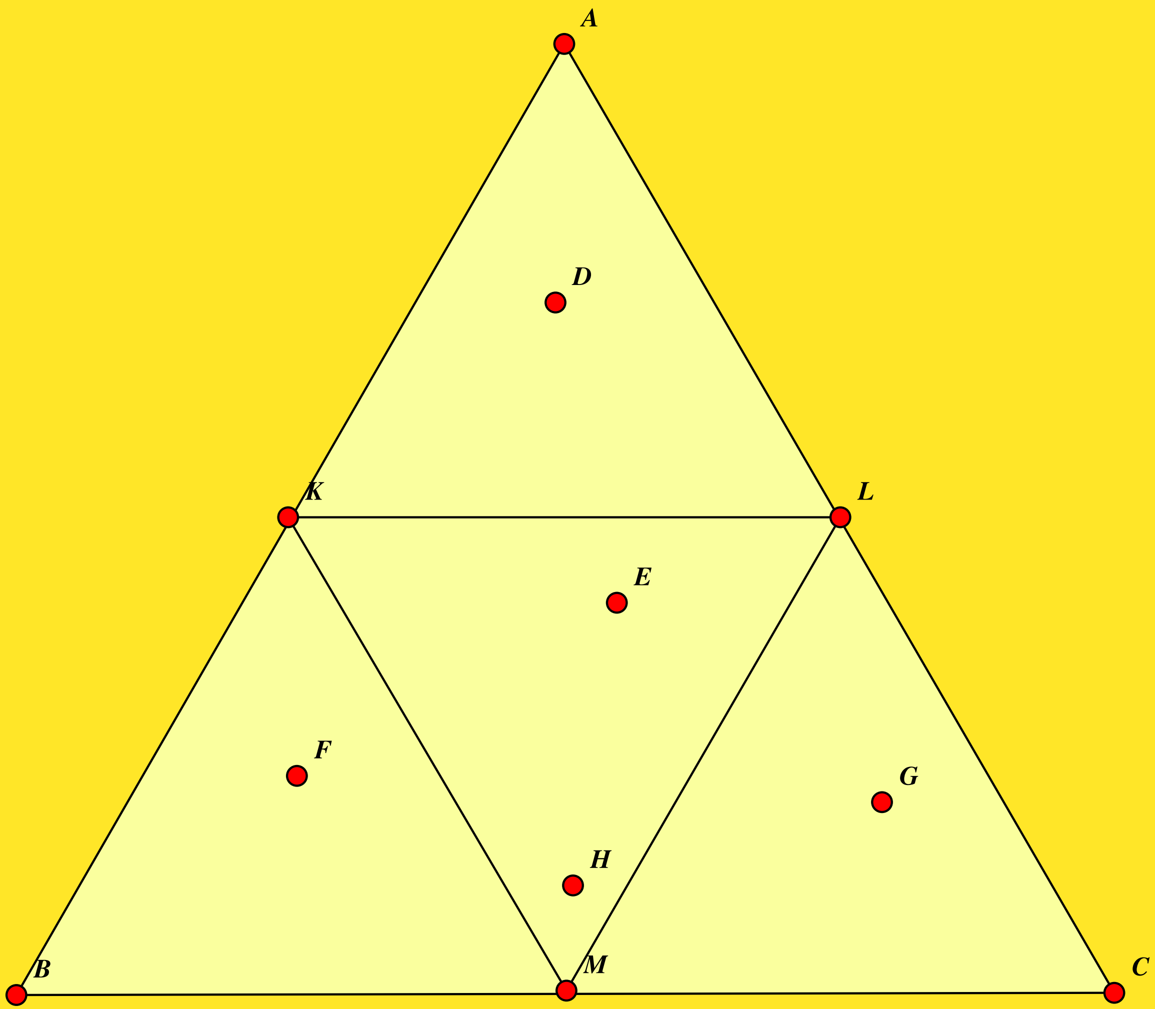
A
D
K
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M
C

$$\lim_{x \to 8^+} \frac{1}{x - 8}$$

$$\lim_{x \to 8^+} \frac{1}{x - 8} = +\infty$$

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$$\lim_{x\to 8^+} \frac{\mathbf{1}}{\boldsymbol{x} - \mathbf{8}} = +\infty$$

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Leonhard Euler (1707-1783)

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Consider the following weights:

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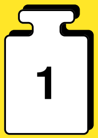

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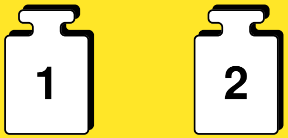

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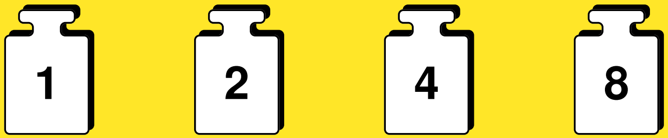

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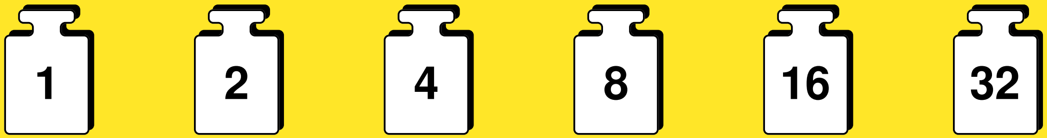

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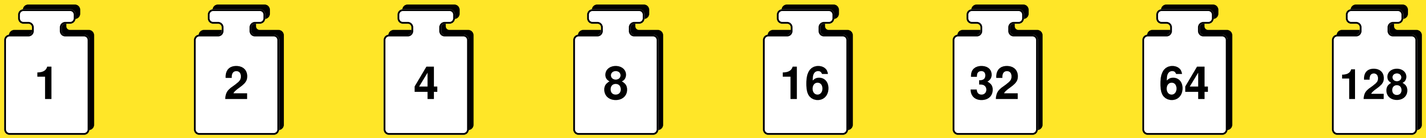

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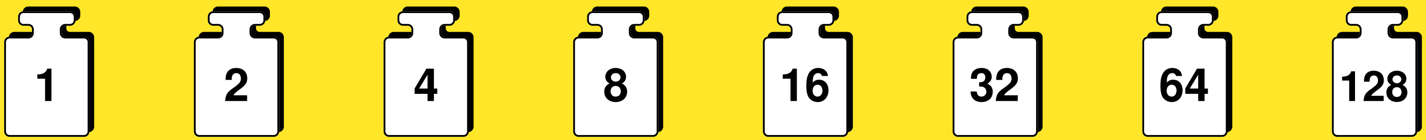

etc.

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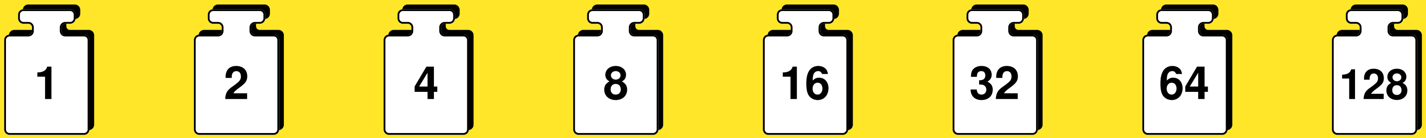

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Which weight can be balanced by them?

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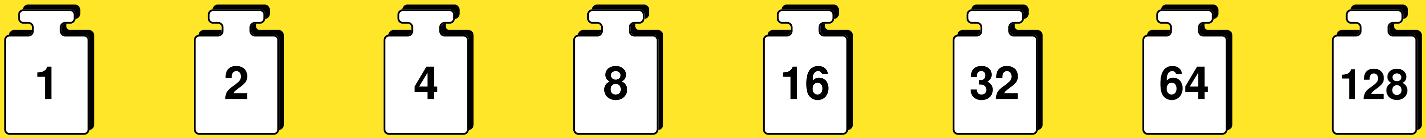

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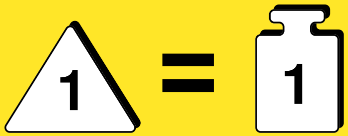

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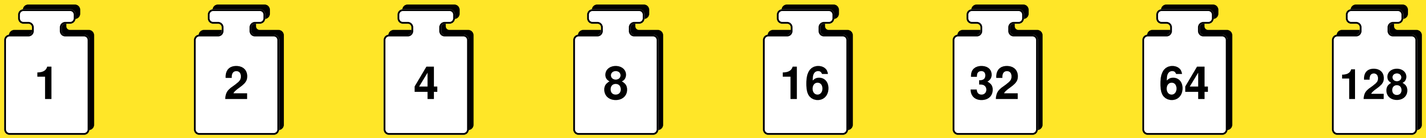

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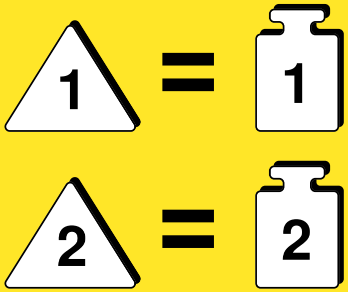

PROBLEM:

Consider the following weights:

1 2 4 8 16 32 64 128 **etc.**

Which weight can be balanced by them?

$1 = 1$

$2 = 2$

$3 = 1 + 2$

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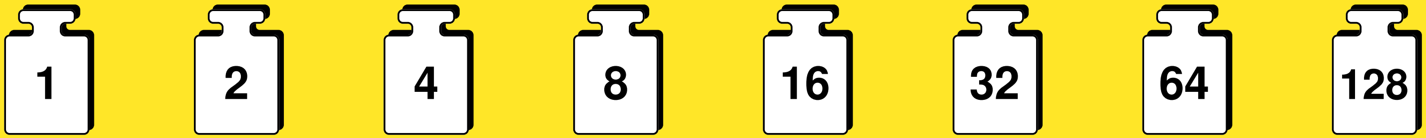

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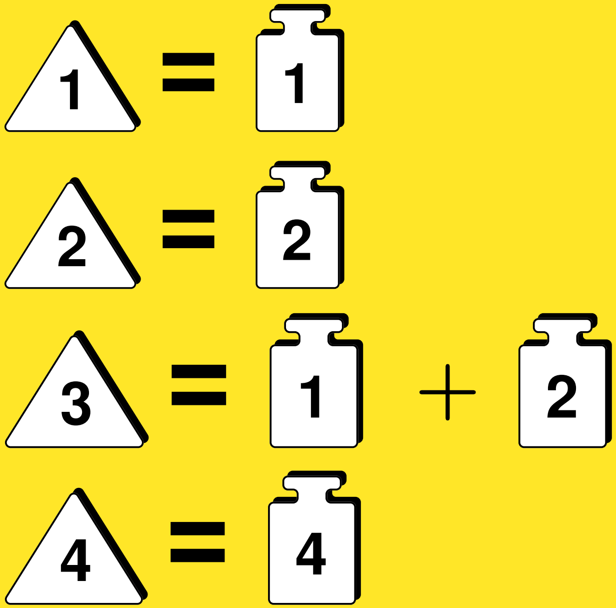

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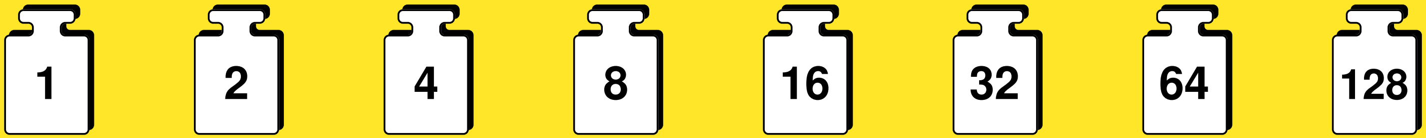

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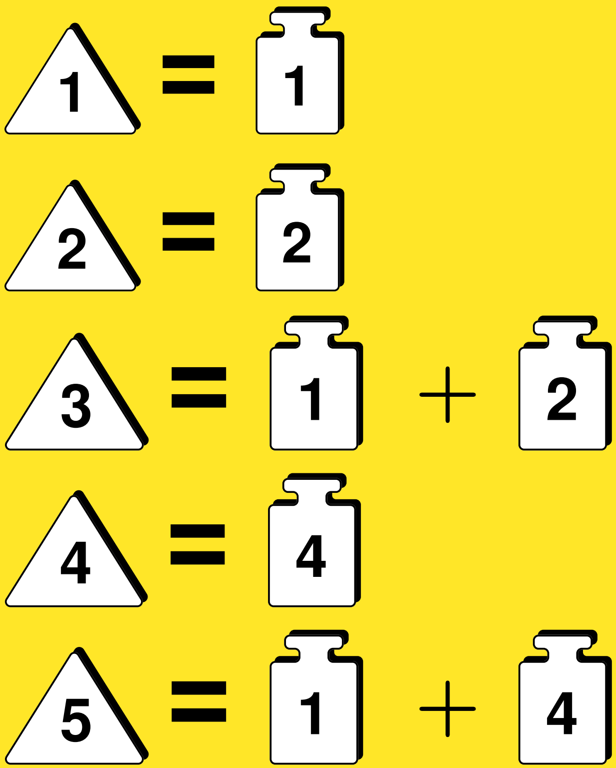

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Which weight can be balanced by them?

1 = 1

2 = 2

3 = 1 + 2

4 = 4

5 = 1 + 4

6 = 2 + 4

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4 = 4

5 = 1 + 4

6 = 2 + 4

7 = 1 + 2 + 4

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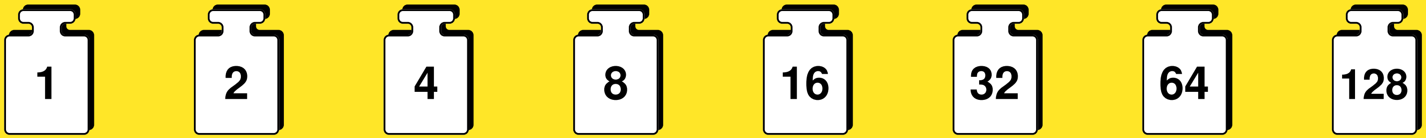

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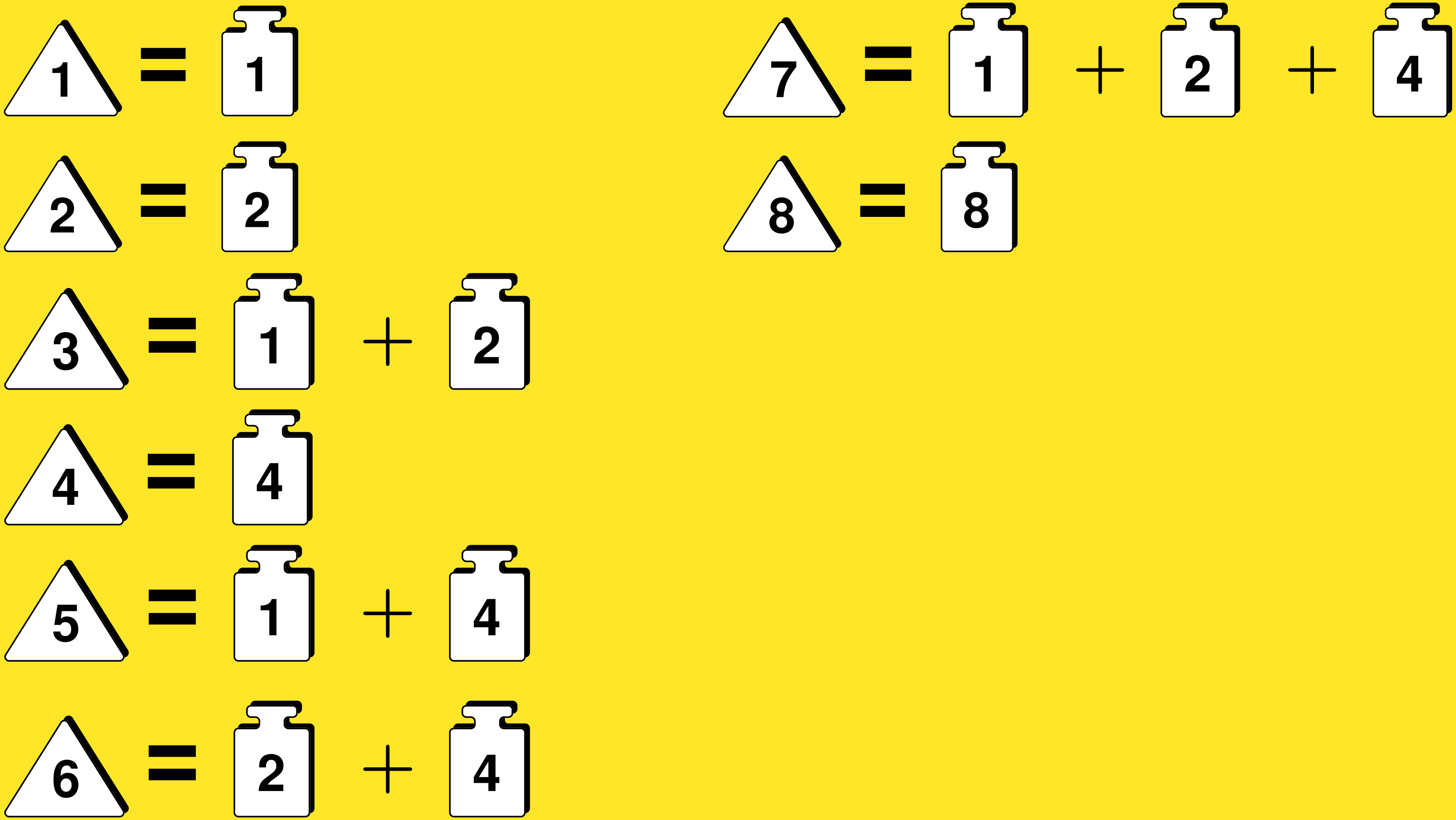

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1 = 1

2 = 2

3 = 1 + 2

4 = 4

5 = 1 + 4

6 = 2 + 4

7 = 1 + 2 + 4

8 = 8

9 = 1 + 8

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3 = 1 + 2

4 = 4

5 = 1 + 4

6 = 2 + 4

7 = 1 + 2 + 4

8 = 8

9 = 1 + 8

10 = 2 + 8

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3 = 1 + 2

4 = 4

5 = 1 + 4

6 = 2 + 4

7 = 1 + 2 + 4

8 = 8

9 = 1 + 8

10 = 2 + 8

11 = 1 + 2 + 8

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2 = 2

3 = 1 + 2

4 = 4

5 = 1 + 4

6 = 2 + 4

7 = 1 + 2 + 4

8 = 8

9 = 1 + 8

10 = 2 + 8

11 = 1 + 2 + 8

12 = 4 + 8

$$1=1$$
$$2=2$$
$$3=1+2$$
$$4=4$$
$$5=1+4$$
$$6=2+4$$
$$7=1+2+4$$
$$8=8$$
$$9=1+8$$
$$10=2+8$$
$$11=1+2+8$$
$$12=4+8$$
$$13=1+4+8$$
$$14=2+4+8$$
$$15=1+2+4+8$$
$$16=16$$
$$17=1+16$$
$$18=2+16$$
$$19=1+2+16$$
$$20=4+16$$
$$21=1+4+16$$
$$22=2+4+16$$
$$23=1+2+4+16$$
$$24=8+16$$
$$25=1+8+16$$
$$26=2+8+16$$
$$27=1+2+8+16$$
$$28=4+8+16$$
$$29=1+4+8+16$$
$$30=2+4+8+16$$
$$31=1+2+4+8+16$$
$$32=32$$
$$33=1+32$$
$$34=2+32$$
$$35=1+2+32$$
$$36=4+32$$
$$37=1+4+32$$
$$38=2+4+32$$
$$39=1+2+4+32$$
$$40=8+32$$
$$41=1+8+32$$
$$42=2+8+32$$
$$43=1+2+8+32$$
$$44=4+8+32$$
$$45=1+4+8+32$$

$$a^{32} - 1$$

$$a^{32} - 1 \equiv (a-1)(a^{31} + a^{30} + a^{29} + \ldots + a^2 + a + 1)$$

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$$a^{32} - 1$$

$$a^{32} - 1 \equiv (a-1)(a^{31} + a^{30} + a^{29} + \ldots + a^2 + a + 1)$$

$$a^{32} - 1 \equiv (a^{16} - 1)(a^{16} + 1)$$

$$a^{32} - 1 \equiv (a-1)(a^{31} + a^{30} + a^{29} + \ldots + a^2 + a + 1)$$

$$a^{32} - 1 \equiv (a^8 - 1)(a^8 + 1)(a^{16} + 1)$$

$$a^{32} - 1 \equiv (a-1)(a^{31} + a^{30} + a^{29} + \ldots + a^2 + a + 1)$$

$$a^{32} - 1 \equiv (a^4 - 1)(a^4 + 1)(a^8 + 1)(a^{16} + 1)$$

$$a^{32} - 1 \equiv (a-1)(a^{31} + a^{30} + a^{29} + \ldots + a^2 + a + 1)$$

$$a^{32} - 1 \equiv (a^2 - 1)(a^2 + 1)(a^4 + 1)(a^8 + 1)(a^{16} + 1)$$

$$a^{32} - 1 \equiv (a-1)(a^{31} + a^{30} + a^{29} + \ldots + a^2 + a + 1)$$

$$a^{32} - 1 \equiv (a-1)(a+1)(a^2+1)(a^4+1)(a^8+1)(a^{16}+1)$$

$$(a-1)(a^{31}+a^{30}+a^{29}+\ldots+a^{2}+a+1)$$

$$(a-1)(a+1)(a^{2}+1)(a^{4}+1)(a^{8}+1)(a^{16}+1)$$

$$(a-1)(a^{31}+a^{30}+a^{29}+\ldots+a^2+a+1)$$

$$(a-1)(a+1)(a^2+1)(a^4+1)(a^8+1)(a^{16}+1)$$

$$a^{31} + a^{30} + a^{29} + \ldots + a^2 + a + 1$$

$$\equiv (a + 1)(a^2 + 1)(a^4 + 1)(a^8 + 1)(a^{16} + 1)$$

$$a^{31} + a^{30} + a^{29} + \ldots + a^2 + a + 1$$

$$\equiv (a+1)(a^2+1)(a^4+1)(a^8+1)(a^{16}+1)$$

$$\equiv 1 + a + a^2 + a^{1+2} + a^4 + a^{1+4} + a^{2+4}$$

$$+ a^{1+2+4} + a^8 + a^{1+8} + a^{2+8} + a^{1+2+8}$$

$$+ a^{4+8} + a^{1+4+8} + a^{2+4+8} + a^{2+4+8+16}$$

$$+ a^{16} + a^{1+16} + a^{2+16} + a^{1+2+16} + a^{4+16}$$

$$+ a^{1+4+16} + a^{2+4+16} + a^{1+2+8+16} + a^{8+16}$$

$$+ a^{1+4+8+16} + a^{2+8+16} + a^{1+8+16} + a^{4+8+16}$$

$$+ a^{1+2+4+8} + a^{1+2+4+16} + a^{1+2+4+8+16}$$

$$a^{31} + a^{30} + a^{29} + \ldots + a^2 + a + 1$$

$$\equiv (a+1)(a^{2}+1)(a^{4}+1)(a^{8}+1)(a^{16}+1)$$

$$\equiv 1 + a + a^2 + a^{1+2} + a^4 + a^{1+4} + a^{2+4}$$

$$+ a^{1+2+4} + a^8 + a^{1+8} + a^{2+8} + a^{1+2+8}$$

$$+ a^{4+8} + a^{1+4+8} + a^{2+4+8} + a^{2+4+8+16}$$

$$+ a^{16} + a^{1+16} + a^{2+16} + a^{1+2+16} + a^{4+16}$$

$$+ a^{1+4+16} + a^{2+4+16} + a^{1+2+8+16} + a^{8+16}$$

$$+ a^{1+4+8+16} + a^{2+8+16} + a^{1+8+16} + a^{4+8+16}$$

$$+ a^{1+2+4+8} + a^{1+2+4+16} + a^{1+2+4+8+16}$$

$$a^{31} + a^{30} + a^{29} + \ldots + a^2 + a + 1$$

$$\equiv (a+1)(a^2+1)(a^4+1)(a^8+1)(a^{16}+1)$$

$$\equiv 1 + a + a^2 + a^{1+2} + a^4 + a^{1+4} + a^{2+4}$$

$$+ a^{1+2+4} + a^8 + a^{1+8} + a^{2+8} + a^{1+2+8}$$

$$+ a^{4+8} + a^{1+4+8} + a^{2+4+8} + a^{2+4+8+16}$$

$$+ a^{16} + a^{1+16} + a^{2+16} + a^{1+2+16} + a^{4+16}$$

$$+ a^{1+4+16} + a^{2+4+16} + a^{1+2+8+16} + a^{8+16}$$

$$+ a^{1+4+8+16} + a^{2+8+16} + a^{1+8+16} + a^{4+8+16}$$

$$+ a^{1+2+4+8} + a^{1+2+4+16} + a^{1+2+4+8+16}$$

$$\boxed{a^{31} + a^{30} + a^{29} + \ldots + a^2 + a + 1}$$

$$\equiv (a+1)(a^2+1)(a^4+1)(a^8+1)(a^{16}+1)$$

$$\equiv 1 + a + a^2 + a^{1+2} + a^4 + a^{1+4} + a^{2+4}$$

$$+ a^{1+2+4} + a^8 + a^{1+8} + a^{2+8} + a^{1+2+8}$$

$$+ a^{4+8} + a^{1+4+8} + a^{2+4+8} + a^{2+4+8+16}$$

$$+ a^{16} + a^{1+16} + a^{2+16} + a^{1+2+16} + a^{4+16}$$

$$+ a^{1+4+16} + a^{2+4+16} + a^{1+2+8+16} + a^{8+16}$$

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$$+ a^{1+2+4+8} + a^{1+2+4+16} + a^{1+2+4+8+16}$$

$$a^{31} + a^{30} + a^{29} + \ldots + a^2 + a + 1$$

$$\equiv (a+1)(a^2+1)(a^4+1)(a^8+1)(a^{16}+1)$$

$$\begin{aligned}\equiv\ & 1 + a + a^2 + a^{1+2} + a^4 + a^{1+4} + a^{2+4} \\ & + a^{1+2+4} + a^8 + a^{1+8} + a^{2+8} + a^{1+2+8} \\ & + a^{4+8} + a^{1+4+8} + a^{2+4+8} + a^{2+4+8+16} \\ & + a^{16} + a^{1+16} + a^{2+16} + a^{1+2+16} + a^{4+16} \\ & + a^{1+4+16} + a^{2+4+16} + a^{1+2+8+16} + a^{8+16} \\ & + a^{1+4+8+16} + a^{2+8+16} + a^{1+8+16} + a^{4+8+16} \\ & + a^{1+2+4+8} + a^{1+2+4+16} + a^{1+2+4+8+16}\end{aligned}$$

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$$a^{2^n-1} + a^{2^n-2} + a^{2^n-3} + \ldots + a^2 + a + 1$$

$$\equiv (a+1)(a^2+1)(a^4+1)\ldots(a^{2^{n-1}}+1)$$

100

$$\mathbf{100 = 64 + 32 + 4}$$

$$
\begin{aligned}
\mathbf{100} &= \mathbf{64 + 32 + 4} \\
&= \mathbf{1 \cdot 64 + 1 \cdot 32 + 0 \cdot 16 + 0 \cdot 8 + 1 \cdot 4 + 0 \cdot 2 + 0 \cdot 1}
\end{aligned}
$$

$$\begin{aligned} 100 &= 64 + 32 + 4 \\ &= 1 \cdot 64 + 1 \cdot 32 + 0 \cdot 16 + 0 \cdot 8 + 1 \cdot 4 + 0 \cdot 2 + 0 \cdot 1 \\ &= (11001001)_2 \end{aligned}$$

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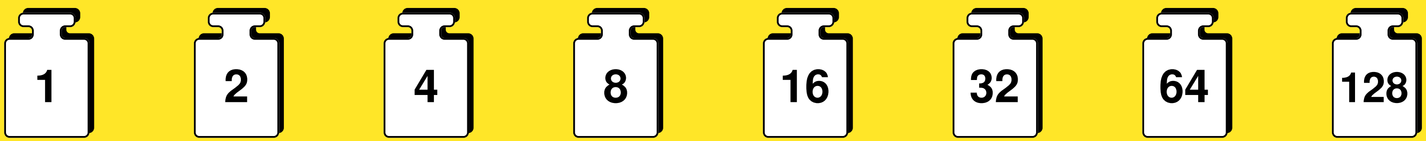
PROBLEM:

Consider the following weights:



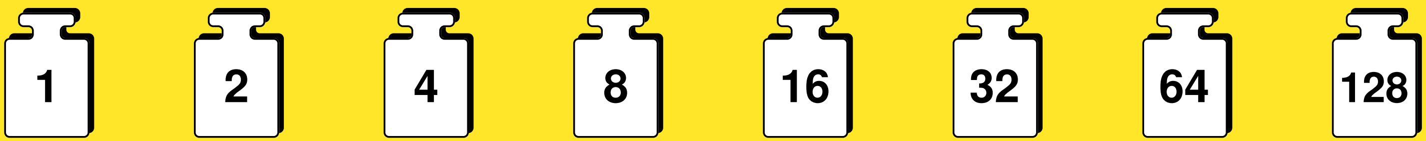
etc.

Which weight can be balanced by them?

$$100 = 64 + 32 + 4$$
$$= 1 \cdot 64 + 1 \cdot 32 + 0 \cdot 16 + 0 \cdot 8 + 1 \cdot 4 + 0 \cdot 2 + 0 \cdot 1$$
$$= (11001001)_2$$

PROBLEM:

Consider the following weights:



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Which weight can be balanced by them?

ANSWER:

$$\begin{aligned} 100 &= 64 + 32 + 4 \\ &= 1 \cdot 64 + 1 \cdot 32 + 0 \cdot 16 + 0 \cdot 8 + 1 \cdot 4 + 0 \cdot 2 + 0 \cdot 1 \\ &= (11001001)_2 \end{aligned}$$

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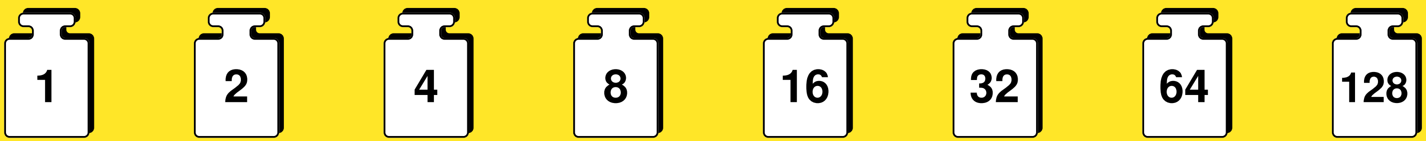
Consider the following weights:



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Which weight can be balanced by them?

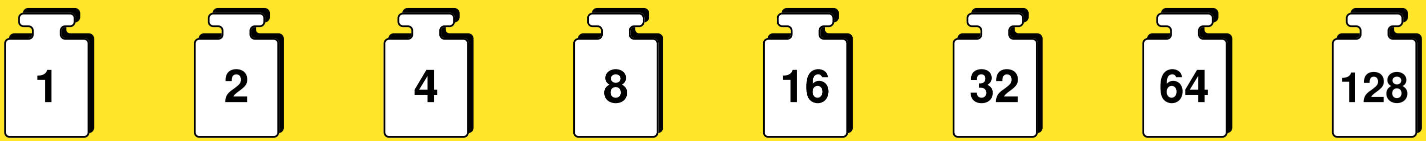
ANSWER:

Any natural number

$$\begin{aligned} 100 &= 64 + 32 + 4 \\ &= 1 \cdot 64 + 1 \cdot 32 + 0 \cdot 16 + 0 \cdot 8 + 1 \cdot 4 + 0 \cdot 2 + 0 \cdot 1 \\ &= (11001001)_2 \end{aligned}$$

PROBLEM:

Consider the following weights:



etc.

Which weight can be balanced by them?

ANSWER:

Any natural number, since any natural number can be converted into the binary system.

Augustin Cauchy (1789 - 1857)

THEOREM:

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$$\frac{a_1 + \ldots + a_n}{n} \geq \sqrt[n]{a_1 \ldots a_n}$$

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THEOREM′: **For any** $a_1, \ldots, a_n \geq 0$ **we have**

$$\left(\frac{a_1 + \ldots + a_n}{n}\right)^n \geq a_1 \ldots a_n$$

THEOREM: For any $a_1, \dots, a_n \geq 0$ we have

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$\Downarrow$

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⇓

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$$\frac{a_1 + \ldots + a_n}{n} \geq \sqrt[n]{a_1 \ldots a_n} \implies \frac{b_1^n + \ldots + b_n^n}{n} \geq \sqrt[n]{b_1^n \ldots b_n^n}$$

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$$\frac{a_1 + \dots + a_n}{n} \geq \sqrt[n]{a_1 \dots a_n} \quad \Longrightarrow \quad \frac{b_1^n + \dots + b_n^n}{n} \geq \sqrt[n]{b_1^n \dots b_n^n}$$

$$= b_1 \dots b_n$$

THEOREM: For any $a_1, \ldots, a_n \geq 0$ we have

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$\Uparrow$

THEOREM$'$: For any $b_1, \ldots, b_n \geq 0$ we have

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$$b_1^n + \ldots + b_n^n \geq n b_1 \ldots b_n \implies a_1 + \ldots + a_n = (\sqrt[n]{a_1})^n + \ldots + (\sqrt[n]{a_n})^n$$

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$$\Uparrow$$

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$$\geq n \sqrt[n]{a_1} \ldots \sqrt[n]{a_n}$$

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$$\geq n \sqrt[n]{a_1} \ldots \sqrt[n]{a_n}$$

$$= n \sqrt[n]{a_1 \ldots a_n}$$

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$\Updownarrow$

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EXAMPLE:

THEOREM: For any $b_1, \ldots, b_n \geq 0$ we have

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EXAMPLE:

For $n = 2$ we have:

THEOREM: **For any** $b_1, \ldots, b_n \geq 0$ **we have**

$$b_1^n + \ldots + b_n^n \geq n b_1 \ldots b_n$$

EXAMPLE:

For $n = 2$ **we have:**

$$b_1^2 + b_2^2 \geq 2 b_1 b_2$$

THEOREM: For any $b_1, \ldots, b_n \geq 0$ we have

$$b_1^n + \ldots + b_n^n \geq n b_1 \ldots b_n$$

EXAMPLE:

For $n = 2$ we have:

$$b_1^2 + b_2^2 \geq 2 b_1 b_2$$

$$b_1^2 - 2 b_1 b_2 + b_2^2 \geq 0$$

THEOREM: For any $b_1, \ldots, b_n \geq 0$ we have

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EXAMPLE:

For $n = 2$ we have:

$$b_1^2 + b_2^2 \geq 2b_1 b_2$$

$$b_1^2 - 2b_1 b_2 + b_2^2 \geq 0$$

$$(b_1 - b_2)^2 \geq 0$$

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EXAMPLE:

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For $n = 3$ we have:

$$b_1^3 + b_2^3 + b_3^3 \geq 3b_1 b_2 b_3$$

THEOREM: For any $b_1, \ldots, b_n \geq 0$ we have

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EXAMPLE:

For $n = 2$ we have:

$$b_1^2 + b_2^2 \geq 2b_1b_2$$

$$b_1 - 2b_1b_2 + b_2 \geq 0$$

$$(b_1 - b_2)^2 \geq 0$$

For $n = 3$ we have:

$$b_1^3 + b_2^3 + b_3^3 \geq 3b_1b_2b_3$$

$$b_1^3 + b_2^3 + b_3^3 - 3b_1b_2b_3 \geq 0$$

THEOREM: For any $b_1, \ldots, b_n \geq 0$ we have

$$b_1^n + \ldots + b_n^n \geq n b_1 \ldots b_n$$

EXAMPLE:

For $n = 2$ we have:

$$b_1^2 + b_2^2 \geq 2b_1 b_2$$

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For $n = 3$ we have:

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$$b_1^3 + b_2^3 + b_3^3 - 3b_1 b_2 b_3 \geq 0$$

?

THEOREM: For any $b_1, b_2, b_3 \geq 0$ we have

$$b_1^3 + b_2^3 + b_3^3 \geq 3b_1b_2b_3$$

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PROOF: We have

$$a^2 + b^2 \geq 2ab$$

$$a^2 - ab + b^2 \geq ab$$

$$(a+b)(a^2 - ab + b^2) \geq (a+b)ab$$

THEOREM: For any $a, b, c \geq 0$ we have

$$a^3 + b^3 + c^3 \geq 3abc$$

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THEOREM: For any $a, b, c, d \geq 0$ we have

$$a^4 + b^4 + c^4 + d^4 \geq 4abcd$$

PROOF: Since $x + y \geq 2\sqrt{xy}$, we have

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$$\begin{aligned} a^4 + b^4 + c^4 + d^4 &\geq 2a^2b^2 + 2c^2d^2 \\ &\geq 2\sqrt{4a^2b^2c^2d^2} \end{aligned}$$

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THEOREM: For any $a_1, a_2, \ldots, a_8 \geq 0$ we have

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$$\begin{aligned} a_1^8 + a_2^8 + \ldots + a_8^8 &\geq 2a_1^4a_2^4 + 2a_3^4a_4^4 + 2a_5^4a_6^4 + 2a_7^4a_8^4 \\ &\geq 2\sqrt{4a_1^4a_2^4a_3^4a_4^4} + 2\sqrt{4a_5^4a_6^4a_7^4a_8^4} \end{aligned}$$

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$$\begin{aligned} a_1^8 + a_2^8 + \ldots + a_8^8 \quad &\geq 2a_1^4a_2^4 + 2a_3^4a_4^4 + 2a_5^4a_6^4 + 2a_7^4a_8^4 \\ &\geq 2\sqrt{4a_1^4a_2^4a_3^4a_4^4} + 2\sqrt{4a_5^4a_6^4a_7^4a_8^4} \\ &= 4a_1^2a_2^2a_3^2a_4^2 + 4a_5^2a_6^2a_7^2a_8^2 \end{aligned}$$

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PROOF: Since $x + y \geq 2\sqrt{xy}$, we have

$$a_1^8 + a_2^8 \geq 2a_1^4 a_2^4$$
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$$a_7^8 + a_8^8 \geq 2a_7^4 a_8^4$$

$$\begin{aligned} a_1^8 + a_2^8 + \ldots + a_8^8 &\geq 2a_1^4 a_2^4 + 2a_3^4 a_4^4 + 2a_5^4 a_6^4 + 2a_7^4 a_8^4 \\ &\geq 2\sqrt{4a_1^4 a_2^4 a_3^4 a_4^4} + 2\sqrt{4a_5^4 a_6^4 a_7^4 a_8^4} \\ &= 4a_1^2 a_2^2 a_3^2 a_4^2 + 4a_5^2 a_6^2 a_7^2 a_8^2 \\ &\geq 2\sqrt{16a_1^2 a_2^2 \ldots a_8^2} = 8a_1 a_2 \ldots a_8 \end{aligned}$$

THEOREM: If $n = 2^k$, then for any $a_1, a_2, \ldots, a_n \geq 0$ we have

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LEMMA:

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LEMMA: Suppose

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for any $a_1, a_2, \dots, a_n \geq 0$. Then

$$b_1^{n-1} + \dots + b_{n-1}^{n-1} \geq (n-1) b_1 \dots b_{n-1}$$

for any $b_1, b_2, \dots, b_{n-1} \geq 0$.

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$$a_1^8 + \ldots + a_8^8 \geq 8 a_1 \ldots a_8$$

for any $a_1, a_2, \ldots, a_8 \geq 0$. Then

$$b_1^7 + \ldots + b_7^7 \geq 7 b_1 \ldots b_7$$

for any $b_1, b_2, \ldots, b_7 \geq 0$.

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$$a_1^7 + \ldots + a_7^7 \geq 7 a_1 \ldots a_7$$

for any $a_1, a_2, \ldots, a_7 \geq 0$. Then

$$b_1^6 + \ldots + b_6^6 \geq 6 b_1 \ldots b_6$$

for any $b_1, b_2, \ldots, b_6 \geq 0$.

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$$a_1^n + a_2^n + \ldots + a_n^n \geq n a_1 a_2 \ldots a_n$$

LEMMA: Suppose

$$a_1^6 + \ldots + a_6^6 \geq 6 a_1 \ldots a_6$$

for any $a_1, a_2, \ldots, a_6 \geq 0$. Then

$$b_1^5 + \ldots + b_5^5 \geq 5 b_1 \ldots b_5$$

for any $b_1, b_2, \ldots, b_5 \geq 0$.

THEOREM: For any $a_1, \ldots, a_n \geq 0$ we have

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PROOF (ELEMENTARY):

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PROOF (ELEMENTARY, SHORT):

THEOREM: For any $a_1, \ldots, a_n \geq 0$ we have

$$\frac{a_1 + \ldots + a_n}{n} \geq \sqrt[n]{a_1 \ldots a_n}$$

PROOF (ELEMENTARY, SHORT): Put

$$f(x_1, \ldots, x_n) = \frac{x_1 + \ldots + x_n}{n} - \sqrt[n]{x_1 \ldots x_n}$$

THEOREM: For any $a_1, \ldots, a_n \geq 0$ **we have**

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Assume to the contrary that $f(x_1, \ldots, x_n) < 0$ **for some** $x_1, \ldots, x_n \in [A, B]$.

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$$f(a_1, a_2, a_3, \dots, a_n) > f\left(\frac{a_1 + a_2}{2}, \frac{a_1 + a_2}{2}, a_3, \dots, a_n,\right) \blacksquare$$

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$f(a_1, a_2, a_3, \ldots, a_n)$

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$$\| \quad \| \quad \| \quad \| \qquad\qquad \wedge \quad \wedge \quad \wedge \quad \wedge$$

$$f\left(\frac{a_1 + a_2}{2}, \frac{a_1 + a_2}{2}, a_3, \ldots, a_n\right) = \frac{\frac{a_1 + a_2}{2} + \frac{a_1 + a_2}{2} + a_3 + \ldots + a_n}{n} - \sqrt[n]{\frac{a_1 + a_2}{2} \frac{a_1 + a_2}{2} a_3 \ldots a_n}$$

THEOREM: For any $a_1, \ldots, a_n \geq 0$ we have

$$\frac{a_1 + \ldots + a_n}{n} \geq \sqrt[n]{a_1 \ldots a_n}$$

PROOF (ELEMENTARY, SHORT): Put

$$f(x_1, \ldots, x_n) = \frac{x_1 + \ldots + x_n}{n} - \sqrt[n]{x_1 \ldots x_n}$$

Assume to the contrary that $f(x_1, \ldots, x_n) < 0$ for some $x_1, \ldots, x_n \in [A, B]$. Suppose that $f(x_1, \ldots, x_n)$ attains its minimal value on $[A, B]$ at some $a_1, \ldots, a_n \in [A, B]$ with $a_1 \neq a_2$. This gives us a contradiction, since

$$f(a_1, a_2, a_3, \ldots, a_n) > f\left(\frac{a_1 + a_2}{2}, \frac{a_1 + a_2}{2}, a_3, \ldots, a_n,\right) \blacksquare$$

$$f(a_1, a_2, a_3, \ldots, a_n) = \frac{a_1 + a_2 + a_3 + \ldots + a_n}{n} - \sqrt[n]{a_1 a_2 a_3 \ldots a_n}$$

$$\vee \quad \vee \quad \vee \quad \vee \quad \vee \quad \vee \quad \vee$$

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THEOREM (Extreme-Value Theorem):

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$$f(x_1, \ldots, x_n) = \frac{x_1 + \ldots + x_n}{n} - \sqrt[n]{x_1 \ldots x_n}$$

Assume to the contrary that $f(x_1, \ldots, x_n) < 0$ for some $x_1, \ldots, x_n \in [A, B]$. Suppose that $f(x_1, \ldots, x_n)$ attains its minimal value on $[A, B]$ at some $a_1, \ldots, a_n \in [A, B]$ with $a_1 \neq a_2$. This gives us a contradiction, since

$$f(a_1, a_2, a_3, \ldots, a_n) > f\left(\frac{a_1+a_2}{2}, \frac{a_1+a_2}{2}, a_3, \ldots, a_n\right) \blacksquare$$

THEOREM (Extreme-Value Theorem): If $f(x_1, \ldots, x_n)$ is continuous on a closed and bounded set R, then f has both an absolute maximum and an absolute minimum on R.