Families of periodic solutions for some Hamiltonian PDEs

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Abstract. We consider the nonlinear wave equation $u_{tt} - u_{xx} = \pm u^3$ and the beam equation $u_{tt} + u_{xxxx} = \pm u^3$ on an interval. Numerical observations indicate that time-periodic solutions for these equations are organized into structures that resemble branches and seem to undergo bifurcations. Besides describing our observations, we prove the existence of time-periodic solutions for various periods (a set of positive measure in the case of the beam equation) along the main nontrivial "branch".

1. Introduction and main results

We prove the existence of time-periodic solutions along what looks like "branches" of solutions for two Hamiltonian partial differential equations: the nonlinear wave equation ($\nu = 1$) and the nonlinear beam equation ($\nu = 2$) in one spatial dimension,

$$\partial_t^2 \mathfrak{u}(t,x) + (-1)^\nu \partial_x^{2\nu} \mathfrak{u}(t,x) = f(\mathfrak{u}(t,x)), \qquad (t,x) \in \mathbb{R} \times (0,\pi), \tag{1.1}$$

with homogeneous Dirichlet ($\nu = 1$) or Navier ($\nu = 2$) boundary conditions.

This work was motivated in part by observations in a new model for suspension bridges [13,14] which consists of two coupled equations: a modified nonlinear beam equation modeling the displacement of the center of the deck, and a modified nonlinear wave equation modeling the torsion of the deck. This model exhibits resonances between longitudinal and torsional modes that depend strongly on the energy (amplitudes). Thus we are interested in families of time-periodic solutions covering a range of different amplitudes; and it is natural to consider first the simpler case of equation (1.1).

The goal is to construct solutions in a way that also yields information about their properties. The computer-aided methods presented in this paper are well suited for such tasks. Similar techniques have been applied successfully to partial differential equations of elliptic and parabolic type [17-27].

Initial results on time-periodic solutions for (1.1) covered periods that are rational multiples of π. For small-amplitude solutions, or if f is near-linear, it is possible to apply perturbative methods; see e.g. [1,2] and references therein. In other cases, existence results have been obtained by variational techniques [3-7]. One of the drawbacks with these techniques is that they yield very limited information about the solutions. The most recent result cover positive-measure sets of periods [8-12], using Nash-Moser schemes or resummation techniques (for divergent series) to deal with the problem of small denominators that arise with irrational periods. These methods are again perturbative.

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Here we consider both the rational and the positive-measure case. We restrict our analysis to the specific nonlinearity $f(\mathfrak{u}) = \sigma\mathfrak{u}^3$ with $\sigma = \pm 1$. Setting $\mathfrak{u}(t,x) = u(\alpha t, x)$, where $2\pi/\alpha$ is the desired period for $\mathfrak{u}$, we arrive at the equation

$$L_\alpha u = \sigma u^3 \,, \qquad L_\alpha = \alpha^2\partial_t^2 + (-1)^\nu \partial_x^{2\nu} \,, \tag{1.2}$$

for a function $u = u(t,x)$ that is 2π-periodic in t and vanishes at $x = 0, \pi$, together with its second x-derivative if $\nu = 2$. Restricting further to solutions u that are invariant under time reversal leads us to consider the vector space $\mathcal{A}^{\mathrm{o}}$ of all real analytic functions u on $\mathbb{R}^2$ that are 2π-periodic in each argument and admit a representation

$$u = \sum_{n,k} u_{n,k} P_{n,k} \,, \qquad P_{n,k}(t,x) = \cos(nt)\sin(kx) \,. \tag{1.3}$$

We are interested mainly in solutions that are dominated by a single mode $P_{a,b}$. In the case $a = b = 1$, one way to characterize such a solution is the following.

Definition 1.1. *A solution u of the equation (1.2) will be called a type $(1,1)$ solution if $|u_{n,k}| < |u_{1,1}|$ whenever $n > 1$ or $k > 1$.*

Other solutions can be obtained via scaling. More specifically, if $u \in \mathcal{A}^{\mathrm{o}}$ satisfies the equation $L_\alpha u = \sigma u^3$, and if we define

$$\tilde{u}(t,x) = b^\nu u(at, bx) \,, \qquad \tilde{\alpha} = \alpha b^\nu / a \,, \tag{1.4}$$

with a and b nonzero integers, then $\tilde{u}$ belongs to $\mathcal{A}^{\mathrm{o}}$ and satisfies $L_{\tilde{\alpha}}\tilde{u} = \sigma\tilde{u}^3$.

As indicated earlier, the nature of the problem (1.2) depends on the arithmetic properties of the frequency α. This can be seen e.g. from the eigenvalues of the operator L_α, which are given by

$$\lambda_{n,k} = k^{2\nu} - (\alpha n)^2 = (k^\nu + \alpha n)(k^\nu - \alpha n) \,, \tag{1.5}$$

with associated eigenfunctions $P_{n,k}$. In particular, if α is irrational, then all eigenvalues are nonzero; although they typically accumulate at zero if $\nu = 1$.

We start by considering rational values of α. In this case L_α can have a nontrivial null space. We avoid this extra complication by restricting our analysis to the subspace $\mathcal{B} \subset \mathcal{A}^{\mathrm{o}}$ consisting of all functions $u \in \mathcal{A}^{\mathrm{o}}$ whose Fourier coefficients $u_{n,k}$ vanish whenever nk is even. Notice that, if u belongs to $\mathcal{B}$ then so do $L_\alpha u$ and u^3. In addition, we restrict to rational values of α that admit a representation $\alpha = p/q$ with p and q nonzero integers of different parity (even or odd). The set of such rationals will be denoted by $\mathbb{Q}_{\mathrm{o}}$.

For the nonlinear wave equation ($\nu = 1$) we consider the sample set

$$Q_1 = \left\{\tfrac{3}{8}, \tfrac{5}{12}, \tfrac{7}{16}, \tfrac{9}{20}, \tfrac{13}{28}, \tfrac{1}{2}, \tfrac{15}{28}, \tfrac{11}{20}, \tfrac{9}{16}, \tfrac{7}{12}, \tfrac{5}{8}, \tfrac{9}{14}, \tfrac{11}{16}, \tfrac{7}{10}, \tfrac{13}{18}, \tfrac{3}{4}, \tfrac{11}{14}, \tfrac{5}{6}, \tfrac{7}{8}, \tfrac{9}{10}, \tfrac{11}{12}, \tfrac{13}{14}, \tfrac{17}{18}\right\}.$$

Theorem 1.2. *For each $\alpha \in Q_1$ the equation $\alpha^2\partial_t^2 - \partial_x^2 u = u^3$ has a solution $u \in \mathcal{B}$ of type $(1,1)$ with $|u_{1,1}| > \sqrt{2(1-\alpha)}$.*

We note that every solution $u \in \mathcal{B}$ of the equation $\alpha^2\partial_t^2 u - \partial_x^2 u = u^3$ with $\alpha \in \mathbb{Q}_{\mathrm{o}}$ yields a solution $\tilde{u} \in \mathcal{B}$ of the equation $\alpha^{-2}\partial_t^2\tilde{u} - \partial_x^2\tilde{u} = -\tilde{u}^3$, and vice-versa. The functions u and $\tilde{u}$ are related via $\tilde{u}(t,x) = \alpha^{-1}u(x - \pi/2, t - \pi/2)$.

Our method used to prove Theorem 1.2 applies in principle to any value $\alpha \in \mathbb{Q}_{\rm o}$. We expect that the given bounds on $u_{1,1}$ hold for all values of α in a subset of $[0,1]\cap\mathbb{Q}_{\rm o}$ whose closure has positive measure. Numerically, the curve $\alpha \mapsto u$ resembles a solution branch of the type known for finite dimensional systems; see Section 2.

Next we consider some irrational values of α. Here we restrict to the beam equation ($\nu = 2$), where the spectrum of L_α is easier to control.

Even in this case it is difficult to construct non-small solutions for any specific irrational value of α. As an example we consider a quadratic irrational $\alpha = 1/\sqrt{m}$, where $m > 1$ is an integer that is not the square of an integer. In this case, Siegel's theorem on integral points on algebraic curves of genus one [16] implies that $|\lambda_{n,k}| \to \infty$ as n or k tends to infinity. Unfortunately we have not been able to find lower bounds on $|\lambda_{n,k}|$ that would be useful for our purpose. We sidestep this problem by making an assumption.

Theorem 1.3. *Let $\alpha = 1/\sqrt{3}$. Assume that $|3k^4 - n^2| \ge 39$ for all $k \ge 9$ and all $n \in \mathbb{N}$. Then the equation $\alpha^2\partial_t^2 u + \partial_x^4 u = u^3$ has a solution $u \in \mathcal{B}$ of type $(1,1)$ with $|u_{1,1}| > 1$.*

We have verified the assumption $\min_n |3k^4 - n^2| \ge 39$ for $9 \le k \le 10^{12}$.

Our next result concerns irrational values of α that are close to the rationals in

$$Q_2 = \left\{\tfrac{1}{4},\ \tfrac{3}{10},\ \tfrac{9}{20},\ \tfrac{1}{2},\ \tfrac{7}{12},\ \tfrac{5}{8},\ \tfrac{3}{4},\ \tfrac{5}{6},\ \tfrac{7}{6},\ \tfrac{5}{4},\ \tfrac{19}{14},\ \tfrac{17}{12},\ \tfrac{31}{20},\ \tfrac{13}{8},\ \tfrac{31}{18},\ \tfrac{61}{34}\right\}.$$

Theorem 1.4. *For each $\kappa \in Q_2$ there exists a set $R_\kappa \subset \mathbb{R}$ of positive measure that includes κ as a Lebesgue density point, such that for each $\alpha \in R_\kappa$, the equation $\alpha^2\partial_t^2 u + \partial_x^4 u = \sigma u^3$ with $\sigma = \mathrm{sign}(1-\alpha)$ has a solution $u \in \mathcal{B}$ of type $(1,1)$ with $|u_{1,1}| > \sqrt{2|1-\alpha|}$.*

Our proofs of Theorems 1.2,3,4 are computer-assisted. The general strategy and main estimates are given in Section 3. This includes a definition of the sets R_κ mentioned in Theorem 1.4. In Section 4 we show that these sets R_κ have positive measure. The computer part is sketched in Section 5 and described in detail in [28].

2. Numerical results

For numerical approximations we truncate the Fourier series (1.3) for a function $u \in \mathcal{B}$ in both variables, using projections $\mathbb{E}_{N}$ and $\mathbb{P}_{K}$ defined by

$$\mathbb{E}_{N} u = \sum_{n\le N}\sum_k u_{n,k}P_{n,k}\,, \qquad \mathbb{P}_{K} u = \sum_n \sum_{k\le K} u_{n,k}P_{n,k}\,. \tag{2.1}$$

However, it is useful to take into account that the equation (1.2) arises from a Hamiltonian flow. The Hamiltonian H is given by

$$H(u,v) = \int_0^\pi \left[\tfrac{1}{2}(\partial_x^\nu u)^2 + \tfrac{1}{2}\alpha^{-2}v^2 - \tfrac{1}{4}\sigma u^4\right] dx\,. \tag{2.2}$$

This Hamiltonian describes not only the full equation $L_\alpha u = \sigma u^3$, but also the truncated equation $L_\alpha u = \mathbb{P}_{K}\sigma u^3$. To be more precise, we can consider H as a function on $\mathcal{B}_{K} \times \dot{\mathcal{B}}_{K}$, where $\mathcal{B}_K$ is the range of $\mathbb{P}_K$, and where $\dot{\mathcal{B}}_K$ is the set of all functions $\partial_t u$ with $u \in \mathcal{B}_K$. The flow defined by the Hamiltonian H is given by $\partial_t u = \nabla_v H$ and $\partial_t v = -\nabla_u H$, using gradients with respect to the L^2 inner product. In particular $\partial_t u = \alpha^{-2} v$. Substituting $v = \alpha^2 \partial_t u$ into the equation $\partial_t v = -\nabla_u H$ yields $L_\alpha u = \mathbb{P}_K \sigma u^3$. If we define $\mathbb{P}_\infty$ to be the identity operator, then the same applies to $K = \infty$.

In what follows we always assume that $\sigma = \operatorname{sign}(1-\alpha)$.

For numerical computations we have to truncate the equation $L_\alpha u = \mathbb{P}_K \sigma u^3$ further to $L_\alpha u = \mathbb{E}_N \mathbb{P}_K \sigma u^3$. This doubly truncated system is no longer Hamiltonian. But this can be remedied partly by choosing $N \gg K$ whenever necessary. The fixed point equation associated with $L_\alpha u = \mathbb{E}_N \mathbb{P}_K \sigma u^3$ is

$$u = \mathcal{F}_\alpha^{NK}(u) \stackrel{\text{def}}{=} L_\alpha^{-1} \mathbb{E}_N \mathbb{P}_K \sigma u^3 , \qquad \sigma = \operatorname{sign}(1-\alpha) . \tag{2.3}$$

Since $\mathcal{F}_\alpha^{NK}$ is a map on the finite dimensional space $\mathbb{E}_N \mathcal{B}_K$, approximate fixed points can be found by standard numerical methods.

We consider values of α in the interval $[0, \nu]$. By the implicit function theorem, the solutions of $\mathcal{F}_\alpha^{NK}(u) = u$ for which $D\mathcal{F}_\alpha^{NK}(u)$ has no eigenvalue 1 are organized into branches where u depends smoothly on the parameter α. The union of all smooth branches that include a solution of type $(1,1)$ will be referred to as the $(1,1)$ branch or "main" branch. Scaling each solution on this main branch via (1.4) yields what we will call the (a,b) branch.

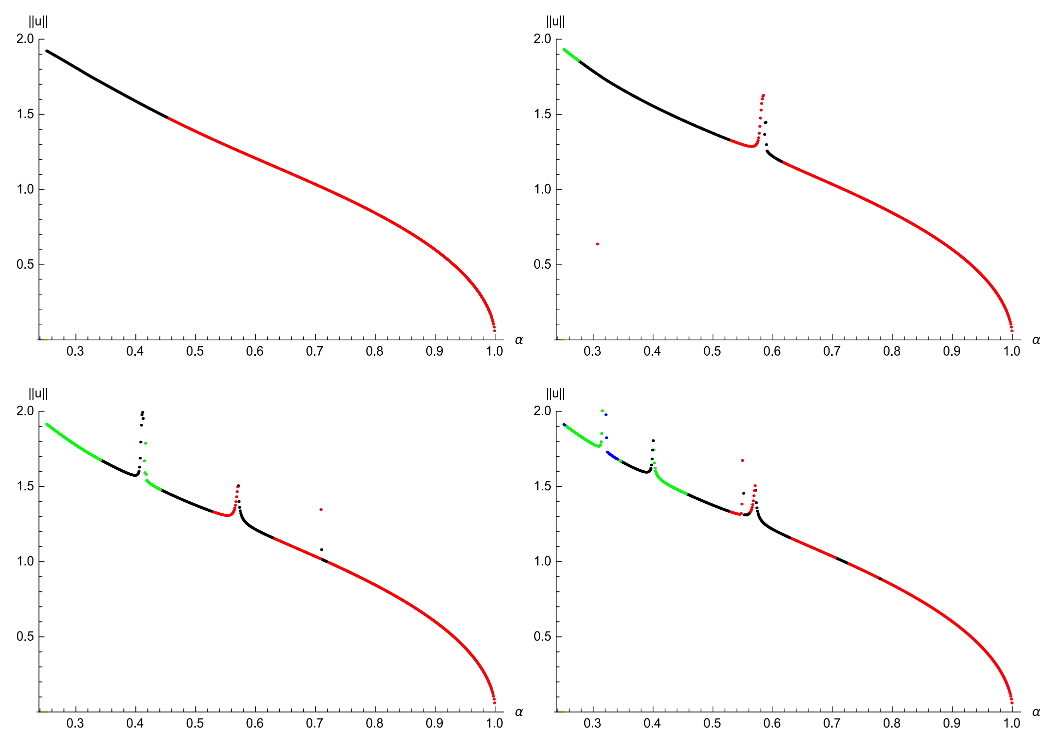

Figure 1. ($\nu = 1$) Norm versus α, for the solutions of (2.3) with $N = K$ and $K = 3, 5, 7, 9$.

We start with the nonlinear wave equation ($\nu = 1$). Our first observation is that the main branch covers a large fraction of the interval $[0,1]$. The four graphs in Figure 1 display the norm $\|u\|_0 = \sum_{n,k} |u_{n,k}|$ as a function of α along the main branch, for $N = K$ and $K = 3, 5, 7, 9$.

As one would expect, the larger K the less regular the graph. But the changes appear rather tame. Spikes appear as K is increased, but they get increasingly narrow and become invisible at any given resolution. To highlight places of possible bifurcations we use colors to indicate the index of the solution u, that is, the number of eigenvalues of $D\mathcal{F}_\alpha^{NK}(u)$.

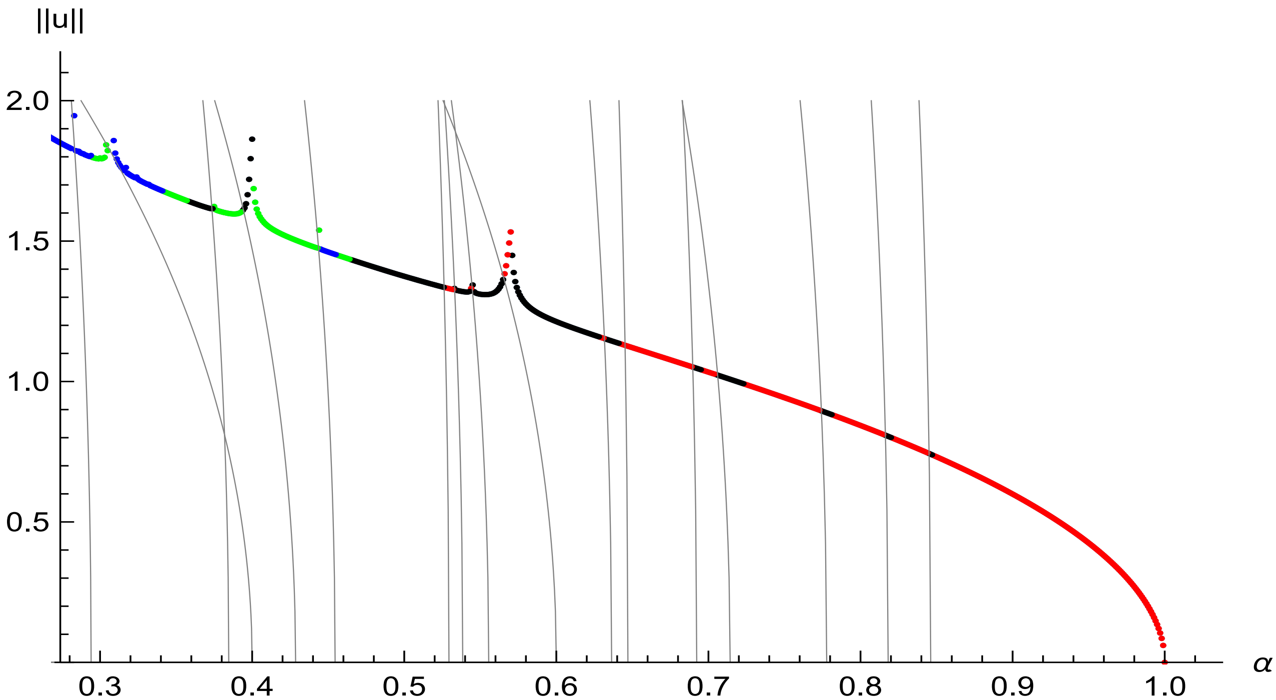

Figure 2. $(\nu = 1)$ The $(1,1)$ branch and some other (a,b) branches (thin lines).

Many of these bifurcations seem to involve other (a,b) branches: Figure 2 shows the main branch together with several (a,b) branches (thin lines). Here the branch (a,b) can be identified by using that it bifurcates out of $u = 0$ at $\alpha = b/a$.

We study this phenomenon in more detail for values of α in the interval $[0.57, 0.59]$ where the first spike appears at truncation $N = K = 5$. To this end we choose $K = 7$, and $N \gg K$ in order to preserve (approximately) the Hamiltonian character of the equation. Figure 3 shows the values of the coefficients $u_{1,1}$, $u_{3,3}$, $u_{5,3}$ and of the norm $\|u\|_0$, for the solutions u that we found on the given α-interval. These graphs indicate clearly that the $(1,1)$ branch coming from higher values of α undergoes a fold bifurcation at $\alpha \simeq 0.571$ and then bends back until it reaches a pitchfork bifurcation at $\alpha \simeq 0.585$. The main branch at the pitchfork bifurcation is the $(5,3)$ branch, which bifurcates out of $u = 0$ at $\alpha = 3/5$. The secondary branch is the continuation of the $(1,1)$ branch.

We did not investigate any of the other bifurcations, but our guess is that all bifurcations of the $(1,1)$ branch involve one of the other (a,b) branches.

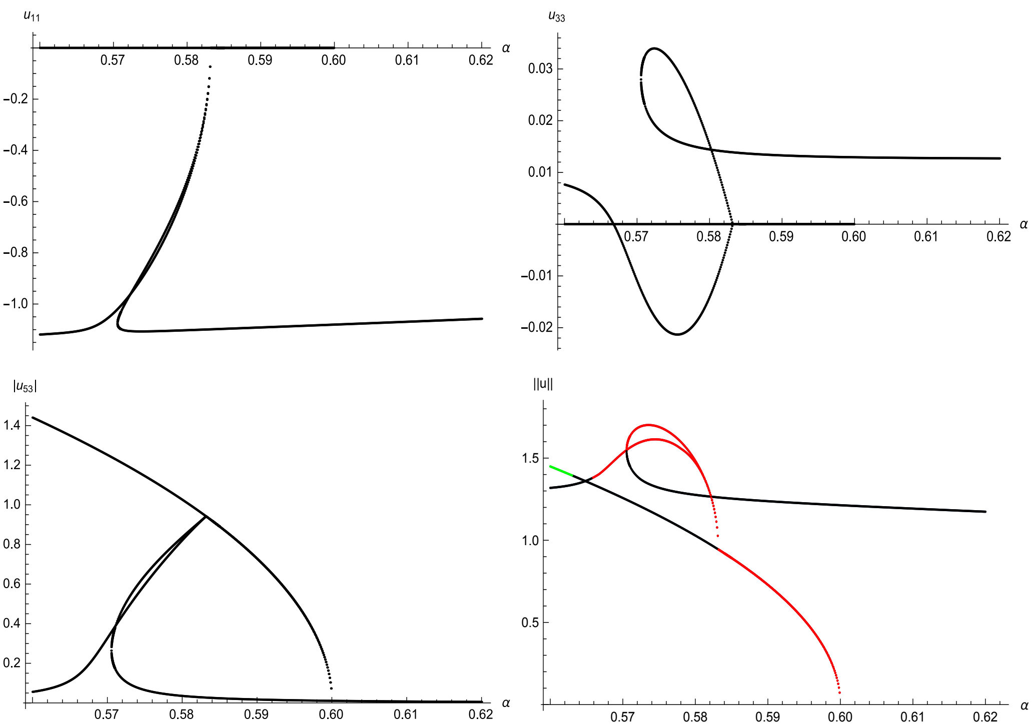

Figure 3. ($\nu = 1$) Coefficients $u_{1,1}$, $u_{3,3}$, $u_{5,3}$ and norm $\|u\|_0$ versus α, for the solutions of the approximate Hamiltonian system, $N \gg K = 7$, near the crossing of the $(1, 1)$ branch and the $(5, 3)$ branch.

Next we consider the nonlinear beam equation ($\nu = 2$). Our numerical results for this equation show similarities with the above-mentioned results for the wave equation ($\nu = 1$). Here we are using $K = 63$ and $N = 127$. The two graphs in Figure 4 depict the $(1, 1)$ branch of solutions of the equation (2.3), for $\alpha < 1$ (first) and for $\alpha > 1$ (second). These graphs appear much more regular than our graphs for $\nu = 1$.

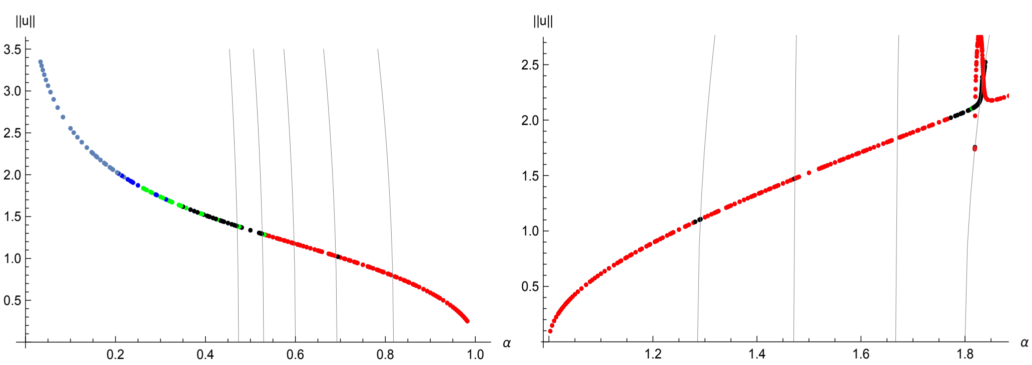

Figure 4. ($\nu = 2$) The $(1, 1)$ branch for $\alpha < 1$ (left) and for $\alpha > 1$ (right). The thin lines represent a few other (a, b) branches.

Still, we observe a crossing between the $(1, 1)$ branch and the $(5, 3)$ branch, with a bifurcation patterns similar to the one for the wave equation; see Figure 5 which shows an enlargement of the spike at $\alpha \simeq 1.83$. The index changes that are visible in Figure 4 indicate that similar bifurcations occur at other branch crossings as well.

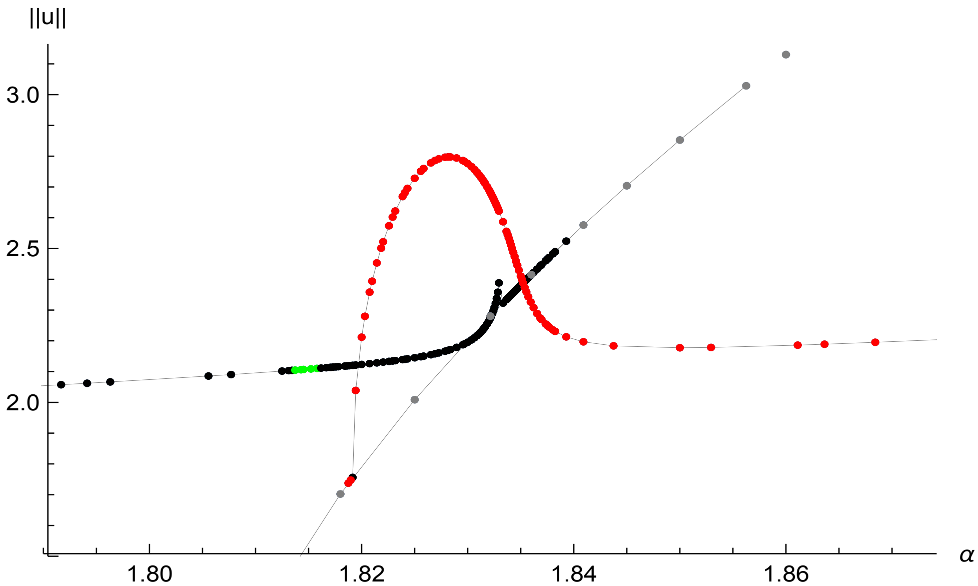

Figure 5. ($\nu = 2$) Enlargement of the spike at $\alpha \simeq 1.83$ in Figure 4.

The values of α used in Figure 4 include the values from the set Q_2 defined before Theorem 1.4. For these values of α there is no visible difference between our numerical values of the norms $\|u\|_0$ and our rigorous norm bounds for the true solutions of $L_\alpha u = \sigma u^3$. We have no doubt that the other points in Figure 4 could be verified in the same sense.

This suggests that the beam equation has families of periodic solutions that, at any finite resolution, are indistinguishable from the solution-branches of a truncated beam equation. The same seems to be true for the wave equation. These families cover a wide range of periods, with varying amplitudes. The existence of such families (of longitudinal modes) was conjectured in [13,14] for a model of a bridge, and we believe that our results for the equation (1.1) lend support to this conjecture.

3. Estimates for linear operators

First we reformulate our main results in terms of contraction mappings. After describing what types of estimates are needed in order to control the linear operators involved, we will give explicit bounds on the operator L_α.

Given a pair $\rho = (\rho_1, \rho_2)$ of positive real numbers, denote by $\mathcal{A}_\rho^{\mathrm{o}}$ the closure with respect to the norm

$$\|u\|_\rho = \sum_{n,k} |u_{n,k}| \varrho_1^n \varrho_2^k \,, \qquad \varrho_j = 1 + \rho_j \,, \tag{3.1}$$

of the space of Fourier polynomials $u \in \mathcal{A}^{\mathrm{o}}$. Notice that the functions in $\mathcal{A}_\rho^{\mathrm{o}}$ are analytic on the domain $\mathcal{D}_\rho$ given by $|\mathrm{Im}\, t| < \ln(\varrho_1)$ and $|\mathrm{Im}\, x| < \ln(\varrho_2)$. In particular $\mathcal{A}_\rho^{\mathrm{o}}$ is a subset of $\mathcal{A}^{\mathrm{o}}$. We also define $\mathcal{B}_\rho = \mathcal{B} \cap \mathcal{A}^{\mathrm{o}}_\rho$. The operator norm of a bounded linear operator $\mathcal{L}: \mathcal{B}_\rho \to \mathcal{B}_\rho$ will be denoted by $\|\mathcal{L}\|_\rho$.

The domain of the operator L_α is defined to be the subspace of all functions $u \in \mathcal{B}_\rho$ for which the sum $\sum_{n,k} |\lambda_{k,n} u_{k,n}| \varrho_1^n \varrho_2^k$ is finite. In the cases considered here, we will show that the eigenvalues of L_α are bounded away from zero, implying that $\mathcal{L}_\alpha$ has a bounded inverse on $\mathcal{B}_\rho$. Consequently the equation (1.2) with $\sigma = \mathrm{sign}(1-\alpha)$ can be written as

$$u = \mathcal{F}_\alpha(u) \stackrel{\mathrm{def}}{=} L_\alpha^{-1} \sigma u^3 \,, \qquad \sigma = \mathrm{sign}(1-\alpha)\,. \tag{3.2}$$

In order to solve the fixed point problem for $\mathcal{F}_\alpha$ we consider an approximate Newton map $\mathcal{N}_\alpha$ associated with $\mathcal{F}_\alpha$. To be more specific, we first determine an approximate fixed point u_0 and write $u = u_0 + Ah$, where A is a suitable linear isomorphism of $\mathcal{B}_\rho$. Then u is a fixed point of $\mathcal{F}_\alpha$ if and only if h is a fixed point of the map $\mathcal{N}_\alpha$ defined by

$$\mathcal{N}_\alpha(h) = \mathcal{F}_\alpha(u_0 + Ah) - u_0 + (\mathrm{I} - A)h\,. \tag{3.3}$$

By choosing A to be an approximate inverse of $\mathrm{I} - D\mathcal{F}_\alpha(u_0)$ we can expect $\mathcal{N}_\alpha$ to be a contraction near u_0.

Given $r > 0$ and $u \in \mathcal{B}_\rho$, denote by $B_r(u)$ the open ball of radius r in $\mathcal{B}_\rho$, centered at u. Theorem 1.4 is proved by verifying the following bounds.

Lemma 3.1. *For each $\kappa \in Q_2$ there exists a set $R_\kappa \subset \mathbb{R}$ of positive measure that includes κ as a Lebesgue density point, a pair ρ of positive real numbers, a Fourier polynomial $u_0 \in \mathcal{B}_\rho$, a linear isomorphism $A: \mathcal{B}_\rho \to \mathcal{B}_\rho$, and positive constants K, δ, ε satisfying $\varepsilon + K\delta < \delta$, such that for every $\alpha \in R_\kappa$ the map $\mathcal{N}_\alpha$ defined by (3.2) and (3.3) is analytic on $B_\delta(0)$ and satisfies*

$$\|\mathcal{N}_\alpha(0)\|_\rho < \varepsilon\,, \qquad \|D\mathcal{N}_\alpha(h)\|_\rho < K\,, \quad h \in B_\delta(0)\,. \tag{3.4}$$

This lemma, together with the contraction mapping principle, implies that for each $\kappa \in K$ and each $\alpha \in R_\kappa$, the map $\mathcal{N}_\alpha$ has a unique fixed point $h_* \in B_\delta(0)$. The corresponding function $u_* = u_0 + Ah_*$ is a fixed point of $\mathcal{F}_\alpha$ and thus solves the equation (1.2). Thus Lemma 3.1 implies Theorem 1.4, after we verify that $u = u_*$ is of type $(1,1)$ and satisfies $|u_{1,1}| > \sqrt{2|1-\alpha|}$. Notice that u_* belongs to the ball $B_r(u_0)$ of radius $r = \delta\|A\|_\rho$.

Similar contraction estimates are used to prove Theorem 1.2 and Theorem 1.3. The corresponding lemmas are completely analogous to Lemma 3.1. Thus we shall not state them explicitly. Our proof of these lemmas is computer-assisted. As a by-product we obtain accurate bounds on the solutions and related quantities.

We will now describe the main estimates involved, which are specific to the problem at hand. More "generic" aspects of the proof are described in Section 5. The general strategy is to approximate a function $u \in \mathcal{A}^{\mathrm{o}}_\rho$ by a Fourier polynomial P and to estimate the difference $\mathcal{E} = u - P$. A typical step in our proof yields much more information about the error $\mathcal{E}$ than just its norm. Keeping track of such information can drastically improve the estimates in subsequent steps.

In order to describe our choice of error terms we need the following: Given positive integers N and K, denote by $\mathcal{A}^{\mathrm{o}}_{\rho,N,K}$ the space of all functions $u \in \mathcal{A}^{\mathrm{o}}_\rho$ whose Fourier coefficients $u_{n,k}$ vanish whenever $n < N$ or $k < K$. Let now N and K be fixed. Then we represent a function $u \in \mathcal{A}_\rho^{\rm o}$ as a finite sum

$$u = P + \mathcal{E}\,, \qquad P = \sum_{\substack{n\le N\\ k\le K}} c_{n,k} P_{n,k}\,, \qquad \mathcal{E} = \sum_{\substack{n\le 2N\\ k\le 2K}} E_{n,k}\,, \tag{3.5}$$

where $E_{n,k}$ is a function $\mathcal{A}_{\rho,n,k}^{\rm o}$. In this context, a bound on u consists in upper and lower bounds on the coefficients $c_{n,k}$ for $n \le N$ and $k \le K$, and an upper bound on $\|E_{n,k}\|_\rho$ for $n \le 2N$ and $k \le 2K$. Notice that the representation (3.5) is highly non-unique. This allows for a wide range of different bounds on functions in $\mathcal{A}_\rho^{\rm o}$.

Our estimates on a continuous linear operator $\mathcal{L} : \mathcal{A}_\rho^{\rm o} \to \mathcal{A}_\rho^{\rm o}$ consist in a bound on $\mathcal{L}P_{n,k}$ for each $n \le N$ and $k \le K$, together with bound of the form

$$\|\mathcal{L}E\|_\rho \le B_{\mathcal{L},k,n} \|E\|_\rho\,, \qquad E \in \mathcal{A}_{\rho,n,k}^{\rm o}\,, \tag{3.6}$$

for each $k \le 2K$ and $n \le 2N$. This also yields a bound on the operator norm on $\mathcal{L}$, namely

$$\|\mathcal{L}\|_\rho \le \Bigl(\bigvee_{\substack{n\le N\\ k\le K}} \varrho_1^{-n}\varrho_2^{-k}\|\mathcal{L}P_{n,k}\|_\rho\Bigr) \vee B_{\mathcal{L},N+1,0} \vee B_{\mathcal{L},0,K+1}\,. \tag{3.7}$$

Here, and in what follows, if s and t are real numbers then $s \vee t$ denotes the maximum value of s and t. The inequality (3.7) is used e.g. to verify the bound $\|D\mathcal{N}_\alpha(h)\|_\rho < K$ in Lemma 3.1.

The following proposition will be useful for estimating the inverse of the operator L_α.

Proposition 3.2. *Let s, t, δ be positive real numbers. If $|t - s| \ge \delta$ then*

$$\bigl|t^2 - s^2\bigr| \ge \bigl(2(s \vee t) - \delta\bigr)\delta\,. \tag{3.8}$$

Proof. Fix $t > 0$. For $s \in \mathbb{R}$ positive define $f(s) = \bigl(t^2 - s^2\bigr)^2$. Then $f(s) \ge f(t) = 0$. The derivative $f'(s) = -4s\bigl(t^2 - s^2\bigr)$ is positive for $s > t$ and negative for $s < t$. Assume that $|t - s| > \delta$. Then s lies outside the interval $[t - \delta, t + \delta]$. So either $f(s) \ge f(t + \delta)$ or $f(s) \ge f(t - \delta)$. But

$$f(t \pm \delta) = \bigl(t + (t \pm \delta)\bigr)^2\bigl(t - (t \pm \delta)\bigr)^2 = \bigl(2t \pm \delta\bigr)^2\delta^2 \ge \bigl(2t - \delta\bigr)^2\delta^2\,. \tag{3.9}$$

The same holds if s and t are exchanged. This proves (3.8). **QED**

First we consider rational values of α. Define $\mathcal{B}_{\rho,n,k} = \mathcal{B} \cap \mathcal{A}_{\rho,n,k}^{\rm o}$.

Proposition 3.3. *Let $\kappa = p/q$ with p odd and q even. Then the operator L_κ with $\nu \ge 1$ has a compact inverse $L_\kappa^{-1} : \mathcal{B}_\rho \to \mathcal{B}_\rho$. If n and k are odd then*

$$\bigl\|L_\kappa^{-1}E\|_\rho \le \frac{q^2}{2(qk^\nu \vee pn) - 1}\|E\|_\rho\,, \qquad \forall E \in \mathcal{B}_{\rho,n,k}\,. \tag{3.10}$$

Proof. If $u\in\mathcal{B}_\rho$ is a Fourier polynomial then $L_\kappa^{-1}u$ is well defined and

$$(L_\kappa^{-1}u)_{n,k}=\frac{q^2}{(qk^\nu+pn)(qk^\nu-pn)}u_{n,k}\,, \tag{3.11}$$

for odd n and k. Here we have used that $qk^\nu-pn$ is odd and thus nonzero. Using Proposition 3.2 with $t=qk^\nu$ and $s=pt$ and $\delta=1$, we obtain

$$\bigl|(L_\kappa^{-1}u)_{n,k}\bigr|\le\frac{q^2}{2(qk^\nu\vee pn)-1}|u_{n,k}|\,. \tag{3.12}$$

Given that the fraction on the right hand side is a decreasing function of both n and k, with a zero limit as $n\vee k\to\infty$, the assertion follows. **QED**

Let $\mathbb{N}_{\rm o}=\{1,3,5,\ldots\}$. For irrational α we need to estimate the quantities

$$\phi(n,k)=\inf_{\substack{x\ge k\\ y\ge n}}\bigl|(\beta x^\nu)^2-y^2\bigr|\,,\qquad n,k\in\mathbb{N}_{\rm o}\,, \tag{3.13}$$

where $\beta=\alpha^{-1}$. Here, and in what follows, x and y always denote odd positive integers. Notice that the function $\phi:\mathbb{N}_{\rm o}\times\mathbb{N}_{\rm o}\to\mathbb{R}$ is non-decreasing in each argument.

The following proposition is trivial, but it gives an explicit expression for the bounds $B_{\mathcal{L},k,n}$ that appear in (3.6) for the operator $\mathcal{L}=L_\alpha^{-1}$.

Proposition 3.4. *Assume that the function ϕ is strictly positive, and unbounded. Then the operator L_α has a compact inverse $L_\alpha^{-1}:\mathcal{B}_\rho\to\mathcal{B}_\rho$. Furthermore, for any $n,k\in\mathbb{N}_{\rm o}$,*

$$\bigl\|L_\alpha^{-1}E\|_\rho\le\frac{\beta^2}{\phi(n,k)}\|E\|_\rho\,,\qquad\forall E\in\mathcal{B}_{\rho,n,k}\,. \tag{3.14}$$

We note that the compactness of L_α^{-1} is not really needed, but just the bound (3.14).

Concerning the case $\beta=\sqrt{3}$, notice that the assumption in Theorem 1.3 says that $\phi(n,9)\ge 39$ for all n. Based on this assumption, it suffices to compute a finite number of terms $\bigl|3x^4-y^2\bigr|$ in order to determine $\phi(n,k)$ for every $k<9$ and every n.

Lemma 3.5. *There exists a set $R\subset\mathbb{R}$ of full measure such that if $\alpha\in R$ then $L_\alpha=\alpha^2\partial_t^2+\partial_x^4$ has a compact inverse $L_\alpha^{-1}:\mathcal{B}_\rho\to\mathcal{B}_\rho$. Let $\kappa=p/q$ with p odd and q even. Given odd positive integers K and N, denote by R_κ the set of all $\alpha\in R$ with the property that*

$$\bigl\|L_\alpha^{-1}E\|_\rho\le\frac{4p^2\alpha^{-2}/7}{p(\alpha^{-1}k^2\vee n)-7/16}\|E\|_\rho\,,\qquad\forall E\in\mathcal{B}_{\rho,n,k}\,, \tag{3.15}$$

holds for all $n\le N$ and $k\le K$. Then κ is a Lebesgue density point for R_κ.

This lemma will be proved in the next section. Notice that if $\alpha=\kappa$ then (3.15) follows from (3.10). Thus $\kappa\in R_\kappa$.

4. Irrational frequencies

Consider the operator $L_\alpha = \alpha^2\partial_t^2 + \partial_x^4$ for irrational values of α. Let $\beta = \alpha^{-1}$. The eigenvalues of $L_\alpha^{-1} : \mathcal{A}_{0,1}^{\circ} \to \mathcal{A}_{0,1}^{\circ}$ are

$$\left(L_\alpha^{-1}\right)_{n,k} = \frac{\beta^2}{(\beta k^2 + n)(\beta k^2 - n)}\,, \tag{4.1}$$

with eigenvectors $P_{n,k}$. Here, and in what follows, $n \geq 0$ and $k \geq 1$. By Proposition 3.2 we have

$$\left|\left(L_\alpha^{-1}\right)_{k,n}\right| \leq \frac{\beta^2}{\bigl(2(\beta k^2 \vee n) - \|\lceil \beta k^2 \rfloor\|\bigr)\|\lceil \beta k^2 \rfloor\|}\,, \tag{4.2}$$

where $\|\lceil s \rfloor\| = \mathrm{dist}(s, \mathbb{Z})$ denotes the distance of $s \in \mathbb{R}$ from the set of integers.

The following proposition is part of a result in [15]. We include a proof since it is short and simple. Let $(\psi_1, \psi_2, \psi_3, \ldots)$ be a summable sequence of nonnegative real numbers.

Proposition 4.1. *Let $m \geq 1$ and $\Psi_m = \sum_{k \geq m} \psi_k$. Consider an interval $I = (a, a+1]$. Then the set of all $\beta \in I$ satisfying*

$$\bigl\|\lceil \beta k^2 \rfloor\bigr\| < \psi_k \quad \text{for some} \quad k \geq m\,, \tag{4.3}$$

has measure at most $2\Psi_m$.

Proof. We may assume that $\psi_k \leq 1/2$. Consider the circle $\mathbb{R}/\mathbb{Z}$. For simplicity we identify this circle with the interval $S = (-1/2, 1/2]$. For $s \in \mathbb{R}$ define $\lceil s \rfloor$ to be the unique real number in S that differs from s by an integer. Notice that $\|\lceil \beta k^2 \rfloor\| < \psi_k$ if and only if $\lceil \beta k^2 \rfloor$ belongs to $(-\psi_k, \psi_k)$.

The map $\beta \mapsto \lceil \beta k^2 \rfloor$ covers the circle S exactly k^2 times as β ranges in I. So the set I_k of all $\beta \in I$ that satisfies $\|\lceil \beta k^2 \rfloor\| < \psi_k$ has measure precisely $2\psi_k$. The set of all $\beta \in I$ that satisfy (4.3) is the union $\bigcup_{k \geq m} I_k$ and thus has measure at most $2\Psi_m$. **QED**

This proposition implies e.g. that for every $\varepsilon > 0$ and almost every $\beta \in I$, we have $\|\lceil \beta k^2 \rfloor\| \geq k^{-1-\varepsilon}$ for all but finitely many values of $k > 0$. Combining this fact with the bound (4.2) implies the

Corollary 4.2. *For almost every $\alpha \in \mathbb{R}$ the operator $L_\alpha = \alpha^2\partial_t^2 + \partial_x^4$ has a compact inverse $L_\alpha^{-1} : \mathcal{A}_{0,1}^{\circ} \to \mathcal{A}_{0,1}^{\circ}$.*

The following extends Proposition 4.1 to smaller intervals, at the cost of imposing a lower bound on m.

Proposition 4.3. *Consider a subinterval $J \subset I$. Assume that $m^{-2} \leq |J|$. Then the set of all $\beta \in J$ satisfying (4.3) has measure less than $4\Psi_m|J|$.*

Proof. We use the notation introduced in the proof of Proposition 4.1. For simplicity assume that $a = 0$.

Let $\psi_k' = k^{-2}\psi_k$. If we identify $(-\psi_k', 0]$ with $(1-\psi_k', 1]$ then I_k is the union of the k^2 intervals $(nk^{-2} - \psi_k', nk^{-2} + \psi_k')$ for $0 \le n < k^2$. Notice that these intervals are centered at integer multiples of k^{-2}, and that each has length $2\psi_k'$.

Assume now that $k^{-2} \le m^{-2} \le |J|$. Then we have $jk^{-2} \le |J| < (j+1)k^{-2}$ for some positive integer j. Thus the set $J_k = J \cap I_k$ has measure

$$|J_k| < (j+1)2\psi_k' = \frac{j+1}{j} jk^{-2} 2\psi_k \le \frac{j+1}{j} |J| 2\psi_k \,. \tag{4.4}$$

Summing over all $k \ge m$ and using that $\frac{j+1}{j} \le 2$ we obtain the desired bound. **QED**

The following will be used when $k \le m$.

For $s \in \mathbb{R}$ define $\|\lceil\beta s\|_{\rm o} = \operatorname{dist}(s, \mathbb{Z}_{\rm o})$, where $\mathbb{Z}_{\rm o}$ denotes the set of odd integers.

Proposition 4.4. *Let $\beta_0 = q/p$ with q even and p odd. Let $0 < r < 1$ and $\beta \in \mathbb{R}$. Then*

$$\big\|\lceil\beta k^2\big\|_{\rm o} \ge \frac{1-r}{p} \quad \text{whenever} \quad |\beta - \beta_0| \le \frac{r}{pk^2}\,, \quad k \in \mathbb{N}_{\rm o}\,. \tag{4.5}$$

Proof. First notice that $p^{-1} \le \big\|\lceil\beta_0 k^2\big\|_{\rm o} \le 1$ for all odd integers k. Clearly

$$\beta k^2 - n = \beta_0 k^2 - n_0 + (\beta - \beta_0)k^2 - (n - n_0)\,. \tag{4.6}$$

Using the odd integer n_0 closest to $\beta_0 k^2$ and the odd integer n closest to βk^2 we get

$$\big\|\lceil\beta k^2\big\|_{\rm o} = \big\|\lceil\beta_0 k^2\big\|_{\rm o} + (\beta - \beta_0)k^2 - (n - n_0)\,. \tag{4.7}$$

Assume that $|\beta - \beta_0| \le \frac{r}{pk^2}$. If $n \ne n_0$ then $|n - n_0| \ge 2$ and thus

$$\big\|\lceil\beta k^2\big\|_{\rm o} \ge |n - n_0| - |\beta - \beta_0|k^2 - \big\|\lceil\beta_0 k^2\big\|_{\rm o} \ge |n - n_0| - \frac{r}{p} - 1 \ge \frac{p-r}{p}\,. \tag{4.8}$$

If $n = n_0$ then

$$\big\|\lceil\beta k^2\big\|_{\rm o} \ge \big\|\lceil\beta_0 k^2\big\|_{\rm o} - |\beta - \beta_0|k^2 \ge \frac{1}{p} - \frac{r}{p} = \frac{1-r}{p}\,. \tag{4.9}$$

In both cases we have (4.5). **QED**

Proof of Lemma 3.5. The compactness of $L_\alpha^{-1} : \mathcal{B}_\rho \to \mathcal{B}_\rho$ for almost all $\alpha \in \mathbb{R}$ follows from Corollary 4.2. In order to prove the remaining part of Lemma 3.5, fix an even integer q and odd integers p, K, N; all positive. Let $\alpha_0 = p/q$ and $\beta_0 = q/p$.

Given $C > 0$ to be specified later, choose $m \in \mathbb{N}_{\rm o}$ larger than $K \vee 4$ and sufficiently large such that

$$\frac{2C}{(m-2)^2} \le \frac{7}{8p}\,, \qquad \beta_0 m^2 > 2N + 4\,, \qquad m^2 - 4\alpha_0 > (m-1)^2\,. \tag{4.10}$$

We define

$$\psi_k = \frac{2C}{k(k+2)}\,, \qquad k \in \mathbb{N}_{\rm o}\,, \quad k \geq m\,, \tag{4.11}$$

and $\psi_k = 0$ if k is even. The sum Ψ_m defined in Proposition 4.1 is given by

$$\Psi_m = \sum_{k\geq m} \psi_k = \frac{C}{m}\,. \tag{4.12}$$

Let $\delta = 1/(8pm^2)$ and $J = [\beta_0 - 4p\delta, \beta_0 + 4p\delta]$. Then by Proposition 4.3 we have a bound $\|\lceil \beta k^2 \rfloor\| \geq \psi_k$ for every $k \geq m$ and for every $\beta \in J$ outside some set of measure $32Cp\delta/m$ or less. Define

$$\psi_k = \frac{7}{8p}\,, \qquad k \in \mathbb{N}_{\rm o}\,, \quad k < m\,. \tag{4.13}$$

Then Proposition 4.4 with $r = 1/8$ implies that $\|\lceil \beta k^2 \rfloor\|_{\rm o} \geq \psi_k$ for all $k < m$ and for all values of β in the interval $J_m = [\beta_0 - \delta/2, \beta_0 + \delta/2]$.

At this point we have proved that $\|\lceil \beta k^2 \rfloor\|_{\rm o} \geq \psi_k$ holds for all k and for every β in a subset $J_m' \subset J_m$ of measure at least $|J_m| - 32Cp\delta/m = |J_m|(1 - 32Cp/m)$. To complete the proof of Lemma 3.5, we will now show that there exists a choice of $C > 0$, such that if $m \in \mathbb{N}_0$ satisfies (4.10) then the bounds (3.15) hold for every $\beta \in J_m'$, every odd $n \leq N$, and every odd $k \leq K$.

To be more precise, we first restrict β to the interval $B = [\beta_0/2, 2\beta_0]$. Then we choose C in such a way that βC is larger than $\varphi_2(N) \vee \varphi_1(K)$ for all $\beta \in B$, where

$$\varphi_1(k) = \frac{7\beta k^2}{4p} - \frac{49}{64p^2}\,, \qquad \varphi_2(n) = \frac{7n}{4p} - \frac{49}{64p^2}\,. \tag{4.14}$$

In addition we require $\beta C \geq 1$. In what follows we always assume that k and n are odd positive integers. We also assume that $\beta \in J_m'$ so that $\|\lceil \beta k^2 \rfloor\|_{\rm o} \geq \psi_k$ for every k.

We now estimate the values $\phi(n,k)$ defined in (3.13). Here $\nu = 2$. First we exploit the fact that $\phi(n,k) \geq \phi(1,k)$. It implies that

$$\phi(n,k) \geq \phi_1(k) \wedge \phi(1,m)\,, \qquad \phi_1(k) = \inf_{\substack{k\leq x<m\\ y\geq 1}} \bigl|\beta^2 x^4 - y^2\bigr|\,, \qquad k < m\,, \tag{4.15}$$

where we have used the notation $s \wedge t = \min(s,t)$. We start by estimating $\phi(1,m)$. If $x \geq m$ then $|\beta x^2 - y| \geq \psi_x \geq Cx^{-2}$ for all y. Thus by Proposition 3.2 we have

$$\phi(1,m) \geq \inf_{x\geq m} \bigl(2\beta x^2 - Cx^{-2}\bigr)Cx^{-2} = 2\beta C - C^2 m^{-4} \geq \beta C\,. \tag{4.16}$$

For the last inequality we have used the first condition in (4.10) and the fact that $\beta C \geq 1$. Now consider $\phi_1(k)$. Using that $|\beta x^2 - y| \geq \frac{7}{8p}$ for $x < m$ and applying Proposition 3.2 we have

$$\phi_1(k) \geq \inf_{k\leq x<m} \bigl(2\beta x^2 - \tfrac{7}{8p}\bigr)\tfrac{7}{8p} = \varphi_1(k)\,, \qquad k < m\,. \tag{4.17}$$

Notice that this applies to any $k \le K$ since we have assumed that $m > K$.

Next we exploit the fact that $\phi(n,k) \ge \phi(n,1)$. Denote by ℓ the largest odd integer not exceeding βm^2. Then

$$\phi(n,k) \ge \phi_2(n) \wedge \phi(\ell,1)\,, \qquad \phi_2(n) = \inf_{\substack{x\ge 1\\ n\le y<\ell}} \bigl|\beta^2 x^4 - y^2\bigr|\,, \qquad n<\ell\,. \tag{4.18}$$

We start by estimating $\phi(\ell,1)$. For every $y \in \mathbb{N}_o$ denote by x_y be the odd integer that minimizes $\bigl|\beta^2 x^4 - y^2\bigr|$. By using the third condition in (4.10) we find that if $y \ge \ell$ then $x_y \ge \sqrt{\alpha y} - 2 > m-3$. Thus $\psi_{x_y} \ge \frac{2C}{(m-2)m}$ whenever $y \ge \ell$, and

$$\phi(\ell,1) \ge \inf_{y\ge\ell}\bigl(2\beta x_y^2 - \psi_{x_y}\bigr)\psi_{x_y} \ge 2\beta C - \psi_{x_\ell}^2 \ge \beta C\,. \tag{4.19}$$

Here we have used Proposition 3.2, the first condition in (4.10), and the fact that $\beta C \ge 1$. Now consider $\phi_2(n)$. If $x \ge m$ then $|\beta x^2 - y| > 1$ for every $y < \ell$. On the other hand, if $x < m$ then $|\beta x^2 - y| \ge \frac{7}{8p}$. In either case, we obtain

$$\phi_2(n) \ge \inf_{n\le y<\ell}\bigl(2y - \tfrac{7}{8p}\bigr)\tfrac{7}{8p} = \varphi_2(n)\,, \qquad n<\ell\,, \tag{4.20}$$

where we have used again Proposition 3.2. Notice that this applies to any $n \le N$ since the second condition in (4.10) implies that $\ell > N$.

By our assumption on C we have $\beta C \ge \varphi_2(n)$ and $\beta C \ge \varphi_1(k)$ for all $n \le N$ and all $k \le K$. Combining (4.15) and (4.16) and (4.17) we find that $\phi(n,k) \ge \varphi_1(k)$ for $n \le N$ and $k \le K$. Similarly, combining (4.18) and (4.19) and (4.20) we find that $\phi(n,k) \ge \varphi_2(n)$ for $n \le N$ and $k \le K$. Thus

$$\phi(n,k) \ge \varphi_2(n) \vee \varphi_1(k)\,, \qquad n \le N\,, \quad k \le K\,, \tag{4.21}$$

holds for every $m \in \mathbb{N}_o$ satisfying (4.10), and for every $\beta \in B \cap J'_m$. Substituting the bound (4.17) for $\varphi_1(k)$ and the bound (4.20) for $\varphi_2(n)$ into the inequality (4.21), and using (3.14), we obtain the bound (3.15). As explained above, this completes the proof of Lemma 3.5. **QED**

5. Estimates done by the computer

What remains to be proved is Lemma 3.1 and two analogous lemmas that imply Theorem 1.2 and Theorem 1.3. We will only describe here the proof of Lemma 3.1. The other two lemmas are proved similarly, and we refer to [28] for the complete details.

Lemma 3.1 involves the choice of parameters $(\rho, u_0, A, K, \delta, \varepsilon)$ for each value of $\kappa \in Q_2$. The precise value of each parameter, as well as the complete set Q_2, is given in [28]. To be more precise, only the set Q_2 and the domain parameter ρ for each $\kappa \in Q_2$ are specified explicitly. The other parameters are computed as specified by our programs. In particular, the approximate fixed point u_0 is first guessed and then improved by iterating a numerical version of $\mathcal{N}_\kappa$. The operator $A : \mathcal{B}_\rho \to \mathcal{B}_\rho$ is the approximate inverse of a numerical approximation for the operator $\mathrm{I} - D\mathcal{F}_\kappa(u_0)$. The constants K, δ, ε are determined a-posteriori to satisfy (3.4). At the end we verify that $\varepsilon + K\delta < \delta$, and that there exists $r \geq \delta\|A\|_\rho$ such that every function $u \in B_r(u_0)$ is of type $(1,1)$ and satisfies $|u_{1,1}| > \sqrt{2|1-\alpha|}$.

This leaves the task of estimating the two norms in (3.4) and the operator norm of A. The norm of the operator $\mathcal{L} = D\mathcal{N}_\alpha(h)$ for $h \in B_\delta(0)$ is estimated by using the inequality (3.7). So all we need are bounds on the function $\mathcal{N}_\alpha(0) = \mathcal{F}_\alpha(u_0) - u_0$, on $\mathcal{L}P_{n,k}$ for $n \leq N$ and $k \leq K$, and on $\mathcal{L}E_{n,k}$ for $n \leq 2N$ and $k \leq 2K$. Here $E_{n,k}$ denotes a function in $\mathcal{A}^{\mathrm{o}}_{n,k}$ which is unknown except for a bound on its norm. The operator norm of A is estimated similarly. These are standard tasks in many computer-assisted proofs, including [26].

At this level our techniques are similar to the techniques used in [26] to find solutions for the boundary value problem $-\Delta u = wu^3$ on the unit square. The functions spaces are in fact the same. As far as estimates are concerned, the main difference is that we now have the operator L_α^{-1} instead of the inverse Laplacean. This is where we use the bounds described in Section 3. But at the level of enclosures (representable sets in $\mathcal{A}_{n,k}$) and data types used to represent such enclosures, we use the same methods as in [26]. Thus, we refer to [26] for a description of the basic principles.

The main goal of such a description is to simplify the reading of our computer programs [26]. The source code of these programs contains the details of how the remaining part of the proof is organized. Our code is written in the programming language Ada [29] and was compiled using a public version of the gcc/gnat compiler [31]. By running the resulting machine code, the computer verifies the inequalities necessary to complete the proof of Lemma 3.1 and Theorem 1.4.

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[29] Ada Reference Manual, ISO/IEC 8652:2012(E),
available e.g. at `http://www.ada-auth.org/arm.html`

[31] A free-software compiler for the Ada programming language, which is part of the GNU Compiler Collection; see `http://gcc.gnu.org/software/gnat/`