Computer-assisted methods for the study of stationary solutions in dissipative systems, applied to the Kuramoto-Sivashinski equation

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Abstract. We develop some computer-assisted techniques for the analysis of stationary solutions of dissipative partial differential equations, their stability, as well as bifurcation diagrams. As a case study, these methods are applied to the Kuramoto-Sivashinski equation. This equation has been investigted extensively, and its bifurcation diagram is well known from a numerical point of view. Here, we describe rigorously the full graph of solutions branching off the trivial branch, complete with all secondary bifurcations, for parameter values between 0 and 80. We also determine the dimension of the unstable manifold for the flow at some stationary solution in each branch.

1. Introduction

The methods described in this paper apply to parabolic equations of the form

$$\partial_t u + (i\partial_x)^m u + H_\alpha(u, \partial_x u, \ldots, \partial_x^{m-1} u) = 0\,, \tag{1.1}$$

with $m > 0$ even, H_α real analytic, and $u = u(t,x)$ periodic in x. The goal is to give a detailed description of stationary solutions, their stability, their dependence on the parameter α, and bifurcation diagrams. Our analysis uses the fact that the stationary solutions of equation (1.1) are real analytic, due to the analyticity of H_α. This allows us to obtain accurate bounds in a relatively straightforward and general way. By comparison, another approach that has been proposed recently requires model-specific a priori bounds, which are hard to come by when dealing with unstable solutions [8].

As a case study, we consider the unidimensional Kuramoto-Sivashinski equation, which has been the focus of numerous analytical and numerical investigations [1–9]. Considering Dirichlet boundary conditions on $[0,\pi]$, this equation can be written in the form

$$\begin{aligned} \partial_t u + 4\partial_x^4 u + \alpha\bigl(\partial_x^2 u + 2u\partial_x u\bigr) &= 0\,, \\ u(t,0) = u(t,\pi) &= 0\,, \end{aligned} \qquad x \in [0,\pi]\,, \quad t \in \mathbb{R}\,. \tag{1.2}$$

Notice that if u is a solution, then so is $R_0 u$, where $(R_0 u)(t,x) = -u(t,\pi - x)$. In addition, the PDE part of equation (1.2) is invariant under translations. Thus, we can (and will) regard (1.2) as an equation for functions u that are odd and 2π-periodic in the variable x.

Another feature of the equation (1.2) is the following. Assume that u is a solution for $\alpha = a$, and define $(S_n u)(t,x) = n u(n^4 t, nx)$. If n is a positive integer, then the rescaled

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function $S_n u$ is again a solution, but for $\alpha = n^2 a$. The same scaling, with $n = a^{-1/2}\alpha$, yields a solution of the Kuramoto-Sivashinski equation for $\alpha = 1$, on the spatial domain $[0, L]$ with $L = a^{1/2}\pi$.

The first part of our analysis focuses on the bifurcation diagram for odd steady states, that is, solutions of the equation

$$4u'''' + \alpha(u'' + 2uu') = 0\,, \qquad u(0) = u(\pi) = 0\,. \tag{1.3}$$

This task is simplified by the fact that many bifurcations in this system involve the breaking of some symmetry. The symmetries are given by the rescaled versions $R_n = S_{2^n} R_0 S_{2^n}^{-1}$ of the reflection R_0. If u is an odd steady state that is invariant under R_n for all $n < k$, then u is periodic with period $2^{1-k}\pi$, and thus $R_k u$ is again an odd steady state. This shows e.g. that the nontrivial steady states come in pairs.

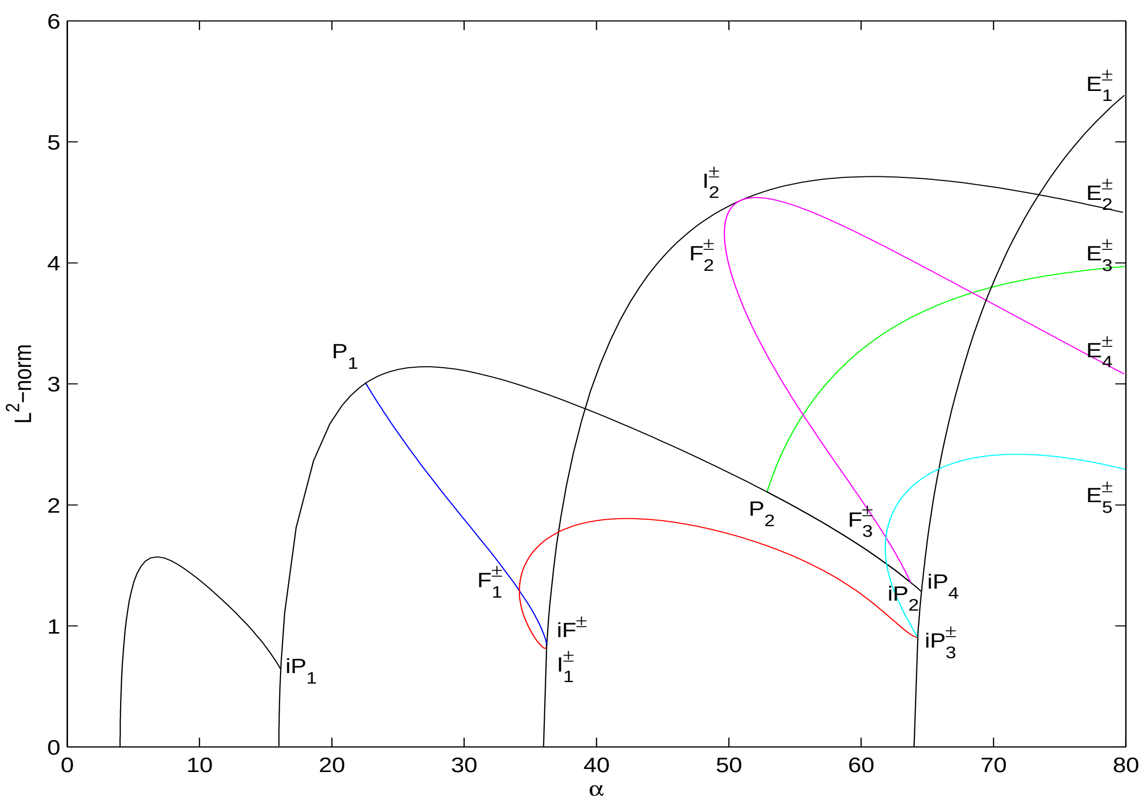

Figure 1. Bifurcation diagram (L^2 norm versus α) for the Kuramoto-Sivashinski equation.

The bifurcations involving the trivial solution $u \equiv 0$ can be described by standard methods and are well known. They occur (only) at those values of the parameter α where the linearized version of equation (1.3), obtained by omitting the term $2uu'$, admits a nontrivial solution; or equivalently, where one of the eigenvalues $4k^4 - \alpha k^2$ of the operator $u \mapsto 4u'''' + \alpha u''$ vanishes. This happens at $\alpha = 4k^2$, with k any positive integer, resulting in a pitchfork bifurcation, where two branches of nontrivial solutions bifurcate off $u \equiv 0$.

These branches are called unimodal ($k = 1$), bimodal ($k = 2$), or k-modal in general. It should be noted that the stability of symmetric solutions need not be identical. In particular, subsequent bifurcations may occurs on both branches, or on one branch only. Numerically, the bifurcation picture is well known, up to values of α around 80; see e.g. [3]. Figure 1 below shows (the L^2 norm for) all solution branches that can be reached via bifurcations from the trivial branch $u \equiv 0$ within the parameter interval $[0, 80]$. The k-modal branches are shown as black lines, while the secondary branches are shown in colored (grey) lines.

One of our goals is to prove that the branches and vertices in this diagram correspond to true solution curves and bifurcations for equation (1.3). The same methods should apply to the steady state equations for many other systems of the type (1.1).

The second part of our analysis is concerned with the flow near stationary states, and invariant manifolds, for equations of the type (1.1). In particular, we determine the dimension of the unstable manifold at several steady states for the Kuramoto-Sivashinski equation (1.2). This involves, among other things, estimates on the eigenvalues of the linearized evolution operator.

The problem of computing rigorously the bifurcation diagram for the stationary solutions of the Kuramoto-Sivashinski equation has been considered before. A partial result, namely the computation of some bifurcation points, was obtained by Zgliczyński and Mischaikow in [8]. Their analysis is based on a priori bounds for the high frequency modes, that are derived beforehand. In our approach, the computer works from the ground up with sets in the full function space and carries out all of the necessary estimates.

2. Results

In this section we present our main results on the Kuramoto-Sivashinski equation. Results and methods that can be applied to other systems are described in subsequent sections.

Figure 2 below defines an abstract graph Γ that will be used to label the bifurcations and solution branches for the Kuramoto-Sivashinski equation (1.3). An edge starting at vertex ω and ending at vertex ω' will be denoted by (ω, ω'). For a linear chain $\{(\omega_1, \omega_2), (\omega_2, \omega_3), \ldots, (\omega_{n-1}, \omega_n)\}$ with endpoints $\omega_1 = \beta$ and $\omega_n = \gamma$, we will use the notation $\langle \beta, \gamma \rangle$.

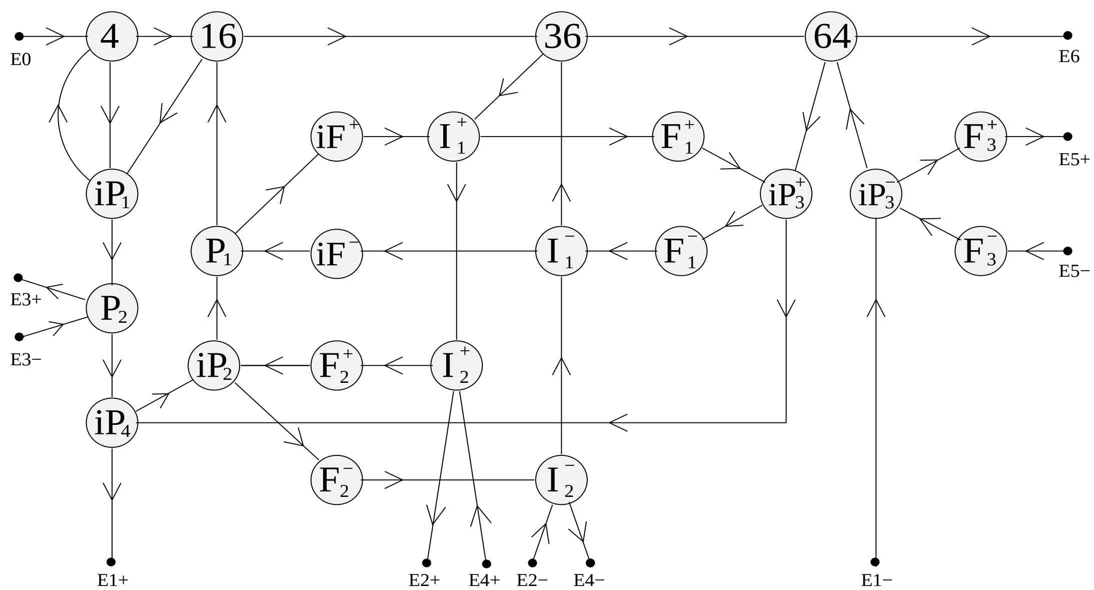

Figure 2. Bifurcation graph Γ for the Kuramoto-Sivashinski equation.

Before we can make a precise statement about solution curves, it is necessary to introduce some functions spaces. Given any positive real number ρ, denote by $\mathcal{H}_\rho$ the space of all functions $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \sum_{k=1}^{\infty} f_k \sin(kx) + \sum_{k=0}^{\infty} f'_k \cos(kx)\,, \tag{2.1}$$

that have a finite norm

$$\|f\|_\rho = \sum_{k=1}^{\infty} |f_k| e^{k\rho} + \sum_{k=0}^{\infty} |f'_k| e^{k\rho}\,. \tag{2.2}$$

The subspace of odd and even functions in $\mathcal{H}_\rho$ will be denoted by $\mathcal{A}_\rho$ and $\mathcal{B}_\rho$, respectively. Notice that the functions in $\mathcal{H}_\rho$ extend to analytic functions in the strip $|\mathrm{Im}(x)| < \rho$.

In the following theorem, a solution of equation (1.3) is understood to be a pair (α, u) in $\mathbb{R} \times \mathcal{A}_\rho$ satisfying (1.3). Here, $\rho = 1/16$.

Theorem 2.1. *There exists an open subset U in $\mathbb{R}\times\mathcal{A}_\rho$, such that the following holds. The solutions in U of equation (1.3) are organized into 44 smooth branches, participating in 23 bifurcations, as described by the lines and vertices of the graph Γ. The branches that connect two bifurcation points are made up of solutions with α in $[3,65]$, while the other branches extend to values of α outside $[1,80]$. At $\alpha=4,16,36,64$, the trivial solution $(\alpha,0)$ undergoes a pitchfork bifurcation. The type and α-value for each of the remaining 19 bifurcations is given in the following table.*

type	α	type	α	type	α
iP_1	$16.13985\ldots$	P_2	$52.89105\ldots$	$F_1^\pm$	$34.16913\ldots$
P_1	$22.55606\ldots$	iP_2	$63.73699\ldots$	$iF^\pm$	$36.23501\ldots$
$I_1^\pm$	$36.23390\ldots$	$iP_3^\pm$	$64.27481\ldots$	$F_2^\pm$	$49.66453\ldots$
$I_2^\pm$	$50.90983\ldots$	iP_4	$64.55942\ldots$	$F_3^\pm$	$61.82232\ldots$

A symbol P, iP, I, F, and iF in this table indicates a bifurcation of type "pitchfork", "inverse pitchfork", "intersection", "fold", and "inverse fold", respectively.

Our proof of this theorem, and related results, will be given in Section 3. It is based on estimates that have been carried out with the aid of a computer, as described in the Appendix. We also show that the restriction of R_k to $U\cap R_k(U)$ acts as follows. Denote by r_k the "pullback" of R_k under the map that associates to each edge (vertex) of the graph its solution branch (bifurcation point) in U. Then $I_n^- = r_2(I_n^+)$ for $n=1,2$; $F_n^- = r_0(F_n^+)$ for $n=1,2,3$; $iF^- = r_0(iF^+)$ and $iP_3^- = r_3(iP_3^+)$.

Next, we consider the stability of stationary solutions $t\mapsto v\in\mathcal{A}_\rho$ under the flow induced by equation (1.2). The time evolution of a function $u=v+h$ is described by the equation

$$\dot h = L_v h + \alpha D(h^2)\,, \qquad L_v h = -\bigl(4D^4+\alpha D^2\bigr)h + 2\alpha D(vh)\,. \tag{2.3}$$

Here, and in what follows, $u\mapsto \dot u$ and D denote differentiation with respect to the variables t and x, respectively. As is the case for general equations of the form (1.1), the local time evolution is well defined on $\mathcal{A}_\rho$, and under certain conditions on the spectrum of the operator L_v, it admits local stable and unstable manifolds that are tangent at v to the corresponding spectral subspaces for L_v; see Section 4 for details.

In the following theorem, we consider all solutions of the Kuramoto-Sivashinski equation (1.2), for $\alpha\in\{10,20,\ldots,80\}$, that lie on the branches described in Theorem 2.1.

Theorem 2.2. *For each triple (α,β,γ) in the table below, equation (1.2) admits two stationary solutions, one on the branch $\langle\beta,\gamma\rangle$, and one on the branch $\langle\gamma,\beta\rangle$, with L^2 norm in the given interval. The unstable manifold at these solutions has dimension p as indicated in this table.*

α	β	γ	p	L^2 norm	α	β	γ	p	L^2 norm
10	4	iP_1	$0,0$	$1.378\ldots$	60	$F_2^\pm$	iP_2	$1,1$	$2.043\ldots$
20	16	iP_4	$0,1$	$2.697\ldots$	60	36	$E_2^\pm$	$0,0$	$4.711\ldots$
30	16	iP_4	$0,0$	$3.109\ldots$	60	$I_2^\pm$	$E_4^\pm$	$1,1$	$4.233\ldots$
30	P_1	$iF^\pm$	$1,1$	$1.880\ldots$	60	P_2	$E_3^\pm$	$0,0$	$3.279\ldots$
40	16	iP_4	$0,2$	$2.757\ldots$	70	$I_2^\pm$	$E_4^\pm$	$1,1$	$3.658\ldots$
40	36	$E_2^\pm$	$1,1$	$3.081\ldots$	70	64	$E_1^\pm$	$2,2$	$3.840\ldots$
40	$F_1^\pm$	iP_3^+	$2,2$	$1.869\ldots$	70	36	$E_2^\pm$	$2,2$	$4.627\ldots$
50	$F_1^\pm$	iP_3^+	$2,2$	$1.761\ldots$	70	P_2	$E_3^\pm$	$0,0$	$3.802\ldots$
50	16	iP_4	$0,2$	$2.266\ldots$	70	$F_3^\pm$	$E_5^\pm$	$3,3$	$2.409\ldots$
50	$F_2^\pm$	iP_2	$1,2$	$3.985\ldots$	80	$I_2^\pm$	$E_4^\pm$	$1,1$	$3.075\ldots$
50	$F_2^\pm$	$I_2^\pm$	$0,2$	$4.423\ldots$	80	64	$E_1^\pm$	$2,1$	$5.394\ldots$
50	36	$I_2^\pm$	$1,1$	$4.469\ldots$	80	36	$E_2^\pm$	$2,2$	$4.411\ldots$
60	$F_1^\pm$	iP_3^+	$2,2$	$1.262\ldots$	80	P_2	$E_3^\pm$	$0,0$	$3.970\ldots$
60	16	iP_4	$1,1$	$1.656\ldots$	80	$F_3^\pm$	$E_5^\pm$	$3,3$	$2.292\ldots$

Our proof of this theorem is given in Section 4, by reducing it to estimates that can be (and have been) carried out with the aid of a computer.

3. Existence of solutions and branches of solutions

3.1. Solutions

A common way of solving differential equations is to convert them to integral equations, where one can take advantage of the compactness property of the inverse derivative. We can achieve the same effect by rewriting equation (1.3) in the form $F_\alpha(u)=u$, where

$$F_\alpha(u)=-\tfrac{\alpha}{4}D^{-2}u+\tfrac{\alpha}{4}D^{-3}\bigl(u^2\bigr)\,. \tag{3.1}$$

Here, D^{-1} denotes the antiderivative operator on the space of continuous 2π-periodic functions with average zero, extended to functions with nonzero average by first subtracting their average value. We will focus on cases where the spectrum of $DF_\alpha(u)$ is bounded away from 1. Then it is relatively easy to find an approximate fixed point u_0 of F_α , and to prove that there exists a true fixed point nearby.

The reason why this approach works well is that F_α is compact, which allows for accurate finite dimensional approximations. It should be noted that the same holds for the maps F_α associated with other equations of the form (1.1). In the present context, it is natural to use approximations based on frequency cutoffs. Given a fixed integer $n>0$, the low frequency part $\mathbb{P}_n^- f$ of a function $f\in\mathcal{H}_\rho$ is defined by restricting its Fourier series (2.2) to frequencies $k\le n$. The corresponding projection onto high frequency modes is defined as $\mathbb{P}_n^+=\mathrm{I}-\mathbb{P}_n^-$. For a linear operator A on $\mathcal{H}_\rho$, we will write $A\asymp 0$ if $\mathbb{P}_n^+A=A\mathbb{P}_n^+=0$ for some $n>0$.

Following the ideas described in [10], we choose an approximation $M\asymp 0$ for the map $\mathrm{I}+[DF_\alpha(u_0)-\mathrm{I}]^{-1}$, and then define

$$\mathcal{C}_\alpha(u)=F_\alpha(u)-M\bigl[F_\alpha(u)-u\bigr]. \tag{3.2}$$

Formally, the map $\mathcal{C}_\alpha$ is close to the Newton map for F_α. Thus, our goal is to prove that $\mathcal{C}_\alpha$ is a contraction. The necessary conditions are given in the following proposition.

Proposition 3.1. *Consider a triple (α, u_0, r), where α and $r > 0$ are real numbers, and u_0 is a function in $\mathcal{A}_\rho$. Assume that there exists a bounded linear operator M on $\mathcal{A}_\rho$, and constants $\varepsilon, K > 0$, such that $M - \mathrm{I}$ has a bounded inverse and*

$$\|\mathcal{C}_\alpha(u_0) - u_0\| < \varepsilon\,, \qquad \|D\mathcal{C}_\alpha(u)\| < K\,, \qquad \varepsilon + Kr < r\,, \tag{3.3}$$

for all functions u in a closed ball B in $\mathcal{A}_\rho$ of radius r, centered at u_0. Then the equation (1.3) has a unique solution $u = u_\alpha$ in B.

The proof of Proposition 3.1 is straightforward: under the given hypothesis, the contraction mapping principle guarantees that the map $\mathcal{C}_\alpha$ has a unique fixed point $u_\alpha \in B$; and since $M - \mathrm{I}$ is invertible, the fixed points of $\mathcal{C}_\alpha$ agree with those of F_α, which are precisely the solutions of equation (1.3).

This proposition is used to prove the existence of a solution u of equation (1.3) near an approximate solution u_0. In particular, we have the following

Lemma 3.2. *For each of the endpoints (vertices of order 1) in the graph Γ, there exist a Fourier polynomial u_0, and a real number $r > 0$, such that the hypotheses of Proposition 3.1 are satisfied for the triple $(80, u_0, r)$. Furthermore, the ball B of radius r centered at u_0 is disjoint from the corresponding balls for the other endpoints.*

Our (computer-assisted) proof of this proposition will be described in the Appendix.

3.2. Bifurcations

For the study of bifurcations, we write equation (1.3) as $\mathcal{F}(\alpha, u) = 0$, where

$$\mathcal{F}(\alpha, u) = -u - \tfrac{\alpha}{4} D^{-2} u + \tfrac{\alpha}{4} D^{-3}(u^2)\,. \tag{3.4}$$

The types of bifurcations considered here take place in two dimensional submanifolds of $\mathcal{A}_\rho$. We will parametrize these surfaces by using the frequency α and the value λ of some coordinate function on $\mathcal{A}_\rho$. As a coordinate function, we choose a suitable one-dimensional projection $\ell \asymp 0$. Then we define a two-parameter family of functions $u(\alpha, \lambda)$ in $\mathcal{A}_\rho$ by solving

$$(\mathrm{I} - \ell)\mathcal{F}\bigl(\alpha, u(\alpha, \lambda)\bigr) = 0\,, \qquad \ell u(\alpha, \lambda) = \lambda \hat{u}\,, \tag{3.5}$$

where $\hat{u}$ is a fixed nonzero function in the range of ℓ. Our goal is to show that for certain rectangles $I \times J$ in parameter space, the equation (3.5) has a smooth and locally unique solution $u : I \times J \to \mathcal{A}_\rho$. Then locally, the solutions of $\mathcal{F}(\alpha, u) = 0$ are determined by the zeros of the function g,

$$g(\alpha, \lambda)\hat{u} = \ell \mathcal{F}\bigl(\alpha, u(\alpha, \lambda)\bigr)\,. \tag{3.6}$$

The equation (3.5) for $u = u(\alpha, \lambda)$ is equivalent to the fixed point equation for the map $F_{\alpha,\lambda}$, defined by $F_{\alpha,\lambda}(u) = (\mathrm{I} - \ell)F_\alpha(u) + \lambda\hat{u}$. As in the last subsection, we use the contraction mapping principle to solve this fixed point problem. (By the implicit function theorem, the solution then depends smoothly on the two parameters α and λ.) After choosing $M \asymp 0$ appropriately, we set

$$\mathcal{C}_{\alpha,\lambda}(u) = F_{\alpha,\lambda}(u) - M\bigl[F_{\alpha,\lambda}(u) - u\bigr]. \tag{3.7}$$

Proposition 3.3. *Consider a quadruple (I, J, u_0, r), where I and J are subintervals of $\mathbb{R}$, u_0 is a function in $\mathcal{A}_\rho$, and r a positive real number. Assume that for some choice of ℓ and $\hat{u}$, there exists a bounded linear operator M on $\mathcal{A}_\rho$, and constants $\varepsilon, K > 0$, such that $M - \mathrm{I}$ has a bounded inverse and*

$$\|\mathcal{C}_{\alpha,\lambda}(u_0) - u_0\| < \varepsilon\,, \qquad \|D\mathcal{C}_{\alpha,\lambda}(u)\| < K\,, \qquad \varepsilon + Kr < r\,, \tag{3.8}$$

for all (α, λ) in an open neighborhood of $I \times J$, and for all functions u in a closed ball B in $\mathcal{A}_\rho$ of radius r, centered at u_0. Then for every (α, λ) in $I \times J$, the equation (3.5) has a unique solution $u(\alpha, \lambda)$ in B, and the corresponding family $(\alpha, \lambda) \mapsto u = u(\alpha, \lambda)$ is smooth. For any given $\alpha \in I$, a function u in $B \cap \ell^{-1}(J\hat{u})$ solves equation (1.3) if and only if $u = u(\alpha, \lambda)$ for some $\lambda \in J$, and $g(\alpha, \lambda) = 0$.

This leaves the problem of verifying that the zeros of g correspond to a specific type of bifurcations. For the sake of definiteness, we will restrict our discussion here to the case of a pitchfork bifurcation. A sufficient set of conditions for the existence of such a bifurcation is given below. A concrete example of a function g that satisfies these conditions (near the origin) is $(\alpha, \lambda) \mapsto \lambda^3 - \alpha\lambda$.

If f is any differentiable function of two variables, denote by $\dot{f}$ and f' the partial derivatives of f with respect to the first and second argument, respectively.

Let $I = [\alpha_1, \alpha_2]$ and $J = [-b, b]$.

Lemma 3.4. (pitchfork bifurcation) *Let g be a real-valued C^3 function on an open neighborhood of $I \times J$, such that $g(\alpha, 0) = 0$ for all $\alpha \in I$, and*
(1) $g''' > 0$ *on* $I \times J$, *(2)* $\dot{g}' < 0$ *on* $I \times J$,
(3) $g'(\alpha_1, 0) \pm \frac{1}{2} b g''(\alpha_1, 0) > 0$, *(4)* $\pm g(\alpha_2, \pm b) > 0$, *(5)* $g'(\alpha_2, 0) < 0$.
Then $g(\alpha, \lambda) = \lambda G(\alpha, \lambda)$ for some C^2 function G, and the solution set of $G(\alpha, \lambda) = 0$ in $I \times J$ is the graph of a C^2 function $\alpha = a(\lambda)$, defined on a proper subinterval J_0 of J. This function takes the value α_2 at the endpoints of J_0, and satisfies $\alpha_1 < a(\lambda) < \alpha_2$ at all interior points of J_0, which includes the origin.

Analogous lemmas for other bifurcations, as well as the proofs, can be found in the Appendix. These lemmas will be referred to collectively as the "basic bifurcation lemmas".

The claims of Theorem 2.1 concerning the bifurcations are now a consequence of the following lemma, whose proof will be described in the Appendix.

Lemma 3.5. *For each ω among the vertices $\{4, 16, 36, 64, P_1, P_2\}$ of the graph Γ, there exists a quadruple (I_ω, u_0, r), such that the hypotheses of Proposition 3.3 are satisfied, for some choices of the projection ℓ and the function $\hat{u}$. Each $\alpha \in I_\omega$ satisfies the bound given for the value of the bifurcation parameter in Theorem 2.1. Let now $C_\omega = B \cap \ell^{-1}(J\hat{u})$, where B denotes the closed ball of radius r in $\mathcal{A}_\rho$, centered at u_0. Then the function g*

which, according to Proposition 3.3, determines the solutions of (1.3) in the region $I_\omega \times C_\omega$, satisfies the hypotheses of Lemma 3.4. Analogous results hold for the other vertices of the graph Γ.

3.3. Derivatives

The derivatives of g that are needed in Lemma 3.4 are obtained by implicit differentiation as follows. Writing the solution of (3.5) as $u = \lambda\hat{u} + h$, we have

$$g(\lambda,\alpha)\hat{u} = \mathcal{F}\bigl(\alpha, \lambda\hat{u} + h(\lambda,\alpha)\bigr)\,, \qquad \ell h = 0\,. \tag{3.9}$$

Given a pair of non-negative integers $n = (n_1, n_2)$, define $\partial^n = \partial_1^{n_1}\partial_2^{n_2}$, where ∂_j denotes partial differentiation with respect to the j-th argument, and let

$$\Gamma_n = \partial^n g\hat{u} - \mathcal{F}'\partial^n h\,. \tag{3.10}$$

Notice that, if we use equation (3.9) to express $\partial^n g$ in terms of derivatives of $\mathcal{F}$ and h, then Γ_n only involves derivatives of h up to order $|n| - 1$. Applying the projection $P = \mathrm{I} - \ell$ to both sides of equation (3.10), we obtain

$$\partial^n h = -P(P\mathcal{F}'P)^{-1}P\Gamma_n\,. \tag{3.11}$$

This identity can now be used to compute the derivatives of h recursively. Substituting it back into (3.10) yields

$$\partial^n g\hat{u} = \bigl\{\ell - \ell\mathcal{F}'P(P\mathcal{F}'P)^{-1}P\bigr\}\Gamma_n\,. \tag{3.12}$$

In order to eliminate the need to invert $\mathcal{F}'$ on the subspace $P\mathcal{A}_\rho$, which is not very practical, consider the function $\gamma = \ell\mathcal{F}'\hat{u}$ and the operator

$$W = \mathcal{F}' + c\ell - P\mathcal{F}'\ell\,, \tag{3.13}$$

where c is some constant such that $\gamma + c$ is nonzero. In matrix notation, corresponding to the splitting $\mathrm{I} = \ell + P$, the inverse of W is given by

$$W^{-1} = \begin{bmatrix} (\ell\mathcal{F}'\ell + c)^{-1} & -(\ell\mathcal{F}'\ell + c)^{-1}\ell\mathcal{F}'P(P\mathcal{F}'P)^{-1} \\ 0 & (P\mathcal{F}'P)^{-1} \end{bmatrix}. \tag{3.14}$$

Thus, we can write the equations (3.11) and (3.12) as follows.

$$\partial^n h = -PW^{-1}\Gamma_n\,, \qquad \partial^n g\hat{u} = (\gamma + c)\ell W^{-1}\Gamma_n\,. \tag{3.15}$$

A straightforward computation now shows that the first few functions Γ_n are given by

$$\begin{aligned}
\Gamma_{(0,1)} &= \mathcal{F}'\hat{u}\,, & \Gamma_{(1,0)} &= \dot{\mathcal{F}}\,,\\
\Gamma_{(0,2)} &= \mathcal{F}''[\hat{u} + h', \hat{u} + h']\,, & \Gamma_{(1,1)} &= \dot{\mathcal{F}}'(\hat{u} + h') + \mathcal{F}''[\hat{u} + h', \dot{h}]\,,\\
\Gamma_{(0,3)} &= 3\mathcal{F}''[\hat{u} + h', h'']\,.
\end{aligned}$$

3.4. Inversion

We have $W = W_1 + W_2$, where

$$\begin{aligned} W_1 f &= -f - \tfrac{\alpha}{4} D^{-2} f\,, \\ W_2 f &= c\ell f + \tfrac{\alpha}{4} P D^{-2} \ell f + \tfrac{\alpha}{2} \ell D^{-3}(u\ell f) + \tfrac{\alpha}{2} D^{-3}(uPf)\,. \end{aligned} \tag{3.16}$$

Let $n > 0$ be a fixed cutoff frequency, such that $\ell = \mathbb{P}_n^- \ell \mathbb{P}_n^-$. In order to estimate the inverse of W, we approximate W at low frequencies by an operator $W_0 = \mathbb{P}_n^- W_0 \mathbb{P}_n^-$ of rank n, and at high frequencies by W_1. To be more precise, let

$$W_3 = \mathbb{P}_n^+ W_1 \mathbb{P}_n^+ + W_0\,, \qquad W_4 = W_0 - \mathbb{P}_n^- W_1 \mathbb{P}_n^- - W_2\,.$$

Then $W = W_3 - W_4$, and

$$W^{-1} = (W_3 - W_4)^{-1} = \bigl[\mathrm{I} - W_3^{-1} W_4\bigr]^{-1} W_3^{-1}\,.$$

The first step is to obtain good estimates on the two operators

$$\begin{aligned} W_3^{-1} &= \mathbb{P}_n^+ W_1^{-1} \mathbb{P}_n^+ + \mathbb{P}_n^- W_0^{-1} \mathbb{P}_n^-\,, \\ W_3^{-1} W_4 &= \mathbb{P}_n^- - \mathbb{P}_n^- W_0^{-1} \mathbb{P}_n^- W - \mathbb{P}_n^+ W_1^{-1} \mathbb{P}_n^+ W_2\,, \end{aligned} \tag{3.17}$$

and in particular, a bound

$$\bigl\| W_3^{-1} W_4 \bigr\| = a < 1\,. \tag{3.18}$$

Then the inverse of W can be estimated by using the identity

$$W^{-1} = \sum_{n=0}^{m-1} \bigl(W_3^{-1} W_4\bigr)^n W_3^{-1} + \bigl[\mathrm{I} - W_3^{-1} W_4\bigr]^{-1} \bigl(W_3^{-1} W_4\bigr)^m W_3^{-1}\,,$$

for some $m > 0$, depending on the value of a in the bound (3.18).

As can be seen from equation (3.14), it suffices to obtain a good estimate on $PW^{-1}P$. In practice, if this operator acts on a function h, we first determine

$$h_k = \bigl(W_3^{-1} W_4\bigr)^k W_3^{-1} P h\,, \qquad k = 0, 1, \ldots, m\,. \tag{3.19}$$

Then

$$PW^{-1}Ph = P\bigl(h_0 + \ldots + h_{m-1}\bigr) + P\bigl[\mathrm{I} - W_3^{-1} W_4\bigr]^{-1} h_m\,, \tag{3.20}$$

and the last term in this equation can be bounded by

$$\bigl\| P\bigl[\mathrm{I} - W_3^{-1} W_4\bigr]^{-1} h_m \bigr\| \le (1-a)^{-1} \|P\| \|h_m\|\,. \tag{3.21}$$

3.5. Branches

The solutions of equation (1.3) that have been discussed so far correspond to the vertices (including endpoints) of the graph Γ. In order to establish the existence of solution branches associated with the edges of this graph, we need a slight generalization of Proposition 3.1, which follows from Proposition 3.1 and the implicit function theorem.

Proposition 3.6. *Consider a triple (I, u_0, r), where I is a subinterval of $\mathbb{R}$, u_0 a function in $\mathcal{A}_\rho$, and r a positive real number. Assume that there exists a bounded linear operator M on $\mathcal{A}_\rho$, and constants $\varepsilon, K > 0$, such that $M - \mathrm{I}$ has a bounded inverse, and such that the bounds (3.3) hold, for all α in an open neighborhood of I, and for all functions u in a closed ball B in $\mathcal{A}_\rho$ of radius r, centered at u_0. Then for every $\alpha \in I$, the equation (1.3) has a unique solution $u = u_\alpha$ in B, and the curve $\alpha \mapsto u_\alpha$ is smooth.*

Next, we consider the problem of linking such local solution curves together, into branches that connect the bifurcations described earlier.

We say that a pair of triples (α_i, u_i, r_i) has property $\mathcal{P}$ if both triples satisfies the hypotheses of Proposition 3.1 and if there exists a function $u \in \mathcal{A}_\rho$, and a real number $R \geq \max_i(\|u_i - u\| + r_i)$, such that (I, u, R) satisfies the hypotheses of Proposition 3.6. Here, I denotes the closed interval with endpoints α_i. Notice that, due to the uniqueness statement in Proposition 3.6, the solution curve associated with (I, u, R) has to pass through the two solutions associated with the triples (α_i, u_i, r_i). Thus, such pairs can be linked together to form a chain which "shadows" a unique solution curve for equation (1.3).

Lemma 3.7. *Let (ω, ω') be an edge of the graph Γ, whose endpoints ω and ω' are vertices of order ≥ 2. Then there exists a monotone sequence of real numbers $\{\alpha_i\}_{i=1}^n$, and a sequence $\{(u_i, r_i)\}_{i=1}^n$ in $\mathcal{A}_\rho \times \mathbb{R}_+$, such that the pair $\{(\alpha_i, u_i, r_i), (\alpha_{i+1}, u_{i+1}, r_{i+1})\}$ has property $\mathcal{P}$, for each positive $i < n$. Furthermore, $\alpha_1 \in I_\omega$ and $b_1 \subset B_\omega$, where I_ω and B_ω are the sets described in Lemma 3.5, and where b_1 is the closed ball in $\mathcal{A}_\rho$ of radius r_1, centered at u_1. Similarly, $\alpha_n \in I_{\omega'}$ and $b_n \subset B_{\omega'}$. In the case of the two edges $(4, iP_1)$ and $(iP_1, 4)$ the given sequences are distinguished e.g. by the sign of the parameter value λ in the corresponding regions b_1 and b_n. Finally, an analogous result holds for the edges of the graph Γ that connect a vertex of order ≥ 2 to an endpoint.*

Our proof of this lemma is computer-assisted and will be described in the Appendix.

Proof of Theorem 2.1. The assertions of Theorem 2.1 follow directly from the results presented in this section, with the possible exception of the claim that the various local solution curves connect up precisely as indicated by the graph Γ.

Concerning this remaining question, we first note that the basic bifurcation lemmas imply that for each of the 23 bifurcations ω, the local solution curves in the region $I_\omega \times B_\omega$ have no self-intersections and meet in a single point (the bifurcation point). Then, by the local uniqueness property described in Proposition 3.3, each of the solution curves obtained from Lemma 3.7, that enters a region $I_\omega \times B_\omega$, is a continuation of one of the $|\omega|$ local solution curves in this region, where $|\omega|$ denotes the order of the vertex ω. Now recall that each of these curves is associated with an edge of the graph Γ. Thus, due to the monotonicity of the sequences $\{\alpha_i\}_{i=1}^n$ in Lemma 3.7, and the uniqueness property described in Proposition 3.6, this curve extends in a unique way to a solution branch that starts and ends at the corresponding vertex solutions. The same uniqueness property also implies that distinct branches can meet only at their endpoints, and only if this point is a bifurcation point. Here, we have also used that the endpoints of the graph Γ correspond to distinct solutions, as implied by Lemma 3.2. Thus, the branches obtained from Lemma 3.7 meet precisely as indicated by the graph Γ, and in particular, each of the $|\omega|$ local solution curves in a bifurcation region $I_\omega \times B_\omega$ is connected to one of these branches.

Denote by S the union of all these branches. By using again the uniqueness properties described in Proposition 3.3 and Proposition 3.6, and the compactness of S, we can find an open set U in $\mathbb{R}\times\mathcal{A}_\rho$ that contains all solutions from S with $\alpha\in[1,79]$, but no solutions of (1.3) not belonging to S, such that the boundary of U intersects S in precisely 12 points; namely one for each of the 12 branches associated with an endpoint of the graph Γ. Thus, Theorem 2.1 is proved. **QED**

4. Stability of Solutions

The time evolution near a stationary solution $t\mapsto v\in\mathcal{A}_\rho$ of the Kuramoto-Sivashinski equation can be described by equation (2.3), where $u=v+h$ is the function that evolves according to (1.2). Denote by p the number of eigenvalues of L_v that have a positive real part. Our goal is to determine this number for some of our solutions v, and to show that L_v has no eigenvalues on the imaginary axis. Then we can use existing results to conclude that the flow defined by equation (1.2) has differentiable local stable and unstable manifolds at v, with the unstable manifold being of dimension p.

To be more precise and more general as well, consider equation (1.1) for functions $u(t,x)$ that are 2π-periodic in x. Let v be a stationary solution of this equation. If H_α is analytic on $\mathbb{C}^m$, which we will assume, then v belongs to $\mathcal{H}_\rho$. The equation for $h=u-v$ takes the form

$$\dot{h}=Lh+N(h)\,, \tag{4.1}$$

where L is linear and N nonlinear, with $N(\varepsilon h)$ of order ε^2. Let $T=-(iD)^m-E$, where E denotes the canonical (averaging) projection onto constant functions. The standard way of solving an equation of the form (4.1) is to convert it to an integral equation, such as

$$h(t)=e^{tT}h(0)+\int_0^t e^{(t-s)T}(A+N)(h(s))ds\,,\qquad t\ge 0\,, \tag{4.2}$$

where $A=L-T$. Define $A_0=|D^2-E|^{(m-1)/2}$. If $h(0)$ is sufficiently close to zero in $\mathcal{H}_\rho$, then for times t less than some $\tau>0$, the solution can be obtained via the contraction mapping principle in $C([0,\tau],\mathcal{H}_\rho)$, by using that $A_0^{-1}(A+N)$ is a differentiable map on $\mathcal{H}_\rho$, and that

$$\bigl\|A_0e^{tT}\bigr\|\le Ct^{-\nu}e^{-\delta t}\,,\qquad t>0\,, \tag{4.3}$$

for some positive constants C, δ, and $\nu<1$. Thus, equation (1.1) defines a continuous flow $t\mapsto(\Phi_t:h(0)\mapsto h(t))$ on a small open neighborhood of v in $\mathcal{H}_\rho$. Some global results on the flow for the Kuramoto-Sivashinski equation (1.2) can be found in [5–7].

Lemma 4.1. *Assume that L has $p \geq 0$ eigenvalues with positive real part, and no eigenvalues on the imaginary axis. Then the flow Φ admits differentiable local stable and unstable manifolds at v, with the unstable (stable) manifold being of dimension (codimension) p and tangent at v to the spectral subspace of L corresponding to the eigenvalues of L with negative (positive) real part.*

Proof. Our aim is to invoke one of the general theorems that applies in such situations; see e.g. [14–16]. Their proofs are based on solving an integral equation analogous to (4.2), where $A+N$ is replaced by N, and T by either $L^{-} = (\mathrm{I}-P)L$ or the negative of $L^{+} = PL$. Here, P denotes the spectral projection associated with the p eigenvalues of L with positive real part. Since the range of P is finite dimensional, we have $\|\exp(-tL^{+})\| \leq C\exp(-\delta t)$ for some constants $C, \delta > 0$. An analogous bound can be obtained for $\exp(tL^{-})$, by writing this operator as a contour integral involving the resolvent of L^{-}, and using that A is a relatively compact perturbation of T; see also [14].

What is needed is a bound of the type (4.3) for the operators L^{-} and $-L^{+}$. Given $\rho' > \rho$, let h be a function in $\mathcal{H}_{\rho'}$ that has norm one in $\mathcal{H}_{\rho}$, and define $f(t) = A_0 e^{tL^{-}} h$. Then

$$f(t) = A_0 e^{tT} h(0) + \int_0^t A_0 e^{(t-s)T} A A_0^{-1} f(s)\,ds\,, \qquad t \geq 0\,. \tag{4.4}$$

By using (4.3), and the fact that AA_0^{-1} is bounded on $\mathcal{H}_{\rho}$, we see that this equation has a unique solution in the space of continuous curves from $(0,\tau)$ to $\mathcal{H}_{\rho}$, with finite norm $\sup_{0\leq t\leq\tau} t^{\nu}\|f(t)\|_{\rho}$, provided that $\tau > 0$ is chosen sufficiently small. Both τ and the resulting bound on the norm of the solution are independent of h. Thus, since $\mathcal{H}_{\rho'}$ is dense in $\mathcal{H}_{\rho}$, we find that L^{-} satisfies a bound analogous to (4.3), for positive $t \leq \tau$. By using that $t \mapsto \exp(tL^{-})$ is a contraction semigroup, as mentioned above, this bound (but with different constants) can be extended to all $t > 0$. A similar bound for $\exp(-tL^{+})$ follows trivially, since P has finite rank.

In addition, as was noted earlier, the nonlinearity N defines a differentiable map $A_0^{-1}N$ on the space $\mathcal{H}_{\rho}$. With the necessary estimates in place, we can now apply e.g. the results of [16] and obtain the conclusions of Lemma 4.1. **QED**

In order to determine the number of eigenvalues of L with positive real part, we will write L as a perturbation of a linear operator whose eigenvalues and eigenvectors are known explicitly. Let $\mathcal{T}$ and $\mathcal{L} = \mathcal{T} + A$ be closed linear operators whose spectrum consists of isolated eigenvalues only. Then the following holds.

Proposition 4.2. *Let Ω be a bounded open subset of $\mathbb{C}$, whose boundary $\partial\Omega$ consists of finitely many rectifiable Jordan curves and avoids the eigenvalues of $\mathcal{T}$. If*

$$\left\|A(z-\mathcal{T})^{-1}\right\| < 1\,, \qquad \forall z \in \partial\Omega, \tag{4.5}$$

then $\mathcal{T}$ and $\mathcal{L} = \mathcal{T} + A$ have the same number of eigenvalues (counting multiplicities) in the region Ω, and in its closure.

A proof of this (well known) fact is based on the integral formula

$$P_\Omega = \frac{1}{2\pi i}\int_\Gamma (z-\mathcal{L})^{-1}dz = \frac{1}{2\pi i}\int_\Gamma (z-\mathcal{T})^{-1}\bigl[\mathrm{I}-A(z-\mathcal{T})^{-1}\bigr]^{-1}dz \tag{4.6}$$

for the spectral projection P_Ω of $\mathcal{L}$, associated with the eigenvalues of $\mathcal{L}$ in Ω; see e.g. [13]. Here, Γ denotes the boundary of Ω, positively oriented with respect to Ω. The operator in square brackets is invertible via Neumann series, due to the condition (4.5). The same remains true if A is replaced by sA, with $|s| \le 1$. This yields a continuous family of projections $P_\Omega(s)$ of finite rank, and continuity implies that the rank is independent of s.

In the remaining part of this section, we consider again the special case (2.3) of the Kuramoto-Sivashinski equation. Proposition 4.2 will be applied not to $L = L_v$ directly, but to the operator

$$\mathcal{L} = E^{-1}L_vE\,, \tag{4.7}$$

where $E \asymp \mathrm{I}$ is a suitable linear isomorphism of $\mathcal{A}_\rho$. Here, and in what follows, $U \asymp V$ stands for $(U-V) \asymp 0$.

Corollary 4.3. *Let $\mathcal{T} \asymp L_0$ and $\mathcal{S} \asymp L_0$ be linear operators on $\mathcal{A}_\rho$, that have no eigenvalues on the imaginary axis. Let $A = \mathcal{L} - \mathcal{T}$. If*

$$\|A\mathcal{S}^{-1}\| < 1\,, \qquad \|\mathcal{S}(\mathcal{T}-iy)^{-1}\| \le 1\,, \quad \forall y \in \mathbb{R}, \tag{4.8}$$

then L_v and $\mathcal{T}$ have the same number of eigenvalues in the halfplane $\mathrm{Im}\,(z) > 0$, *and in its closure.*

Proof. Since $\mathcal{S} \asymp \mathcal{T}$, the operator norm of $\mathcal{S}(\mathcal{T}-z)^{-1}$ tends to zero as $|z| \to \infty$. Thus, if Ω denotes the intersection of the halfplane $\mathrm{Im}\,(z) > 0$ with the disk $|z| < R$, then (4.8) implies (4.5) for sufficiently large $R > 0$. Thus, the conclusion of Proposition 4.2 holds for all such regions Ω, and since L_v has the same spectrum as $\mathcal{L}$, the assertion follows. **QED**

In our proof of Theorem 2.2, the isomorphism E is chosen in such a way that $\mathcal{L}$ is close to an operator $\mathcal{T}$ that is block-diagonal, in the sense that all eigenvectors of $\mathcal{T}$ are either Fourier monomials (for real eigenvalues) or a linear combination of two Fourier monomials (for pairs of complex conjugate eigenvalues). In order to simplify our estimates, the operator $\mathcal{S}$ is taken to be diagonal, meaning that $\mathcal{S}$ commutes with the projections $\mathbb{P}_n^-$. Given $\mathcal{T}$, it is easy to find such an operator $\mathcal{S}$ that satisfies the second inequality in (4.8), without being smaller than necessary.

In order to prove the first bound in (4.8), we split $A\mathcal{S}^{-1}$ into a low order part

$$A\mathcal{S}^{-1}\mathbb{P}_n^- = E^{-1}L_vE\mathcal{S}^{-1}\mathbb{P}_n^- - \mathcal{T}\mathcal{S}^{-1}\mathbb{P}_n^-\,, \tag{4.9}$$

which can be controlled explicitly, and a higher order part, which simplifies to

$$A\mathcal{S}^{-1}\mathbb{P}_n^+ = E^{-1}(L_v - L_0)L_0^{-1}\mathbb{P}_n^+\,, \tag{4.10}$$

if n is sufficiently large. Notice that the norm of the higher order part is of order $\mathcal{O}(n^{-3})$.

Theorem 2.2 is now a consequence of the following lemma.

Lemma 4.4. *For each of the triples* (α, β, γ) *listed in Theorem 2.2, and for each of the branches* $\langle\beta, \gamma\rangle$ *and* $\langle\gamma, \beta\rangle$, *there exists a Fourier polynomial* u_0 *and a real number* $r > 0$, *such that the following holds. The triple* (α, u_0, r) *belongs to the sequence mentioned in Lemma 3.7, associated with the appropriate edge* (ω, ω') *of the given branch (and agrees with the triple mentioned in Lemma 3.2 in the cases where* $\alpha = 80$*). Every function* $u \in \mathcal{A}_\rho$ *that lies within a distance* r *or less of* u_0 *satisfies the* L^2 *bound given in Theorem 2.2. Furthermore, there exist operators* E, $\mathcal{S}$ *and* $\mathcal{T}$ *as described above, such that the hypotheses of Corollary 4.3 are satisfied and the number of eigenvalues of* $\mathcal{T}$ *in the halfplane* $\mathrm{Im}\,(z) > 0$ *agrees with the number* p *listed in Theorem 2.2.*

Our proof of this lemma is computer assisted; see part 2 of the Appendix below. We note that the given bounds on the L^2 norm also imply the last claim in Lemma 3.2.

5. Appendix

5.1. Basic bifurcations

In this Section, we prove Lemma 3.4 concerning the pitchfork bifurcation, and analogous lemmas for the intersection and fold bifurcation. These are adaptations of classic bifurcation results, very similar to those given in [9]. The main purpose for including them here is to state the assumptions precisely in the form needed; and the proofs are sufficiently short to be included as well.

If G is any differentiable function of two variables, denote by $\dot{G}$ and G' the partial derivatives of G with respect to the first and second argument, respectively. Let $I = [\alpha_1, \alpha_2]$ and $J = [\lambda_0 - b, \lambda_0 + b]$.

The basic starting point is the following

Proposition 5.1. (monotone solution) *Let* G *be a real-valued* C^1 *function on an open neighborhood of* $I \times J$, *satisfying*

(1) $G' > 0$ *on* $I \times J$, *(2)* $\dot{G} < 0$ *on* $I \times J$,
(3) $G(\alpha_1, \lambda_0) < 0$, *(4)* $G(\alpha_2, \lambda_0) > 0$.

Then the solution set of $G(\alpha, \lambda) = 0$ *in* $I \times J$ *is the graph of a strictly monotone* C^1 *function* $\alpha = a(\lambda)$, *defined on a closed subinterval* J_0 *of* J *containing an open neighborhood of* λ_0.

The proof of this proposition is straightforward and will be omitted. A simple example of a function satisfying these conditions (near the origin) is $G(\alpha, \lambda) = \lambda - \alpha$.

Consider now $\lambda_0 = 0$.

Lemma 5.2. (intersection bifurcation) *Let* g *be a real-valued* C^2 *function on an open neighborhood of* $I \times J$, *such that* $g(\alpha, 0) = 0$ *for all* $\alpha \in I$, *and*

(1) $g'' > 0$ *on* $I \times J$, *(2)* $\dot{g}' < 0$ *on* $I \times J$,
(3) $g'(\alpha_1, 0) < 0$, *(4)* $g'(\alpha_2, 0) > 0$.

Then $g(\alpha,\lambda)=\lambda G(\alpha,\lambda)$, *with* G *satisfying the conclusion of Proposition 5.1.*

Proof. Assume that g satisfies the given hypotheses. Define $G(\alpha,\lambda)=g(\alpha,\lambda)/\lambda$ for $\lambda>0$, and $G(\alpha,0)=g'(\alpha,0)$. Clearly, G is of class C^1 and satisfies conditions (3) and (4) of Proposition 5.1. The first two conditions of Proposition 5.1 follow from the identity $G(\alpha,\lambda)=\int_0^1 ds\, g'(\alpha,s\lambda)$. **QED**

The basic example for this proposition is $g(\alpha,\lambda)=\lambda^2-\lambda\alpha$ near the origin.

Lemma 5.3. (fold bifurcation) *Let* G *be a real-valued* C^2 *function on an open neighborhood of* $I\times J$, *satisfying*

(1) $G''>0$ *on* $I\times J$, *(2)* $\dot{G}<0$ *on* $I\times J$,
(3) $G(\alpha_1,\lambda_0)\pm bG'(\alpha_1,\lambda_0)>0$, *(4)* $G(\alpha_2,\lambda_0\pm b)>0$, *(5)* $G(\alpha_2,\lambda_0)<0$.

Then the solution set of $G(\alpha,\lambda)=0$ *in* $I\times J$ *is the graph of a* C^2 *function* $\alpha=a(\lambda)$, *defined on a proper subinterval* J_0 *of* J. *This function takes the value* α_2 *at the endpoints of* J_0, *and satisfies* $\alpha_1<a(\lambda)<\alpha_2$ *at all interior points of* J_0, *which includes* λ_0.

Proof. Assume that G satisfies the given conditions. By condition (1), the function $\lambda\mapsto G(\alpha,\lambda)$ has at most two zeros. For $\alpha=\alpha_2$, there are exacly two, by conditions (4) and (5). They define an interval J_0 as described above. By (2) and (4) we now have $G(\alpha_2,\lambda)>0$ whenever λ lies outside J_0. And for λ in J_0, there can be at most one α for which $G(\alpha,\lambda)=0$, since $\dot{G}<0$. For λ near the left endpoint of J_0, the existence of such an $\alpha=a(\lambda)$ follows from the Implicit Function Theorem (using again that $\dot{G}<0$). The same theorem guarantees that a is of class C^2, and that the solution can be continued as λ increases, provided that $a(\lambda)$ stays withing I. But $a(\lambda)$ cannot reach the boundary of I, as long as λ lies in the interior of J_0, since $G(\alpha_2,\lambda)<0$ for these λ, and $G(\alpha_1,\lambda)>0$. The latter follows the fact that by conditions (1) and (3)

$$G(\alpha_1,\lambda)\ge G(\alpha_1,\lambda_0)+(\lambda-\lambda_0)G'(\alpha_1,\lambda_0)\ge G(\alpha_1,\lambda_0)-b|G'(\alpha_1,\lambda_0)|>0\,.$$

QED

Here, the standard example is $G(\alpha,\lambda)=\lambda^2-\alpha$, with α and λ near zero.

Proof of Lemma 3.4. Assume that g satisfies the given hypotheses. Define $G(\alpha,\lambda)=g(\alpha,\lambda)/\lambda$ for $\lambda>0$, and $G(\alpha,0)=g'(\alpha,0)$. Clearly, G is of class C^2 and satisfies conditions (4) and (5) of Lemma 5.3. The first two conditions of Lemma 5.3 follow from the identity $G(\alpha,\lambda)=\int_0^1 ds\, g'(\alpha,s\lambda)$, and the third is obtained by using that $G'(\alpha,0)=\int_0^1 ds\, sg''(\alpha,0)=\frac{1}{2}g''(\alpha,0)$. **QED**

5.2. Description of the computer-assisted proofs

All of the remaining proofs involve finding first approximate solutions of some functional equation, and then proving that there exists a true solution nearby. The approximate solutions used in this process are determined with purely numerical methods, e.g., by following branches. The existence of true solutions is then obtained by solving a fixed point equation, using the contraction mapping principle in a ball centered at the approximate solution.

Our proofs are based on a discretization of the problem, carried out and controlled with the aid of a computer. They involve a systematic reduction of the desired bounds to a finite number of trivial inequalities between (representable) real numbers. This is made possible by the fact that our maps F_α and $F_{\alpha,\lambda}$ are compact.

The general approach is quite standard by now. We start by associating to a space X an appropriate collection $\mathrm{std}(X)$ of subsets of X. A "bound" on an element $u \in X$ is then a set $U \in \mathrm{std}(X)$ containing u, while a bound on a map $g : X \to Y$ is a map $G : D_G \to \mathrm{std}(Y)$, with domain $D_G \subset \mathrm{std}(X)$, such that $g(u) \in G(U)$ whenever $u \in U \in D_G$. Notice that the composition of two bounds, if defined, is a bound on the corresponding composed map. This and other properties allow us to combine bounds on elementary maps into bounds on more complex maps like F_α, and thus to mechanize the necessary estimates.

The basic bounds used in the present proof have been developed already in [10], up to the level of bounds on basic maps (like norms, products, antiderivatives, etc.) between the spaces $\mathbb{R}$, $\mathcal{A}_\rho$ and $\mathcal{B}_\rho$, as well as between products of these spaces. We also use the same steps as in [10] to estimate linear operators and derivatives of maps like $\mathcal{C}_\alpha$. Thus, in order to avoid undue repetition, the reader is refereed to [10] for a description of the bounds used at this level. These bounds are now combined as described in Sections 3 and 4, in order to verify the various inequalities like (3.3), (3.8), and (4.8), that are needed to verify the claims of Lemma 3.2, Lemma 3.7, Lemma 3.5, and Lemma 4.4.

We shall now give a brief description of the steps used in the proof of Lemma 3.7, since this algorithm is a bit more complex than others. In what follows, words in `this font` are names of procedures that are part of our computer programs [12].

`USmall`. Given a function u_0 and a parameter value α, this procedure chooses M as described in Subsection 3.1, finds an upper bound ε on $\|C_\alpha(u_0)-u_0\|$, and then verifies the remaining inequalities in (3.3), with a radius $r > \varepsilon$ close to ε. The value of r is returned.

`BigR`. Given a function u_0 and a parameter interval $\Delta\alpha$, this procedure verifies the same inequalities as `USmall`, but with α replaced by the interval $\Delta\alpha$, and with a radius that is as large as possible, within a certain range. The value R of the radius is returned.

`SetInterval`. Given a triple (α_1, u, r) for which the assumptions of Proposition 3.1 are satisfied, `SetInterval` determines $\alpha_2 > \alpha_1$ as large as feasible, such that `BigR` succeeds with $\Delta\alpha = [\alpha_1, \alpha_2]$. The value R from `BigR`, and the interval $\Delta\alpha$ are returned.

These procedures are now combined as follows. Starting with a pair (α_1, u_1) in the region $I_\omega \times B_\omega$, we first use `USmall` to find a ball b_1 in B_ω containing a true solution for $\alpha = \alpha_1$. Next, we choose $R > 0$ and $\alpha_2 > \alpha_1$ using `SetInterval`. Then a `Newton` algorithm is used to find approximate solutions (α_2, u_2) and $((\alpha_1+\alpha_2)/2, u)$. After verifying that the assumptions of Proposition 3.1 are satisfied at $([\alpha_1, \alpha_2], u, R)$, we apply `USmall` to (α_2, u_2), and then check that $b_2 = B_r(u_2)$ is contained in $B_R(u)$. Now, we start over with a choice of $\alpha_3 > \alpha_2$, etc. This algorithm is repeated until we end up with $b_n \subset B_{\omega'}$.

For a detailed and complete description of these proofs, we refer to the source code (in Ada95) and input data for our computer programs [12]. These programs were run successfully on several different types of machines, using public versions of the GNAT compiler [11]).

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[11] The GNU NYU Ada 9X Translator, available at `ftp://cs.nyu.edu/pub/gnat` and many other places.

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