UNITARY COMPLETIONS OF LOCALLY ALGEBRAIC AND LOCALLY ANALYTIC PRINCIPAL SERIES OF p-ADIC GL_2

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Abstract. We generalize a result of Emerton on the relationship between unitary completions of locally $\mathbf{Q}_p$-analytic and locally $\mathbf{Q}_p$-algebraic principal series representations induced from certain locally $\mathbf{Q}_p$-algebraic characters of the diagonal torus of $\mathrm{GL}_2(L)$, where L is a finite extension of $\mathbf{Q}_p$. Namely, under a non-critical slope hypothesis on the character being induced, the map on universal unitary completions arising from the inclusion of the locally algebraic induction into the locally analytic induction is a topological isomorphism (Theorem 3.1). (Emerton proved this result for $L = \mathbf{Q}_p$.) The main ingredients in carrying out a "several-variable" version of Emerton's argument are the description of the local convex space of locally $\mathbf{Q}_p$-analytic functions on the group $\mathscr{O}_L$ in terms of the embeddings of L into our p-adic coefficient field, given in §2, and a generalization by Breuil of the classical result of Amice-Vélu and Vishik on "tempered distributions" on $\mathbf{Z}_p$ (Lemma 2.6).

1. Introduction

In this note, we generalize a result of Emerton [5, Proposition 2.5] (proved independently in [1]) on continuous homomorphisms from locally analytic principal series representations of $\mathrm{GL}_2(L)$ into unitary Banach space representations. Here L is a finite extension of $\mathbf{Q}_p$, and all representations are over some sufficiently large p-adic field E. For certain locally algebraic characters χ of T, the subgroup of diagonal matrices in $\mathrm{GL}_2(L)$, we define the locally algebraic and locally $\mathbf{Q}_p$-analytic $\overline{B}$-inductions $I(\chi)$ and $I^{\mathrm{la}}(\chi)$ of χ, where $\overline{B}$ is the subgroup of lower triangular matrices in $\mathrm{GL}_2(L)$ and χ is viewed as a character of $\overline{B}$ via the projection $\overline{B}$ to T. These are admissible locally $\mathbf{Q}_p$-analytic representations of $\mathrm{GL}_2(L)$, and there is a canonical closed embedding $I(\chi) \hookrightarrow I^{\mathrm{la}}(\chi)$, allowing us to view $I(\chi)$ as a locally algebraic subrepresentation of $I^{\mathrm{la}}(\chi)$. Let U be a Banach space representation of $\mathrm{GL}_2(L)$ admitting a $\mathrm{GL}_2(L)$-invariant norm and assume that $\chi|_{Z(\mathrm{GL}_2(L))}$ takes values in $\mathscr{O}_E^\times$. Our main result (Theorem 3.1) states that, under a "non-critical slope" hypothesis on χ, any continuous $\mathrm{GL}_2(L)$-equivariant linear map $I(\chi) \to U$ extends uniquely to a continuous $\mathrm{GL}_2(L)$-equivariant linear map $I^{\mathrm{la}}(\chi) \to U$. This result is equivalent to the assertion that $I(\chi)$ and $I^{\mathrm{la}}(\chi)$ have the same universal unitary completion (in the sense of [5, Definition 1.1]). Emerton proved this result for $L = \mathbf{Q}_p$. Breuil has proved a similar result [2, Theorem 7.1] covering injective linear maps out of locally J-analytic principal series of $\mathrm{GL}_2(L)$, for subsets $J \subseteq \mathrm{Hom}_{\mathrm{Alg}/\mathbf{Q}_p}(L, E)$ ($J = \emptyset$ corresponds to $I(\chi)$, while $J = \mathrm{Hom}_{\mathrm{Alg}/\mathbf{Q}_p}(E, L)$ corresponds to $I^{\mathrm{la}}(\chi)$). Breuil's result is stated without the hypothesis of injectivity in [3, Proposition 5.5], and it is asserted there that Breuil's proof can be made to work in the more general case. We closely follow Emerton's method of proof, which is different from Breuil's, although we do make use of a generalization of a classical result of Amice-Vélu and Vishik, Lemma 2.6 below, proved by Breuil in [2, Lemma 6.1]. We should emphasize

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that we make no attempt here to address the question of whether or not the principal series representations we consider admit *non-zero* universal unitary completions. This was studied for $L = \mathbf{Q}_p$ in the work of Berger-Breuil and Emerton ([1] and [5], respectively) in certain cases, and is completely understood for $L = \mathbf{Q}_p$ as a consequence of known properties of the p-adic local Langlands correspondence. For general extensions $L/\mathbf{Q}_p$, this problem is studied in [3]. For a unitary χ, it is easy to see that the universal unitary completion of $I^{\mathrm{la}}(\chi)$ is non-zero, since in this case $I^{\mathrm{la}}(\chi)$ admits a continuous $\mathrm{GL}_2(L)$-equivariant injection into the continuous induction of χ, which is a unitary Banach space representation.

We introduce our notation below, mostly retaining that of [5]. In §2 we describe the locally convex space $\mathscr{C}^{\mathrm{la}}(\mathscr{O}_L, E)$ of E-valued locally $\mathbf{Q}_p$-analytic functions on $\mathscr{O}_L$ using the set of embeddings $\mathrm{Hom}_{\mathrm{Alg}/\mathbf{Q}_p}(L, E)$ (which is assumed to have $[L : \mathbf{Q}_p]$ elements). Although a description of this space along these lines has been used (somewhat implicitly) in other places (e.g. [2]), as far as we know, there is no published proof that it coincides (set-theoretically and topologically) with the standard description of this space given in [9, §10]. Although this may be clear to experts, because of the fact that this description of $\mathscr{C}^{\mathrm{la}}(\mathscr{O}_L, E)$ is crucial to our argument, we felt it was worthwhile to give a detailed proof of the equivalence between our description and the more generally applicable one found in other parts of the literature on p-adic functional analysis. We define the parabolic inductions $I(\chi)$ and $I^{\mathrm{la}}(\chi)$ in §3, and state the main result which is proved (following Emerton's proof in [5, §3]) in §4. The argument is primarily representation-theoretic and functional-analytic, and most of the work is dedicated to reducing the statement of Theorem 3.1 to Lemma 2.6 by reinterpreting the condition of Definition 2.4 in terms of a continuity condition with respect to the action of a submonoid B^+ of B, the group of upper triangular matrices in $\mathrm{GL}_2(L)$ (this reinterpretation is provided by Lemma 4.3).

Notation Fix a prime p. Let L and E be finite extensions of $\mathbf{Q}_p$ with respective rings of integers $\mathscr{O}_L$ and $\mathscr{O}_E$, and denote by ϖ_L a choice of uniformizer for $\mathscr{O}_L$. Set $r = [L : \mathbf{Q}_p]$, and assume that $\mathrm{Hom}_{\mathrm{Alg}/\mathbf{Q}_p}(L, E)$ has r distinct elements that we order for convenience: $\sigma_1, \ldots, \sigma_r$ (nothing we do will depend on the choice of ordering, and it is only made to ease notation). The field E will serve as the coefficient field of our representations.

We normalize the discrete valuation of E, $\mathrm{ord} = \mathrm{ord}_E$, by $\mathrm{ord}_E(p) = e(L/\mathbf{Q}_p)$ (the ramification index of L over $\mathbf{Q}_p$) and use the absolute value $|\cdot| = |\cdot|_E$ defined by $|\alpha| = q^{-\mathrm{ord}(\alpha)}$, where q is the cardinality of the residue field of L. If we use the same normalizations for the discrete valuation and absolute value on L, then L is endowed with its canonical absolute value, i.e., the one giving ϖ_L absolute value q^{-1}, and each σ_i is an isometry. We will therefore denote the discrete valuation on either E or L simply by ord, and the absolute value by $|\cdot|$.

We denote by G the group $\mathrm{GL}_2(L)$, viewed as the group of $\mathbf{Q}_p$-points of the connected reductive linear algebraic $\mathbf{Q}_p$-group $\mathbf{G} = \mathrm{Res}_{L/\mathbf{Q}_p}(\mathrm{GL}_{2/L})$. Thus we regard $\mathrm{GL}_2(L)$ as a locally $\mathbf{Q}_p$-analytic group, and by "locally analytic," we will always mean "locally $\mathbf{Q}_p$-analytic." We apply the same convention to all other groups that we consider. We let B and $\overline{B}$ denote the groups of $\mathbf{Q}_p$-points of the upper triangular and lower triangular, respectively, Borel subgroups of $\mathbf{G}$, N and $\overline{N}$ the groups of $\mathbf{Q}_p$-points of their unipotent radicals, and T the group of $\mathbf{Q}_p$-points of the diagonal torus in $\mathbf{G}$. Setting $G_0 = G_0(0) = \mathrm{GL}_2(\mathscr{O}_L)$, we define, for each integer $s \geq 1$,

$$G_0(s) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_0 : c \equiv 0 \pmod{\varpi_L^s \mathscr{O}_L} \right\}.$$

These are compact open subgroups of G admitting an Iwahori decomposition with respect to B and $\overline{B}$, meaning that if $T_0 = T \cap G_0(s) = T \cap G_0$, $N_0 = N \cap G_0(s) = N \cap G_0$, and $\overline{N}(s) = \overline{N} \cap G_0(s)$ for $s \geq 1$, then the natural multiplication map $N_0 T_0 \overline{N}(s) \to G_0(s)$ is a bijection. If $T^+ = \{t \in T : tN_0t^{-1} \subseteq N_0\}$, then T^+ is a submonoid of T containing T_0, and for each $t \in T^+$ and each integer $s \geq 1$, $t^{-1}\overline{N}(s)t \subseteq \overline{N}(s)$. Explicitly, T^+ consists of all matrices $\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right) \in T$ with $ad^{-1} \in \mathscr{O}_L$. One can verify then that for $s \geq 1$, $G^+(s) = N_0 T^+ \overline{N}(s)$ is an open submonoid of G containing $G_0(s)$. We write B^+ for $G^+(1) \cap B = N_0 T^+$ (the equality holds because $\overline{N} \cap B = \{1\}$, and shows we could replace the integer 1 in the definition of B^+ by any integer $s \geq 1$ without changing the result); this is a submonoid of B which (by inspection) generates B as a group. Each of these subgroups (respectively submonoids) of G will be regarded as a subgroup (respectively submonoid) of the group of E-points of $\mathbf{G} \times_{\mathbf{Q}_p} E = \prod_{i=1}^r \mathrm{GL}_{2/L} \times_{\sigma_i} E$ via the continuous injection

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \left(\begin{pmatrix} \sigma_i(a) & \sigma_i(b) \\ \sigma_i(c) & \sigma_i(d) \end{pmatrix} \right)_{1 \leq i \leq r}.$$

We will generally identify N_0 with $\mathscr{O}_L$ via the locally analytic isomorphism $\left(\begin{smallmatrix} 1 & z \\ 0 & 1 \end{smallmatrix}\right) \mapsto z$.

Elements of $\mathbf{Z}^r_{\geq 0}$ will be denoted by underlined Roman letters, e.g. $\underline{m} = (m_1, \ldots, m_r)$, and we set $|\underline{m}| = \sum_{i=1}^r m_i$. For each integer $n \geq 0$, we write $\mathscr{A}_n$ for the affinoid E-algebra of formal power series

$$F = F(X_1, \ldots, X_r) = \sum_{\underline{m} \in \mathbf{Z}^r_{\geq 0}} a_{\underline{m}} X_1^{m_1} \cdots X_r^{m_r} \in E[[X_1, \ldots, X_r]]$$

satisfying $\lim_{|\underline{m}| \to \infty} |a_{\underline{m}}| q^{-n|\underline{m}|} = 0$. This is an E-Banach algebra with the multiplicative Gauss norm $\| \cdot \|_n$ given by $\|F\|_n = \max_{\underline{m}} |a_{\underline{m}}| q^{-n|\underline{m}|}$. When $n = 0$, we will write $\mathscr{A}$ (respectively $\| \cdot \|_{\mathscr{A}}$) instead of $\mathscr{A}_0$ (respectively $\| \cdot \|_0$). If $\underline{k} \in \mathbf{Z}^r_{\geq 0}$, then we will denote by $\mathscr{A}^{\underline{k}}$ the finite-dimensional (hence closed) subspace of $\mathscr{A}$ consisting of all polynomials in $E[X_1, \ldots, X_r]$ whose degree in X_i is at most k_i for $1 \leq i \leq r$ (note that in fact $\mathscr{A}^{\underline{k}}$ is a closed subspace of $\mathscr{A}_n$ for all $n \geq 0$). We will refer to $\mathscr{A}^{\underline{k}}$ as the space of polynomials in $\mathscr{A}$ "of degree at most $\underline{k}$."

If H is a locally $\mathbf{Q}_p$-analytic group, $\mathscr{C}^{\mathrm{la}}(H, E)$ denotes the locally convex space of locally analytic E-valued functions on H (see [9, §10] for a detailed description of the locally convex topology on this space, and §2 below for an alternative description in the case $H = \mathscr{O}_L$) and $\mathscr{C}^{\mathrm{sm}}(H, E)$ denotes the space of smooth (i.e. locally constant) E-valued functions on H. For an open subset U of H, $\mathbf{1}_U$ denotes the characteristic function of U (so $\mathbf{1}_U \in \mathscr{C}^{\mathrm{sm}}(H, E) \subseteq \mathscr{C}^{\mathrm{la}}(H, E)$). The isomorphism $N_0 \cong \mathscr{O}_L$ yields a topological isomorphism $\mathscr{C}^{\mathrm{la}}(N_0, E) \cong \mathscr{C}^{\mathrm{la}}(\mathscr{O}_L, E)$.

If V and W are locally convex spaces over E, $\mathcal{L}(V, W)$ denotes the space of continuous E-linear maps from V to W. If moreover each of V, W is endowed with an action of a topological monoid H by E-linear (topological) automorphisms, then $\mathcal{L}_H(V, W)$ denotes the subspace of $\mathcal{L}(V, W)$ consisting of continuous H-equivariant E-linear maps. We will not need to consider any locally convex topologies on the space $\mathcal{L}(V, W)$, so an isomorphism between spaces of continuous linear maps is simply intended as an isomorphism of E-vector spaces. An E-Banach space representation U of H is *unitary* if the topology of U can be defined by a norm that is invariant under H. Thus an E-valued character of H is unitary if and only if it takes values in $\mathscr{O}_E^\times$.

2. The locally convex space $\mathscr{C}^{\mathrm{la}}(\mathscr{O}_L, E)$ and tempered linear maps

In accordance with our convention regarding locally analytic structures mentioned in §1, we regard the locally L-analytic group $\mathscr{O}_L$ as a locally $\mathbf{Q}_p$-analytic group by restriction of scalars. Explicitly, if we choose a $\mathbf{Z}_p$-basis for $\mathscr{O}_L$, then the induced $\mathbf{Z}_p$-linear isomorphism $\mathscr{O}_L \cong \mathbf{Z}_p^r$ is a global chart for the locally $\mathbf{Q}_p$-analytic structure on $\mathscr{O}_L$. By definition, a function $f : \mathscr{O}_L \to E$ is locally analytic if, upon choosing an isomorphism $\mathscr{O}_L \cong \mathbf{Z}_p^r$, the resulting function $\mathbf{Z}_p^r \to E$ admits a power expansion (in r variables, with coefficients in E) in a sufficiently small ball around each point of $\mathbf{Z}_p^r$. Thus, given a choice of coordinates $\mathscr{O}_L \cong \mathbf{Z}_p^r$, elements of the affinoid algebras $\mathscr{A}_n$ give rise to locally analytic functions on $\mathscr{O}_L$. In fact it is somewhat more intrinsic (but equivalent) to consider functions which are locally given by convergent power series in the embeddings $\sigma_i : L \hookrightarrow E$, as we now explain.

If $w \in \mathscr{O}_L$ and $n \geq 0$, then because each $\sigma_i : L \hookrightarrow E$ is an isometry for our choice of absolute values on E and L, an element $F \in \mathscr{A}_n$ gives rise to a continuous function $F_{n,w} : w + \varpi_L^n \mathscr{O}_L \to E$ defined by $F_{n,w}(z) = F(\sigma_1(z-w), \ldots, \sigma_r(z-w))$ (we will abuse notation by sometimes writing the right-hand side of this definition as $F(z-w)$, and will use the same notation to denote the function on $\mathscr{O}_L$ obtained by extending $F_{n,w}$ by zero). It turns out that the E-valued locally analytic functions on $\mathscr{O}_L$ are precisely the functions $\mathscr{O}_L \to E$ which locally arise from this construction in the sense of the following proposition.

Proposition 2.1. *A function $f : \mathscr{O}_L \to E$ is locally analytic if and only if for each $w \in \mathscr{O}_L$ there exists an integer $n \geq 0$ and a series $F \in \mathscr{A}_n$ such that $f|_{w+\varpi_L^n \mathscr{O}_L} = F_{n,w}$.*

Proof. Let $\{z_1, \ldots, z_r\}$ be a $\mathbf{Z}_p$-basis for $\mathscr{O}_L$ and let $\pi : \mathscr{O}_L \cong \mathbf{Z}_p^r$ be the $\mathbf{Z}_p$-linear isomorphism defined by this choice of basis. Then π is an isomorphism of locally analytic groups. For $1 \leq i \leq r$ let $\pi_i : \mathscr{O}_L \to \mathbf{Z}_p$ be the $\mathbf{Z}_p$-linear map given by $\pi_i(z_j) = \delta_{ij}$, so that $\pi(z) = (\pi_1(z), \ldots, \pi_r(z))$ for each $z \in \mathscr{O}_L$. The π_i form an E-basis for the space $M = \mathrm{Hom}_{\mathrm{Mod}/\mathbf{Z}_p}(\mathscr{O}_L, E)$, and we have

$$\sigma_j = \sum_{i=1}^{r} \sigma_j(z_i)\pi_i.$$

for $1 \leq j \leq r$. As the σ_i also form an E-basis for M, we can write

$$\pi_j = \sum_{i=1}^{r} \beta_{ij}\sigma_i$$

for some $\beta_{ij} \in E$, $1 \leq i, j \leq r$. The $r \times r$ matrices (β_{ij}) and $(\sigma_j(z_i))$ are then mutually inverse in $\mathrm{GL}_r(E)$, and $(\sigma_j(z_i))$ has coefficients in $\mathscr{O}_E$ (though it need not have unit determinant, i.e., (β_{ij}) might not have integral coefficients). This is the essential point of the argument which shows that both the σ_i and the π_i can serve as "coordinates" for locally analytic functions on $\mathscr{O}_L$. A similar idea is used in the proof of [7, Lemma 4.1]. However, since the σ_i do not constitute an actual chart for $\mathscr{O}_L$, and since $L/\mathbf{Q}_p$ is not assumed Galois as in *loc. cit.*, we make the details of going from one set of "coordinates" to the other explicit below.

Define polynomials $g_j = \sum_{i=1}^r \sigma_j(z_i)X_i$ and $h_j = \sum_{i=1}^r \beta_{ij}X_i$ in $E[X_1, \ldots, X_r]$ for $1 \leq j \leq r$, noting that

$$g_j(\pi_1(z), \ldots, \pi_r(z)) = \sigma_j(z) \tag{2.1}$$

and

$$h_j(\sigma_1(z),\dots,\sigma_r(z)) = \pi_j(z) \tag{2.2}$$

for all $z \in \mathscr{O}_L$ and $1 \le j \le r$. Suppose $f : \mathscr{O}_L \to E$ is locally-analytic and fix $w \in \mathscr{O}_L$. We may then choose an integer $k \ge 0$ and a power series $F_0 \in \mathscr{A}_k$ such that

$$(f \circ \pi^{-1})(x_1,\dots,x_r) = F_0(x_1 - \pi_1(w),\dots,x_r - \pi_r(w))$$

for each $(x_1,\dots,x_r) \in \mathbf{Z}_p^r$ with $\max_i |x_i - \pi_i(w)| \le q^{-k}$. Now choose an integer $n \ge 0$ large enough to ensure that $\|h_j\|_n = q^{-n}\max_i |\beta_{ij}|$ is less than or equal to q^{-k} for $1 \le j \le r$ (the coefficients β_{ij} have valuation depending on the ramification of L over $\mathbf{Q}_p$). There is then a unique continuous E-algebra homomorphism $\mathscr{A}_k \to \mathscr{A}_n$ satisfying $X_j \mapsto h_j$ for $1 \le j \le r$, and this E-algebra homomorphism is compatible with evaluation of series in $\mathscr{A}_n$ on points of the closed ball around 0 in E^r of radius q^{-n}. (This is the universal property of Tate algebras, and details can be found in [9, Proposition 5.4]. Implicit in the statement about evaluation on points is that each h_j maps the closed ball of radius q^{-n} around 0 in E^r into the closed ball of radius q^{-k} around 0 in E.) Let F be the image of F_0 under this homomorphism (so we think of F as $F_0(h_1,\dots,h_r)$). Given $z \in w + \varpi_L^n\mathscr{O}_L$, we have $|\sigma_i(z-w)| \le q^{-n}$ for $1 \le i \le r$. Using the aforementioned compatibility of $F_0 \mapsto F$ with evaluation on points, we find that

$$\begin{aligned} &F(\sigma_1(z-w),\dots,\sigma_r(z-w)) \\ &\quad = F_0(h_1(\sigma_1(z-w),\dots,\sigma_r(z-w)),\dots,h_r(\sigma_1(z-w),\dots,\sigma_r(z-w))) \\ &\quad = F_0(\pi_1(z-w),\dots,\pi_r(z-w)) = F_0(\pi_1(z)-\pi_1(w),\dots,\pi_r(z)-\pi_r(w)) \\ &\quad = (f\circ\pi^{-1})(\pi_1(z),\dots,\pi_r(z)) \end{aligned}$$

where, in going from the second to the third line, we have used Equation (2.2), and in the final equality, we have used the parenthetical remark above explaining why $|\pi_i(z) - \pi_i(w)| \le q^{-k}$ for $1 \le i \le r$. Thus f has the desired local form.

The converse is similar but more straightforward because the g_j have integral coefficients, which gives $\|g_j\|_n \le q^{-n}$ for $1 \le j \le r$ and for any n. We thus have a unique continuous E-algebra homomorphism $\mathscr{A}_n \to \mathscr{A}_n$ satisfying $X_j \mapsto g_j$ for $1 \le j \le r$. This map is compatible with evaluation on points as before, and we may use it (together with Equation (2.1)) to prove that a function satisfying the condition in the statement of the proposition is locally analytic by converting a local power series expansion in the σ_i to a local power series expansion in the π_i. □

Remark 2.2. It is actually not necessary to check the condition in Proposition 2.1 at every $w \in \mathscr{O}_L$. In fact, the condition in the proposition is equivalent to the condition that there exists a single integer $n \ge 0$ such that for each w in a (necessarily finite) set of coset representatives for $\varpi_L^n\mathscr{O}_L$ in $\mathscr{O}_L$, there exists $F \in \mathscr{A}_n$ (depending on w) such that $f|_{w+\varpi_L^n\mathscr{O}_L} = F_{n,w}$. This follows from [9, Corollary 5.5], which shows that, for any $w' \in w + \varpi_L^n\mathscr{O}_L$, the function $F_{n,w}$ for $F \in \mathscr{A}_n$ coincides with $F'_{n,w'}$ for some $F' \in \mathscr{A}_n$ (and one even necessarily has $\|F\|_n = \|F'\|_n$.)

We now describe the locally convex topology on $\mathscr{C}^{\mathrm{la}}(\mathscr{O}_L, E)$ in terms of the description of this vector space provided by Proposition 2.1. For each $n \ge 0$, let T_n be a set of coset representatives in $\mathscr{O}_L$ for $\varpi_L^n\mathscr{O}_L$, and let $\iota_n : \prod_{w\in T_n}\mathscr{A}_n \to \mathscr{C}^{\mathrm{la}}(\mathscr{O}_L, E)$ be given by sending a

tuple $(F_w)_{w\in T_n}$ of q^{-n}-convergent power series to the function $\mathscr{O}_L \to E$ that is given on the ball $w + \varpi_L^n \mathscr{O}_L$ by $z \mapsto F_w(\sigma_1(z-w), \dots, \sigma_r(z-w))$ (that this function is in fact locally analytic follows from Proposition 2.1 coupled with Remark 2.2). The Zariski density of the L-valued points of an affinoid ball over L ensure that each ι_n is injective, and both the image $\mathscr{F}_n(\mathscr{O}_L, E)$ of ι_n and the norm induced on $\mathscr{F}_n(\mathscr{O}_L, E)$ from the maximum of the Gauss norms on each factor of the source of ι_n are independent of the choice of T_n (again by Remark 2.2). Thus $\mathscr{F}_n(\mathscr{O}_L, E)$ is canonically an E-Banach space. We have $\mathscr{F}_n(\mathscr{O}_L, E) \subseteq \mathscr{F}_{n+1}(\mathscr{O}_L, E)$ for each $n \geq 0$, a continuous inclusion, and Remark 2.2 shows that the natural E-linear injection $\varinjlim_n \mathscr{F}_n(\mathscr{O}_L, E) \to \mathscr{C}^{\mathrm{la}}(\mathscr{O}_L, E)$ is an isomorphism of E-vector spaces. We may therefore endow $\mathscr{C}^{\mathrm{la}}(\mathscr{O}_L, E)$ with the locally convex inductive limit topology coming from this isomorphism and the Banach space structure on each $\mathscr{F}_n(\mathscr{O}_L, E)$. Thus if U is a locally convex space over E, a linear map $\mathscr{C}^{\mathrm{la}}(\mathscr{O}_L, E) \to U$ is continuous if and only if the restriction of the map to $\mathscr{F}_n(\mathscr{O}_L, E)$ is continuous for every $n \geq 0$.

Proposition 2.3. *The locally convex topology just defined on $\mathscr{C}^{\mathrm{la}}(\mathscr{O}_L, E)$ coincides with the locally convex topology defined in* [9, §10].

Proof. Let $\pi : \mathscr{O}_L \cong \mathbf{Z}_p^r$ be as in the proof of Proposition 2.1 and let $e = e(L/\mathbf{Q}_p)$ be the ramification index of L over $\mathbf{Q}_p$. Recall that, with our normalizations, $|p| = q^{-e}$. Thus, if $n \geq 0$, π induces a locally analytic isomorphism $\varpi_L^{ne}\mathscr{O}_L = p^n\mathscr{O}_L \cong p^n\mathbf{Z}_p^r$ between the balls around 0 of radius $|\varpi_L^{ne}| = |p^n| = q^{-ne}$ in $\mathscr{O}_L$ and $\mathbf{Z}_p^r$ (where we use the norm $\|x\| = \max_i |x_i|$ on $\mathbf{Z}_p^r$). In particular, if T_{ne} is a set of coset representatives for $\varpi_L^{ne}\mathscr{O}_L$ in $\mathscr{O}_L$, then $\pi(T_{ne})$ is a set of coset representatives for $p^n\mathbf{Z}_p^r$ in $\mathbf{Z}_p^r$. We have a diagram

$$\begin{array}{ccc} \prod_{w\in T_{ne}} \mathscr{A}_{ne} & \xrightarrow{\iota_{ne}} & \mathscr{F}_{ne}(\mathscr{O}_L, E) \\ \downarrow & & \downarrow \subseteq \\ \prod_{w\in T_{ne}} \mathscr{A}_{ne} & \longrightarrow & \mathscr{C}^{\mathrm{la}}(\mathscr{O}_L, E)_\pi \end{array}$$

where the left-hand vertical map is given in each factor by $X_i \mapsto g_i$ for $1 \leq i \leq r$ (in the notation of the proof of Proposition 2.1), the bottom horizontal map sends a tuple of q^{-ne}-convergent power series $(F_w)_{w\in T_{ne}}$ to the function given on $w + \varpi_L^{ne}\mathscr{O}_L$ by $z \mapsto F(\pi_1(z-w), \dots, \pi_r(z-w))$, and $\mathscr{C}^{\mathrm{la}}(\mathscr{O}_L, E)_\pi$ denotes the E-vector space $\mathscr{C}^{\mathrm{la}}(\mathscr{O}_L, E)$ endowed with the topology of [9, §10]. The commutativity of the diagram holds by the definition of the g_i. The map ι_{ne} is a topological isomorphism by the definition of $\mathscr{F}_{ne}(\mathscr{O}_L, E)$, the continuity of the left-hand vertical map is built into its construction, and the bottom horizontal map is continuous by the definition of the topology on the target. Thus the right-hand vertical inclusion $\mathcal{F}_{ne}(\mathscr{O}_L, E) \subseteq \mathscr{C}^{\mathrm{la}}(\mathscr{O}_L, E)_\pi$ is continuous, from which it follows that $\mathscr{F}_n(\mathscr{O}_L, E) \subseteq \mathscr{C}^{\mathrm{la}}(\mathscr{O}_L, E)_\pi$ is continuous for all $n \geq 0$. Therefore the identity map $\mathscr{C}^{\mathrm{la}}(\mathscr{O}_L, E) \to \mathscr{C}^{\mathrm{la}}(\mathscr{O}_L, E)_\pi$ is continuous. But the source is of compact type (by a straightforward generalization of the argument in the example at the end of [8, §16]), while the target is of compact type by [10, Lemma 2.1]; as bijective continuous linear maps between spaces of compact type are necessarily topological isomorphisms [4, Theorem 1.1.17], the topologies coincide. □

Let $\underline{k} \in \mathbf{Z}_{\geq 0}^r$. The image under ι_n of the finite-dimensional subspace $\prod_{w\in T_n} \mathscr{A}^{\underline{k}} \subseteq \prod_{w\in T_n} \mathscr{A}_n$ will be denoted $\mathscr{F}_n^{\underline{k}}(\mathscr{O}_L, E)$. The inductive limit $\varinjlim_n \mathscr{F}_n^{\underline{k}}(\mathscr{O}_L, E)$ inside $\mathscr{C}^{\mathrm{la}}(\mathscr{O}_L, E)$

is the subspace $\mathscr{C}^{\mathrm{lp}\leq\underline{k}}(\mathscr{O}_L, E)$ of "locally polynomial functions of degree at most $\underline{k}$." Since finite-dimensional (Hausdorff) locally convex spaces over E are necessarily equipped with their finest locally convex topologies, the locally convex inductive limit topology on the space $\mathscr{C}^{\mathrm{lp}\leq\underline{k}}(\mathscr{O}_L, E)$ is its finest locally convex topology. Thus if U is a locally convex space over E, any linear map $\mathscr{C}^{\mathrm{lp}\leq\underline{k}}(\mathscr{O}_L, E) \to U$ is continuous. The inclusion $\mathscr{C}^{\mathrm{lp}\leq\underline{k}}(\mathscr{O}_L, E) \subseteq \mathscr{C}^{\mathrm{la}}(\mathscr{O}_L, E)$ is then a homeomorphism onto its image, which is closed in $\mathscr{C}^{\mathrm{la}}(\mathscr{O}_L, E)$.

We now wish to state a result of Breuil generalizing classical work of Amice-Vélu and Vishik, but must first introduce a slight variation on the construction of functions via convergent power series. We have already explained how a series $F \in \mathscr{A}_n$ gives rise to a locally analytic function on $\mathscr{O}_L$ by (roughly) substituting the embeddings σ_i for the variables X_i (on choosing a "center point" for each ball of radius q^{-n}). By composing with certain continuous homomorphisms $\mathscr{A} \to \mathscr{A}_n$, we can essentially use $\mathscr{A} = \mathscr{A}_0$ to produce the locally analytic functions arising from all the $\mathscr{A}_n$. Namely, if $w \in \mathscr{O}_L$, $n \geq 0$, and $F \in \mathscr{A}$, we will denote by $F((z-w)/\varpi_L^n)$ the locally analytic function on $\mathscr{O}_L$ that is given on $w + \varpi_L^n \mathscr{O}_L$ by $z \mapsto F(\sigma_1((z-w)/\varpi_L^n), \ldots, \sigma_r((z-w)/\varpi_L^n))$, and is extended by zero to the rest of $\mathscr{O}_L$. Note that this construction can also be described as the composite of the map $F \mapsto F_{n,w} : \mathscr{A}_n \to \mathscr{C}^{\mathrm{la}}(\mathscr{O}_L, E)$ with the continuous E-algebra homomorphism $\mathscr{A} \to \mathscr{A}_n$ given by sending X_i to $\sigma_i(\varpi_L)^{-n} X_i$ for $1 \leq i \leq r$. The resulting linear map $\mathscr{A} \to \mathscr{C}^{\mathrm{la}}(\mathscr{O}_L, E)$ is continuous by the definition of the locally convex topology on the target. Note that we can in particular restrict this construction to the finite-dimensional subspace $\mathscr{A}^{\underline{k}} \subseteq \mathscr{A}$ of polynomials of degree at most $\underline{k}$.

Let U be an E-Banach space and let $\|\cdot\|_U$ denote a choice of norm on U inducing its topology. The following definition is independent of this choice in the sense that if the condition in the definition holds for one norm defining the topology of U, it holds for any other. The definition is an immediate translation of [5, Definition 3.12] from the case $L = \mathbf{Q}_p$.

Definition 2.4. Let $\alpha \in E^\times$. An element

$$\varphi \in \mathcal{L}(\mathscr{C}^{\mathrm{la}}(\mathscr{O}_L, E), U) \quad (\text{respectively } \varphi \in \mathcal{L}(\mathscr{C}^{\mathrm{lp}\leq\underline{k}}(\mathscr{O}_L, E), U))$$

is said to be α-tempered if there is a constant $C > 0$ such that for each $F \in \mathscr{A}$ (respectively $F \in \mathscr{A}^{\underline{k}}$), $w \in \mathscr{O}_L$, and $n \in \mathbf{Z}_{\geq 0}$, we have

$$\left\| \varphi\left(F\left(\frac{z-w}{\varpi_L^n} \right) \right) \right\|_U \leq C|\alpha|^{-n} \|F\|_{\mathscr{A}}. \tag{2.3}$$

The notation $\mathcal{L}(\mathscr{C}^{\mathrm{la}}(\mathscr{O}_L, E), U)^\alpha$ (respectively $\mathcal{L}(\mathscr{C}^{\mathrm{lp}\leq\underline{k}}(\mathscr{O}_L, E), U)^\alpha$) indicates the subspace of $\mathcal{L}(\mathscr{C}^{\mathrm{la}}(\mathscr{O}_L, E), U)$ (respectively of $\mathcal{L}(\mathscr{C}^{\mathrm{lp}\leq\underline{k}}(\mathscr{O}_L, E), U)$) consisting of α-tempered maps.

The following lemma equates the condition given in Definition 2.4, which is more suited for our argument, with the condition used in [2].

Lemma 2.5. *Let $\alpha \in E^\times$ and let $c = \mathrm{ord}(\alpha)$. An element $\varphi \in \mathcal{L}(\mathscr{C}^{\mathrm{la}}(\mathscr{O}_L, E), U)$ (respectively $\varphi \in \mathcal{L}(\mathscr{C}^{\mathrm{lp}\leq\underline{k}}(\mathscr{O}_L, E), U)$) is α-tempered if and only if there is a constant $C > 0$ such that for each $w \in \mathscr{O}_L$, $n \in \mathbf{Z}_{\geq 0}$, and $\underline{m} \in \mathbf{Z}_{\geq 0}^r$ (respectively $\underline{m} \in \mathbf{Z}_{\geq 0}^r$ with $m_i \leq k_i$ for $1 \leq i \leq r$), we have*

$$\left\| \varphi\left(\mathbf{1}_{w+\varpi_L^n \mathscr{O}_L}(z) \prod_{i=1}^r \sigma_i(z-w)^{m_i} \right) \right\|_U \leq C q^{-n(|\underline{m}|-c)}. \tag{2.4}$$

Proof. Suppose $\varphi \in \mathcal{L}(\mathscr{C}^{\mathrm{la}}(\mathscr{O}_L, E), U)$ is α-tempered in the sense of Definition 2.4 and let C be a constant for which (2.3) holds. Given $w \in \mathscr{O}_L$, $n \geq 0$, and $\underline{m} \in \mathbf{Z}^r_{\geq 0}$, consider the element $F = \prod_{i=1}^r X_i^{m_i}$ of $\mathscr{A}$, noting that $\|F\|_{\mathscr{A}} = 1$. We have, by definition,

(2.5)
$$F\left(\frac{z-w}{\varpi_L^n}\right) = \mathbf{1}_{w+\varpi_L^n\mathscr{O}_L}(z)\prod_{i=1}^r \sigma_i\left(\frac{z-w}{\varpi_L^n}\right)^{m_i} = \prod_{i=1}^r \sigma_i(\varpi_L)^{-nm_i}\mathbf{1}_{w+\varpi_L^n\mathscr{O}_L}(z)\prod_{i=1}^r \sigma_i(z-w)^{m_i}.$$

Therefore, because φ is α-tempered,

$$\begin{aligned}\left\|\varphi\left(\mathbf{1}_{w+\varpi_L^n\mathscr{O}_L}(z)\prod_{i=1}^r \sigma_i(z-w)^{m_i}\right)\right\|_U &= \left\|\varphi\left(\prod_{i=1}^r \sigma_i(\varpi_L)^{nm_i}F\left(\frac{z-w}{\varpi_L^n}\right)\right)\right\|_U \\ &= \prod_{i=1}^r |\varpi_L|^{nm_i}\left\|\varphi\left(F\left(\frac{z-w}{\varpi_L^n}\right)\right)\right\|_U \\ &\leq \prod_{i=1}^r q^{-nm_i}C|\alpha|^{-n}\|F\|_{\mathscr{A}} = Cq^{-n|\underline{m}|}q^{nc} = Cq^{-n(|\underline{m}|-c)}.\end{aligned}$$

Thus (2.4) holds for φ. The same proof applies to an α-tempered $\varphi \in \mathcal{L}(\mathscr{C}^{\mathrm{lp}\leq\underline{k}}(\mathscr{O}_L, E), U)$, except that we only take $\underline{m} \in \mathbf{Z}^r_{\geq 0}$ with $m_i \leq k_i$ for $1 \leq i \leq r$.

Now assume conversely that (2.4) holds for φ and all relevant data, with constant C, and let F be as above. Taking (2.5) into account along with the fact that $q^{nc} = |\alpha|^{-n}$, (2.4) becomes

$$\left\|\prod_{i=1}^r \sigma_i(\varpi_L)^{nm_i}\varphi\left(F\left(\frac{z-w}{\varpi_L^n}\right)\right)\right\| \leq Cq^{-n|\underline{m}|}|\alpha|^{-n}.$$

The absolute value of the quantity multiplying $F((z-w)/\varpi_L^n)$ is $q^{-n|\underline{m}|}$, so we may cancel this factor from both sides of the above inequality and recall that $\|F\|_{\mathscr{A}} = 1$ to obtain

$$\left\|\varphi\left(F\left(\frac{z-w)}{\varpi_L^n}\right)\right)\right\|_U \leq C|\alpha|^{-n}\|F\|_{\mathscr{A}}.$$

The strong triangle inequality then implies the desired inequality for any $F \in \mathscr{A}$ that is a linear combination of monomials as above. The inequality then holds for a general $F \in \mathscr{A}$ because the polynomials in $\mathscr{A}$ are dense, and the association

$$F \mapsto F((z-w)/\varpi_L^n) : \mathscr{A} \to \mathscr{C}^{\mathrm{la}}(\mathscr{O}_L, E)$$

is continuous. The case of $\varphi \in \mathcal{L}(\mathscr{C}^{\mathrm{lp}\leq\underline{k}}(\mathscr{O}_L, E), U)$ follows from the same argument (except that the final step involving density of the polynomials in $\mathscr{A}$ is not necessary). □

We now state Breuil's generalization to arbitrary L of the result of Amice-Vélu and Vishik (whose result was stated for $L = \mathbf{Q}_p$).

Lemma 2.6. *For $\underline{k} = (k_1, \ldots, k_r) \in \mathbf{Z}^r_{\geq 0}$ and $\alpha \in E^\times$ satisfying $\mathrm{ord}(\alpha) < k_i + 1$ for $1 \leq i \leq r$, the restriction map*

$$\mathcal{L}(\mathscr{C}^{\mathrm{la}}(\mathscr{O}_L, E), U) \to \mathcal{L}(\mathscr{C}^{\mathrm{lp}\leq\underline{k}}(\mathscr{O}_L, E), U)$$

induces an isomorphism

$$\mathcal{L}(\mathscr{C}^{\mathrm{la}}(\mathscr{O}_L, E), U)^\alpha \cong \mathcal{L}(\mathscr{C}^{\mathrm{lp}\leq\underline{k}}(\mathscr{O}_L, E), U)^\alpha.$$

Proof. This is a special case of [2, Lemma 6.1] (where, in the notation of that reference, $J = \emptyset$, so that $J' = \mathrm{Hom}_{\mathrm{Alg}/\mathbf{Q}_p}(L, E)$), taking into account the fact that the condition on linear maps imposed there is equivalent to the condition in Definition 2.4 by Lemma 2.5. □

3. Locally algebraic and locally analytic principal series

Fix $\underline{k} \in \mathbf{Z}^r_{\geq 0}$. We are interested in locally analytic representations of G induced from locally algebraic characters of T (regarded as characters of $\overline{B}$ via the projection $\overline{B} \to T$). More precisely, we consider characters of the form $\chi = \theta\psi_{\underline{k}}$, where $\theta : T \to E^\times$ has the form

$$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \mapsto \theta_1(a)\theta_2(d)$$

for smooth characters $\theta_1, \theta_2 : L^\times \to E^\times$, and $\psi_{\underline{k}} : T \to E^\times$ denotes the character

$$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \mapsto \prod_{i=1}^{r} \sigma_i(d)^{-k_i}.$$

If $V_{\underline{k}}$ is the irreducible algebraic representation $\bigotimes_{i=1}^r \mathrm{Sym}^{k_i}_E(E^2)$ of $\mathbf{G}_E$, and $W_{\underline{k}}$ denotes its contragredient, then $\psi_{\underline{k}}$ is the restriction to T of the highest weight of $W_{\underline{k}}$ (relative to the upper triangular Borel subgroup of $\mathbf{G}_E$).

For χ as above, we define the locally algebraic parabolic induction $I(\chi) = W_{\underline{k}} \otimes_E \mathrm{Ind}_{\overline{B}}^G(\theta)_{\mathrm{sm}}$, where the right tensor factor is the smooth parabolic induction of θ. Letting G act on $W_{\underline{k}}$ via the inclusion $G \hookrightarrow \mathbf{G}_E(E)$ and on $\mathrm{Ind}_{\overline{B}}^G(\theta)_{\mathrm{sm}}$ via the right regular representation on $\mathscr{C}^{\mathrm{sm}}(G, E)$, $I(\chi)$ becomes a $W_{\underline{k}}$-locally algebraic representation of G in the sense of [4, Proposition-Definition 4.2.6]. We also define $I^{\mathrm{la}}(\chi) = \mathrm{Ind}_{\overline{B}}^G(\chi)$, the locally analytic parabolic induction of χ [5, Example C], which consists of all functions $f \in \mathscr{C}^{\mathrm{la}}(G, E)$ satisfying $f(\overline{b}g) = \chi(\overline{b})f(g)$ for each $\overline{b} \in \overline{B}$ and $g \in G$. Letting G act on $I^{\mathrm{la}}(\chi)$ via its right regular representation on the locally convex space $\mathscr{C}^{\mathrm{la}}(G, E)$, and endowing $I^{\mathrm{la}}(\chi)$ with the induced topology, $I^{\mathrm{la}}(\chi)$ becomes a strongly admissible locally analytic representation of G [5, Proposition 1.21].

We may view $I(\chi)$ as a closed subrepresentation of $I^{\mathrm{la}}(\chi)$ in the following way. Let $\mathscr{O}(\mathbf{G}_E)$ denote the affine coordinate ring of $\mathbf{G}_E$, and let $\mathscr{C}^{\mathrm{alg}}(G, E)$ denote the image of the restriction map $\mathscr{O}(\mathbf{G}_E) \hookrightarrow \mathscr{C}^{\mathrm{la}}(G, E)$ (the restriction map is injective because $G \subseteq \mathbf{G}_E(E)$ is Zariski dense in $\mathbf{G}_E(E)$). This is the space of algebraic E-valued functions on G. With $e_{1,i}, e_{2,i}$ the standard elements in the i-th tensor factor of $V_{\underline{k}}$, $e_2 = \otimes_{i=1}^r e_{2,i}^{k_i}$ is a highest weight vector in $V_{\underline{k}}$ relative to the lower triangular Borel subgroup of $\mathbf{G}_E$, and the map $W_{\underline{k}} \to \mathscr{O}(\mathbf{G}_E)$ given by $w \mapsto (g \mapsto w(g^{-1}e_2))$ is a $\mathbf{G}_E(E)$-equivariant E-linear injection which, when composed with the isomorphism $\mathscr{O}(\mathbf{G}_E) \cong \mathscr{C}^{\mathrm{alg}}(G, E)$, allows us to view $W_{\underline{k}}$ as a subrepresentation of $\mathscr{C}^{\mathrm{alg}}(G, E)$ (for the right regular action of G). Tensoring the injection

$$W_{\underline{k}} \hookrightarrow \mathscr{C}^{\mathrm{alg}}(G, E)$$

with the inclusion $\mathrm{Ind}_{\overline{B}}^G(\theta)_{\mathrm{sm}} \subseteq \mathscr{C}^{\mathrm{sm}}(G, E)$ yields an injection

$$I(\chi) \hookrightarrow \mathscr{C}^{\mathrm{alg}}(G, E) \otimes_E \mathscr{C}^{\mathrm{sm}}(G, E),$$

and following this with the map $\mathscr{C}^{\mathrm{alg}}(G, E) \otimes_E \mathscr{C}^{\mathrm{sm}}(G, E) \to \mathscr{C}^{\mathrm{la}}(G, E)$ given by multiplication of algebraic functions and smooth functions gives $I(\chi) \hookrightarrow \mathscr{C}^{\mathrm{la}}(G, E)$ (the injectivity of this multiplication map is again a consequence of the Zariski density of G in $\mathbf{G}_E$) . Writing down the map $W_{\underline{k}} \hookrightarrow \mathscr{C}^{\mathrm{alg}}(G, E)$ explicitly (using the definition given above and the action

of $g \in G$ on e_2) one finds that the image of $I(\chi)$ in $\mathscr{C}^{\mathrm{la}}(G, E)$ is contained in $I^{\mathrm{la}}(\chi)$. Thus $I(\chi)$ is canonically a G-stable subspace of $I^{\mathrm{la}}(\chi)$. Moreover, $I(\chi)$ is closed in $I^{\mathrm{la}}(\chi)$, and its subspace topology coincides with its finest locally convex topology (with respect to which it is an admissible locally $W_{\underline{k}}$-algebraic representation of G by [4, Proposition 6.3.10]).

We may now state our main result. The proof will be given in §4.

Theorem 3.1. *Assume that*

(i) $\mathrm{ord}(\theta_1(\varpi_L)) < k_i + 1$ *for* $1 \leq i \leq r$, *and that*
(ii) $\chi|_{Z(G)}$ *is unitary.*

Then for any unitary E*-Banach space representation* U *of* G, *the restriction map*

$$\mathcal{L}_G(I^{\mathrm{la}}(\chi), U) \to \mathcal{L}_G(I(\chi), U) \tag{3.1}$$

is an isomorphism.

Remark 3.2. The statement of Theorem 3.1 can be reformulated as the assertion that the map $I(\chi) \hookrightarrow I^{\mathrm{la}}(\chi)$ induces a topological isomorphism on universal unitary completions in the sense of [5, Definition 1.1], under the stated hypotheses.

When $L = \mathbf{Q}_p$, this is essentially Proposition 2.5 of [4]. A version of this result (for general L) is also proved as Theorem 7.1 of [2]. Breuil's result applies to more general locally J-analytic parabolic inductions, where J is a subset of $\mathrm{Hom}_{\mathrm{Alg}/\mathbf{Q}_p}(L, E)$, but he restricts attention to injective linear maps. (See [2, p. 10] for the definition of the locally J-analytic induction; locally algebraic induction corresponds to $J = \emptyset$, while locally analytic induction corresponds to $J = \mathrm{Hom}_{\mathrm{Alg}/\mathbf{Q}_p}(L, E)$). We follow Emerton's argument from [5, §3], which is somewhat more representation-theoretic than Breuil's (although we do make crucial use [2, Lemma 6.1], stated as Lemma 2.6 in the previous section, in place of Emerton's appeal to the classical result of Amice-Vélu and Vishik, of which [2, Lemma 6.1] is a natural generalization).

4. Proof of Theorem 3.1

In this section we prove Theorem 3.1. We therefore assume that $\mathrm{ord}(\theta_1(\varpi_L)) < k_i + 1$ for $1 \leq i \leq r$ and that $\chi|_{Z(G)}$, the central character of $I^{\mathrm{la}}(\chi)$ (and that of $I(\chi)$), is unitary. It will be clear in the argument where these hypotheses are invoked. Our proof closely follows that of Emerton in [5, §3]. The key input to make Emerton's argument go through in the general case is provided by the description of $\mathscr{C}^{\mathrm{la}}(\mathscr{O}_L, E)$ from §2 and the accompanying Lemma 2.6 (which takes the place of the result of Amice-Vélu and Vishik used by Emerton).

If V is one of $I^{\mathrm{la}}(\chi), I(\chi)$, denote by $V(N_0)$ the closed subspace of functions in V whose support lies in $\overline{B}N_0$. This is a $G^+(1)$-invariant closed subspace of $I^{\mathrm{la}}(\chi)$ (see §1 for the definition of the monoid $G^+(1)$, and more generally for the monoids $G^+(s)$ used below). The following result is proved for $L = \mathbf{Q}_p$ in [5, Lemma 3.1], but the argument given there applies to an arbitrary finite extension L of $\mathbf{Q}_p$ (in fact the argument there is extremely general and would work for any reductive group over any finite extension of $\mathbf{Q}_p$, and any parabolic subgroup, with appropriate analogues of the submonoids $G^+(s)$).

Lemma 4.1. *For any* E*-Banach space representation* U *of* G, *the restriction maps*

$$\mathcal{L}_G(I^{\mathrm{la}}(\chi), U) \to \mathcal{L}_{G^+(s)}(I^{\mathrm{la}}(\chi)(N_0), U)$$

and

$$\mathcal{L}_G(I(\chi), U) \to \mathcal{L}_{G^+(s)}(I(\chi)(N_0), U)$$

are isomorphisms for all integers $s \geq 1$.

Thus, for an E-Banach space representation U of G and each $s \geq 1$, there is a commutative diagram of restriction maps

$$\begin{array}{ccc} \mathcal{L}_G(I^{\mathrm{la}}(\chi), U) & \longrightarrow & \mathcal{L}_G(I(\chi), U) \\ \downarrow & & \downarrow \\ \mathcal{L}_{G^+(s)}(I^{\mathrm{la}}(\chi)(N_0), U) & \longrightarrow & \mathcal{L}_{G^+(s)}(I(\chi)(N_0), U) \end{array}$$

where the vertical maps are isomorphisms. To prove Theorem 3.1, which is the assertion that the top horizontal arrow is an isomorphism, it therefore suffices to prove that the bottom horizontal arrow is an isomorphism for some $s \geq 1$. We ultimately reduce this to Lemma 2.6. By [6, Lemma 2.3.3], restricting functions in $I^{\mathrm{la}}(\chi)$ to N_0 yields a topological isomorphism $I^{\mathrm{la}}(\chi)(N_0) \cong \mathscr{C}^{\mathrm{la}}(N_0, U)$, and composing this with the topological isomorphism $\mathscr{C}^{\mathrm{la}}(N_0, E) \cong \mathscr{C}^{\mathrm{la}}(\mathscr{O}_L, E)$ (see the discussion of function spaces in §1), we obtain an isomorphism

$$I^{\mathrm{la}}(\chi)(N_0) \cong \mathscr{C}^{\mathrm{la}}(\mathscr{O}_L, E). \tag{4.1}$$

Restricting (4.1) to $I(\chi)(N_0)$, and using the explicit description of the embedding

$$I(\chi) \hookrightarrow I^{\mathrm{la}}(\chi)$$

of §3, we obtain an induced isomorphism

$$I(\chi)(N_0) \cong \mathscr{C}^{\mathrm{lp}\leq \underline{k}}(\mathscr{O}_L, E). \tag{4.2}$$

Since $I^{\mathrm{la}}(\chi)(N_0)$ is a $G^+(1)$-stable subspace of $I^{\mathrm{la}}(\chi)$, we may use the isomorphism (4.1) to transfer the action of $G^+(1)$ on $I^{\mathrm{la}}(\chi)(N_0)$ to an action of $G^+(1)$ on $\mathscr{C}^{\mathrm{la}}(\mathscr{O}_L, E)$. A computation shows that if $f \in \mathscr{C}^{\mathrm{la}}(\mathscr{O}_L, E)$ and $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G^+(1)$, then we have, for any $z \in \mathscr{O}_L$,

(4.3)

$$(gf)(z) = \begin{cases} 0 & \text{if } \dfrac{b+dz}{a+cz} \notin \mathscr{O}_L \\ \left(\prod_{i=1}^r \sigma_i \left(\dfrac{a+cz}{\det(g)} \right)^{k_i} \right) \theta_1(a+cz) \theta_2\left(\dfrac{\det(g)}{a+cz} \right) f\left(\dfrac{b+dz}{a+cz} \right) & \text{if } \dfrac{b+dz}{a+cz} \in \mathscr{O}_L. \end{cases}$$

As $I(\chi)(N_0)$ is a $G^+(1)$-stable subspace of $I^{\mathrm{la}}(\chi)(N_0)$, in light of (4.2), $\mathscr{C}^{\mathrm{lp}\leq \underline{k}}(\mathscr{O}_L, E)$ is a $G^+(1)$-stable subspace of $\mathscr{C}^{\mathrm{la}}(\mathscr{O}_L, E)$ for the action defined in (4.3).

We also need to define an action of $G_0(1)$ on $\mathscr{A}$, which we now explain. Let $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ be an element of $G_0(1)$. By the definition of $G_0(1) \subseteq \mathrm{GL}_2(\mathscr{O}_L)$, each of a, b, c, d is in $\mathscr{O}_L$ and $c \in \varpi_L \mathscr{O}_L$, so $a, d \in \mathscr{O}_L^\times$. Thus, for $1 \leq i \leq r$, the series

$$\frac{\sigma_i(b) + \sigma_i(d) X_i}{\sigma_i(a) + \sigma_i(c) X_i} = \frac{\sigma_i(b) + \sigma_i(d) X_i}{\sigma_i(a)} \sum_{m=0}^{\infty} (-1)^m \left(\frac{\sigma_i(c)}{\sigma_i(a)} \right)^m X_i^m$$

is an element of $\mathscr{A}$ of norm 1. Therefore, there is a unique continuous E-algebra endomorphism $\nu_g : \mathscr{A} \to \mathscr{A}$ with $\nu_g(X_i) = (\sigma_i(b) + \sigma_i(d) X_i)/(\sigma_i(a) + \sigma_i(c) X_i)$ for $1 \leq i \leq r$, and the operator norm of ν_g is at most 1. A (slightly messy but straightforward) computation shows

that $\nu_{g_1 g_2} = \nu_{g_1} \circ \nu_{g_2}$, and it follows that in fact each ν_g is an isometry of $\mathscr{A}$. We now define the $G_0(1)$-action on $\mathscr{A}$ by

$$g(F(X_1, \dots, X_r)) = \left(\prod_{i=1}^{r} \left(\frac{\sigma_i(a) + \sigma_i(c) X_i}{\sigma_i(\det(g))} \right)^{k_i} \right) \theta_2(\det(g)) \nu_g(F(X_1, \dots, X_r)). \tag{4.4}$$

Because the factor multiplying $\nu_g(F(X_1, \dots, X_r))$ in (4.4) is of Gauss norm 1 (since $\det(g)$ is a unit in $\mathscr{O}_L$, $\theta_2(\det(g)) \in \mathscr{O}_E^\times$), this $G_0(1)$-action on $\mathscr{A}$ is unitary. Moreover, the factor ensures that $\mathscr{A}^{\underline{k}}$ is a $G_0(1)$-stable subspace of $\mathscr{A}$ for this action.

Comparing the formulas above, we find that, if $g \in G_0(1) \subseteq G^+(1)$, $F \in \mathscr{A}$, and $z \in \mathscr{O}_L$, then

$$g(F(z)) = \frac{\theta_1(a + cz)}{\theta_2(a + cz)} ((gF)(z)), \tag{4.5}$$

where on the left-hand side g acts on $\mathscr{C}^{\mathrm{la}}(\mathscr{O}_L, E)$ via (4.3), and on the right-hand side g acts on $\mathscr{A}$ via (4.4) (recall that for $F \in \mathscr{A}$, the notation $F(z)$ indicates the function $z \mapsto F(\sigma_1(z), \dots, \sigma_r(z))$ on $\mathscr{O}_L$, and note that if $g \in G_0(1)$, then $(b + dz)/(a + cz) \in \mathscr{O}_L$ for any $z \in \mathscr{O}_L$).

We now follow Emerton in relating the actions just defined to the notion of an α-tempered linear map (Definition 2.4), where $\alpha = \theta_1(\varpi_L)$. In preparation, we introduce the subset B' of B defined by

$$B' = \left\{ \begin{pmatrix} \varpi_L^n & -w \\ 0 & 1 \end{pmatrix} : n \in \mathbf{Z}_{\geq 0}, w \in \mathscr{O}_L \right\}.$$

This is a submonoid of $B^+ = N_0 T^+$ (see §1 for the notation) since we can write, for any $n \geq 0$ and $w \in \mathscr{O}_L$,

$$\begin{pmatrix} \varpi_L^n & -w \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -w \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \varpi_L^n & 0 \\ 0 & 1 \end{pmatrix}.$$

Lemma 4.2. *Any element $b \in B^+$ may be written as $zb't$ with $z \in Z(G)$, $b' \in B'$, and $t \in T_0$.*

Proof. As $b \in B^+ = N_0 T^+$, we may write

$$b = \begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} = \begin{pmatrix} a & wd \\ 0 & d \end{pmatrix}$$

with $w \in \mathscr{O}_L$ and $ad^{-1} \in \mathscr{O}_L$. Then

$$b = \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} \varpi_L^{\mathrm{ord}(a) - \mathrm{ord}(d)} & w \\ 0 & 1 \end{pmatrix} \begin{pmatrix} ad^{-1} \varpi_L^{\mathrm{ord}(d) - \mathrm{ord}(a)} & 0 \\ 0 & 1 \end{pmatrix}$$

is a decomposition of b of the form $zb't \in Z(G) B' T_0$. □

Lemma 4.3. *Let $\mathscr{C}$ denote either $\mathscr{C}^{\mathrm{la}}(\mathscr{O}_L, E)$ or $\mathscr{C}^{\mathrm{lp} \leq \underline{k}}(\mathscr{O}_L, E)$, with $\mathscr{A}_{\mathscr{C}}$ denoting $\mathscr{A}$ in the former case and $\mathscr{A}^{\underline{k}}$ in the latter case. If U is an E-Banach space and $\varphi \in \mathcal{L}(\mathscr{C}, U)$, then φ is $\theta_1(\varpi_L)$-tempered if and only if there exists a positive constant C such that*

$$\|\varphi(b(F(z)))\|_U \leq C \|F\|_{\mathscr{A}}$$

for all $F \in \mathscr{A}_{\mathscr{C}}$ and $b \in B^+$ (where $\| \cdot \|_U$ is any choice of norm on U defining its topology).

Proof. Given $F \in \mathscr{A}_{\mathscr{C}}$, Equation (4.3) gives

$$\begin{pmatrix} \varpi_L^n & -w \\ 0 & 1 \end{pmatrix} F(z) = \theta_1(\varpi_L)^n F((z-w)/\varpi_L^n).$$

It then follows from Definition 2.4 that φ is $\theta_1(\varpi_L)$-tempered if and only if there is a constant $C > 0$ such that $\|\varphi(b'(F(z)))\|_U \leq C\|F\|_{\mathscr{A}}$ for all $F \in \mathscr{A}_{\mathscr{C}}$ and $b' \in B'$. If the condition in the statement of the lemma holds, then certainly this condition holds, since $B' \subseteq B^+$. Conversely, suppose $\|\varphi(b'(F(z)))\|_U \leq C\|F\|_{\mathscr{A}}$ for all $F \in \mathscr{A}_{\mathscr{C}}$ and $b' \in B'$, and let $b \in B^+$. In accordance with Lemma 4.2, we may write $b = z'b't$ with $z' \in Z(G)$, $b' \in B'$, and $t = \left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right)$ with $a, d \in \mathscr{O}_L^\times$. Noting that the action of $Z(G) \subseteq G^+(1)$ on $\mathscr{C}^{\mathrm{la}}(\mathscr{O}_L, E)$ is given by the character χ (because this is the character by which $Z(G)$ acts on $I^{\mathrm{la}}(\chi)(N_0)$), which is assumed unitary, we have

$$\begin{aligned} \|\varphi(b(F(z)))\|_U &= \|\varphi(z'b't(F(z)))\|_U \\ &= \|\chi(z')\varphi(b't(F(z)))\|_U \\ &= \left\|\chi(z')\frac{\theta_1(a)}{\theta_2(a)}\varphi(b'((tF)(z)))\right\|_U \\ &= \|\varphi(b'((tF)(z)))\|_U \leq C\|tF\|_{\mathscr{A}} = \|F\|_{\mathscr{A}}. \end{aligned}$$

(We have used (4.5) in going from the second to the third line and the unitarity of the action of $G_0(1)$ on $\mathscr{A}$ in the final equality.) Thus the condition in the statement of the lemma holds. □

Remark 4.4. The preceding proof is the only point where the unitarity of the central character $\chi|_{Z(G)}$ is used. However, if this hypothesis fails to hold, one could not really hope to learn much about unitary completions of $I^{\mathrm{la}}(\chi)$ from $I(\chi)$ anyway, because unitarity of $\chi|_{Z(G)}$ is an obvious necessary condition for the unique irreducible constituent of $I(\chi)$ to admit a non-zero G-equivariant map into a unitary Banach representation of G.

Lemma 4.5. *In the notation of Lemma 4.3, if U admits a unitary action of B^+, then $\mathcal{L}_{B^+}(\mathscr{C}, U) \subseteq \mathcal{L}(\mathscr{C}, U)^{\theta_1(\varpi_L)}$.*

Proof. Let $\|\cdot\|_U$ be a B^+-invariant norm on U and let $\varphi \in \mathcal{L}_{B^+}(\mathscr{C}, U)$. By the definition of the topology on $\mathscr{C}^{\mathrm{la}}(\mathscr{O}_L, E)$ (see the discussion following Remark 2.2) and the continuity of φ, the restriction of φ to the image of the map $F \mapsto F(z) : \mathscr{A}_{\mathscr{C}} \to \mathscr{C}^{\mathrm{la}}(\mathscr{O}_L, E)$ is bounded, i.e., there is a constant $C > 0$ such that $\|\varphi(F(z))\|_U \leq C\|F\|_{\mathscr{A}}$ for all $F \in \mathscr{A}_{\mathscr{C}}$. Therefore, if $b \in B^+$ and $F \in \mathscr{A}_{\mathscr{C}}$, we have

$$\|\varphi(b(F(z)))\|_U = \|b\varphi(F(z))\|_U = \|\varphi(F(z))\|_U \leq C\|F\|_{\mathscr{A}},$$

where the first equality follows from the assumed B^+-equivariance of φ and the second follows from the B^+-invariance of $\|\cdot\|_U$. Thus the condition in Lemma 4.3 holds, so φ is $\theta_1(\varpi_L)$-tempered. □

We may now complete the proof of Theorem 3.1. Thus we assume that U is a unitary E-Banach space representation of G with $\|\cdot\|_U$ a G-invariant norm defining the topology of U. Recall that our goal was to show that, for some integer $s \geq 1$, the restriction map

$$\mathcal{L}_{G^+(s)}(I^{\mathrm{la}}(\chi)(N_0), U) \to \mathcal{L}_{G^+(s)}(I(\chi)(N_0), U)$$

is an isomorphism. Using the $G^+(1)$-equivariant isomorphisms (4.1) and (4.2), it is equivalent to prove that

$$\mathcal{L}_{G^+(s)}(\mathscr{C}^{\mathrm{la}}(\mathscr{O}_L,E),U)\to\mathcal{L}_{G^+(s)}(\mathscr{C}^{\mathrm{lp}\leq\underline{k}}(\mathscr{O}_L,E),U) \tag{4.6}$$

is an isomorphism for some $s\geq 1$. We will show that it is enough to take s equal to the conductor exponent of the restrictions of θ_1,θ_2 to $\mathscr{O}_L^\times$, i.e., we assume that θ_1,θ_2 are trivial when restricted to $1+\varpi_L^s\mathscr{O}_L$. (There is some such s because the θ_i are smooth.) Now, by Lemma 4.5 (and the fact that $B^+\subseteq G^+(s)$, so that $G^+(s)$-equivariant maps are also B^+-equivariant), we have

$$\mathcal{L}_{G^+(s)}(\mathscr{C}^{\mathrm{la}}(\mathscr{O}_L,E),U)\subseteq\mathcal{L}(\mathscr{C}^{\mathrm{la}}(\mathscr{O}_L,E),U)^{\theta_1(\varpi_L)} \tag{4.7}$$

and

$$\mathcal{L}_{G^+(s)}(\mathscr{C}^{\mathrm{lp}\leq\underline{k}}(\mathscr{O}_L,E),U)\subseteq\mathcal{L}(\mathscr{C}^{\mathrm{lp}\leq\underline{k}}(\mathscr{O}_L,E),U)^{\theta_1(\varpi_L)}. \tag{4.8}$$

As we are assuming that $\mathrm{ord}(\theta_1(\varpi_L))<k_i+1$ for all i, Lemma 2.6 implies that the restriction map

$$\mathcal{L}(\mathscr{C}^{\mathrm{la}}(\mathscr{O}_L,E),U)^{\theta_1(\varpi_L)}\to\mathcal{L}(\mathscr{C}^{\mathrm{lp}\leq\underline{k}}(\mathscr{O}_L,E),U)^{\theta_1(\varpi_L)} \tag{4.9}$$

is an isomorphism. Thus, in light of the inclusions (4.7) and (4.8), and the injectivity of (4.9), we conclude that (4.6) is injective. (We have not yet used the assumption on s.)

To prove the surjectivity of (4.6), fix $\varphi_0\in\mathcal{L}_{G^+(s)}(\mathscr{C}^{\mathrm{lp}\leq\underline{k}}(\mathscr{O}_L,E),U)$. Because of the inclusion (4.8) and the surjectivity of (4.9), there is an element $\varphi\in\mathcal{L}(\mathscr{C}^{\mathrm{la}}(\mathscr{O}_L,E),U)$ that is $\theta_1(\varpi_L)$-tempered and restricts to φ_0 on $\mathscr{C}^{\mathrm{lp}\leq\underline{k}}(\mathscr{O}_L,E)$. It remains to prove that φ is $G^+(s)$-equivariant. To do this, we consider, for a fixed $g\in G^+(s)$, the continuous linear map

$$\varphi' : f\mapsto g^{-1}\varphi(gf) : \mathscr{C}^{\mathrm{la}}(\mathscr{O}_L,E)\to U.$$

Since φ_0 is $G^+(s)$-equivariant, the restriction of φ' to $\mathscr{C}^{\mathrm{lp}\leq\underline{k}}(\mathscr{O}_L,E)$ coincides with that of φ, so if φ' can be shown to be tempered, the injectivity of (4.9) will give $\varphi'=\varphi$, proving the desired equivariance.

We will show that φ' satisfies the condition in Lemma 4.5. Because φ is $\theta_1(\varpi_L)$-tempered, φ satisfies this condition, i.e., there is a constant $C>0$ such that $\|\varphi(b(F(z)))\|_U\leq C\|F\|_{\mathscr{A}}$ for all $b\in B^+$ and $F\in\mathscr{A}$. If $b\in B^+$, then $gb\in G^+(s)=B^+\overline{N}(s)$, so we may write $gb=b_1\bar{n}$ for some $b_1\in B^+$ and $\bar{n}=\left(\begin{smallmatrix}1&0\\w&1\end{smallmatrix}\right)\in\overline{N}(s)$ (so $w\in\varpi_L^s\mathscr{O}_L$). Then, using the G-invariance of $\|\cdot\|_U$, Equation (4.5), the assumption that θ_1 and θ_2 are trivial on $1+\varpi_L^s\mathscr{O}_L$, and the $\overline{N}(s)$-invariance of $\|\cdot\|_{\mathscr{A}}$, we find that

$$\begin{aligned}\|\varphi'(b(F(z)))\|_U&=\|g^{-1}\varphi(gb(F(z)))\|_U\\&=\|\varphi(b_1\bar{n}(F(z)))\|_U\\&=\left\|\frac{\theta_1(1+wz)}{\theta_2(1+wz)}\varphi(b_1((\bar{n}F)(z)))\right\|_U\\&=\|\varphi(b_1((\bar{n}F)(z)))\|_U\leq C\|\bar{n}F\|_{\mathscr{A}}=C\|F\|_{\mathscr{A}}\end{aligned}$$

for any $F\in\mathscr{A}$. Thus, by Lemma 4.5, φ' is tempered, as desired.

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