Asymptotic scaling and universality for skew products with factors in SL(2,ℝ)

Hans Koch [1]

Abstract. We consider skew-product maps over circle rotations $x \mapsto x + \alpha \pmod 1$ with factors that take values in $\mathrm{SL}(2,\mathbb{R})$. In numerical experiments, with α the inverse golden mean, Fibonacci iterates of maps from the almost Mathieu family exhibit asymptotic scaling behavior that is reminiscent of critical phase transitions. In a restricted setup that is characterized by a symmetry, we prove that critical behavior indeed occurs and is universal in an open neighborhood of the almost Mathieu family. This behavior is governed by a periodic orbit of a renormalization transformation. An extension of this transformation is shown to have a second periodic orbit as well, and we present some evidence that this orbit attracts supercritical almost Mathieu maps.

1. Introduction

We consider the asymptotic behavior of skew products

$$A^{*q}(x) \stackrel{\text{def}}{=} A(x+(q-1)\alpha)\cdots A(x+2\alpha)A(x+\alpha)A(x)\,, \tag{1.1}$$

as $q \to \infty$ along certain subsequences, where A is a real analytic function from the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ to the group $\mathrm{SL}(2,\mathbb{R})$. Here, α is a given irrational number and $x \mapsto x+\alpha$ is considered modulo 1. Our main results concern the inverse golden mean $\alpha = \sqrt{5}/2 - 1/2$, but we expect analogous results to hold for arbitrary quadratic irrationals.

The products (1.1) arise when iterating a map G on $\mathbb{T}\times\mathbb{R}^2$ of the form

$$G(x,y) = \bigl(x+\alpha, A(x)y\bigr)\,, \qquad x \in \mathbb{T}\,, \quad y \in \mathbb{R}^2\,. \tag{1.2}$$

Such maps will be called skew-product maps. We will use the notation $G = (\alpha, A)$ and refer to A as the factor of G. In this notation, the q-th iterate of G is $G^q = (q\alpha, A^{*q})$, with A^{*q} given by (1.1). We note that the map G is invertible, with inverse $G^{-1} = \bigl(-\alpha, A(\,\boldsymbol{.}-\alpha)^{-1}\bigr)$. The q-th iterate of G^{-1} will be denoted by G^{-q}.

Two dynamical quantities associated with such a skew-product map G are its Lyapunov exponent $L(G)$ and its fibered rotation number $\varrho(G)$. They are defined by

$$L(G) = \lim_{q\to\infty} \frac{1}{q}\log\bigl\|A^{*q}(x)\bigr\|\,, \qquad \varrho(G) = \lim_{q\to\infty}\frac{1}{2\pi q}\arg\mathcal{G}^q(x,\vartheta)\,, \tag{1.3}$$

where $\arg(x,\vartheta) = \vartheta$. Here, $\mathcal{G}$ denotes a lift of the map $(x,y) \mapsto \bigl(x+\alpha, \|A(x)y\|^{-1}A(x)y\bigr)$ from $\mathbb{T}\times\mathbb{S}$ to $\mathbb{T}\times\mathbb{R}$, where $\mathbb{S}$ denotes the unit circle $\|y\| = 1$ in $\mathbb{R}^2$. Assuming that $A : \mathbb{T} \to \mathrm{SL}(2,\mathbb{R})$ is continuous, homotopic to the function $x \mapsto \mathbf{1}$, and that α irrational, the limit for $\varrho(G)$ does not depend on x or ϑ, and convergence is uniform. Furthermore, it is independent modulo 1 of the choice of the lift $\mathcal{G}$. Under the same assumptions, the limit for $L(G)$ exists and is a.e. constant in x. For proofs of these and related facts we refer to [12,13,16,39].

[1] Department of Mathematics, The University of Texas at Austin, Austin, TX 78712.

A skew-product map G is said to be of Schrödinger type if its factor is of the form

$$A(x)=A((E,s),x)=\begin{bmatrix} E-\lambda v(x) & -1 \\ 1 & 0 \end{bmatrix}\,, \qquad \lambda=e^s\,. \tag{1.4}$$

A bi-infinite orbit $n\mapsto(x_n,y_n)$ for such a map G has the property that $y_n=\left[\begin{smallmatrix} u_n \\ u_{n-1}\end{smallmatrix}\right]$ for some sequence $n\mapsto u_n$ of real numbers. If $x_0=0$, then this sequence u is a solution of the equation $H_\lambda^\alpha u=Eu$, where H_λ^α is the Schrödinger operator given by the equation

$$(H_\lambda^\alpha u)_n=u_{n+1}+u_{n-1}+\lambda v(n\alpha)u_n\,, \qquad n\in\mathbb{Z}\,. \tag{1.5}$$

The choice of potential $v(x)=2\cos(2\pi(x+\xi))$ defines the family of operators H_λ^α that are knows as almost Mathieu (AM) operators. They describe the motion of an electron on $\mathbb{Z}^2$ under the influence of a magnetic flux $2\pi\alpha$ per unit cell, if one restricts to wave functions $\phi(n,m)=e^{-2\pi i m\xi}u_n$. The full Hamiltonian for this system is known as the Hofstadter Hamiltonian [1,5]. These operators have been studied extensively over the past 20 years. Three reviews can be found in [24,38,51].

Some of the most interesting phenomena in physics arise from the fact that asymptotic quantities can depend in a nontrivial way on model parameters. In the AM family, the main parameters (besides α) are the coupling constant λ and the energy E. The asymptotic quantities include the Lyapunov exponent L and the fibered rotation number ϱ. Among the many known properties are the following [30,43,24,47,38,51]. Here, we suppress the dependence on the parameter ξ, since it is trivial, as was mentioned after (1.3).

Assume that α is irrational. Then the spectrum Σ_λ^α of the operator H_λ^α on $\ell^2(\mathbb{Z})$ is a Cantor set of measure $2-2\min\{\lambda,1/\lambda\}$. For all energies in the spectrum, the Lyapunov exponent of the corresponding AM map G is given by $L(G)=\max\{0,\log\lambda\}$. The fibered rotation number ϱ is a continuous decreasing function of the energy E, and it is constant on each spectral gap $\bigl($a connected component of $\mathbb{R}\setminus\Sigma_\lambda^\alpha\bigr)$. As was described first in [10,13], this resonance phenomenon has an interesting arithmetic aspect: each gap can be labeled canonically by an integer k, known as the Hall conductance. On the gap with index k, the fibered rotation number is constant and satisfies $1-2\varrho(G)\equiv k\alpha\ (\mathrm{mod}\,1)$. The left hand side of this congruence can also be identified with the integrated density of states [12,16,26,39] for the Hamiltonian H_λ^α.

Regions where asymptotic quantities depend analytically on model parameters are also called phases. By varying the parameters, it is possible to induce phase transitions. A common phenomenon observed in such transitions is universality: within a large class of systems, the type of singularity is independent of the system being considered, down to precise values of observable quantities. The theory of critical phenomena aims to explain situations where the singularities involve power laws. Power law behavior represents asymptotic scale invariance, and the quantities that describe such universal scaling are known as critical exponents.

Similar phenomena have been observed in comparatively simple systems. Some examples will be mentioned below. Based on numerical observations and partial results [52,53], we conjecture that skew-product maps exhibit such universal scaling as well.

To be more specific, we consider the inverse golden mean $\alpha_* = \sqrt{5}/2 - 1/2$. Denote by p_k/q_k the k-th continued fraction approximant for α_*. That is, p_k is the k-th Fibonacci number, and $q_k = p_{k+1}$.

Conjecture 1.1. *There exists a "large" class $\mathcal{A}$ of real analytic functions $A : \mathbb{T} \to \mathrm{SL}(2,\mathbb{R})$, which includes the* AM *factors for $\xi = \alpha_*/2$, for which the following holds. Let ϱ be a rational number in $[0, 1/2]$. Then for every real analytic two-parameter family $\beta \mapsto A(\beta, .)$ of functions in $\mathcal{A}$ that satisfies a certain transversality condition, there exists a parameter value β_* where the map $(\alpha_*, A(\beta_*, .))$ has fibered rotation number ϱ, as well as three matrices $L, C, M \in \mathrm{GL}(2,\mathbb{R})$, such that the limits*

$$\begin{aligned} B_*(\beta, x) &= \lim_{n\to\infty} L^{-n} A^{*p_{\ell n}}\bigl(\beta_* + CM^{-n}\beta, \alpha_*^{\ell n} x\bigr) L^n\,, \\ A_*(\beta, x) &= \lim_{n\to\infty} L^{-n} A^{*q_{\ell n}}\bigl(\beta_* + CM^{-n}\beta, \alpha_*^{\ell n} x\bigr) L^n\,, \end{aligned} \tag{1.6}$$

exist for all $x \in \mathbb{R}$ and are independent of the given family. Here ℓ is some positive integer that depends only on ϱ. Furthermore, L is conjugate to some fixed matrix $L_\ell \in \mathrm{GL}(2,\mathbb{R})$, and M is conjugate to some fixed diagonal matrix $\mathrm{diag}(\mu_1, \mu_2)$.

This conjecture has motivated the work presented in this paper as well as our earlier work in [52,53,55]. The integers ℓ that appear in (1.6) can be obtained by considering the map on the torus $\mathbb{T}^2$ given by the matrix $\left[\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right]$. Every point $(0, \varrho)$ with ϱ rational lies on a periodic orbit for this map. The number $\ell = \ell(\varrho)$ is the shortest such period. For a class $\mathcal{A}$ that includes the AM factors, one finds that ℓ must be a multiple of 3, due to a symmetry of this family. For more details we refer to [53].

We believe that Conjecture 1.1 holds for any irrational α_* that has a periodic continued fraction expansion. (For general quadratic irrationals, the same should apply after finitely many steps of the transformation $\mathfrak{R}$ defined below.) An extended version could include α as a parameter. Based on renormalization arguments, we expect a similar scaling in the difference $\alpha - \alpha_*$. But we have not investigated this situation.

An important aspect of Conjecture 1.1 is universality: near its critical point β_*, the behavior of a family can be described accurately in terms of just two parameters. The limits in (1.6), as well as the conjugacy class of the matrices L and M are independent of the family. Universality of this type plays an important role in the description of critical phenomena in condensed matter physics, where it is impossible to know a system precisely.

The points β_* represent phase transitions for the chosen family. In the AM family parametrized by $\beta = (E, s)$, it is known that the system described by the Hamiltonian H_λ^α undergoes a transition from a conducting phase (a.c. spectrum) for $\lambda = e^s < 1$ to an insulation phase (p.p. spectrum) for $\lambda = e^s > 1$. For proofs and references we refer to [33]. So in this case, we expect that $s_* = 0$ at each critical point $\beta_* = (E_*, s_*)$. Furthermore, our numerical computations suggest that the scaling M is diagonal.

The "phase portrait" for $\lambda = 1$, obtained by plotting the spectrum H_1^α as a set-valued function of α, is known as the Hofstadter butterfly [5]. A detailed topological description of the Hofstadter butterfly can be found in [35]. One of its striking features, aside from the gap labeling, is a local self-similarity property: successive magnifications about certain points seem to yield an asymptotic limit set [31,49,52,53,54].

Cases where the expected phase transitions have been studied from the point of view of critical phenomena cover the rotation numbers $\varrho = {}^1\!/_2, {}^3\!/_8, {}^2\!/_6, {}^1\!/_4, {}^1\!/_6, {}^1\!/_8, 0$. The values $\varrho = {}^1\!/_2$ and $\varrho = 0$ correspond to the energies $E_* = \mp 2.5975\ldots$ at the bottom and top of the spectrum, respectively. The value $\varrho = {}^1\!/_4$ corresponds to $E_* = 0$, for symmetry reasons. These three cases have been considered in [52,53]. In particular, rough numerical computations indicate that the Hofstadter butterfly is asymptotically invariant under a scaling about the point (α_*, E_*), and that the scaling factor in the energy direction is given by the constant μ_1.

A framework that has been extremely successful in describing critical phenomena is renormalization. In the area of dynamical systems, this includes period-doubling cascades for interval maps [7,8,18,21,22,25,32] or area-preserving maps [17,37,42], critical circle mappings [15,19,20,36], and the breakup of invariant tori in area-preserving maps [9,11,46,50], to name just a few.

In the problem at hand, we expect renormalization to work as follows. The functions B_n and A_n whose limits are being considered in (1.6) are the factors associated with two skew-product maps F_n and G_n that depend on a parameter β. In the renormalization framework, the sequence of pairs $P_n = (F_n, G_n)$ lie on an orbit of a transformation $\mathfrak{R}$ that acts on a space of pairs. The accumulation property (1.6) describes convergence (modulo re-parametrization) $P_n \to P_*$ to a fixed point P_* of $\mathfrak{R}^\ell$. The fixed point P_* and the observed accumulation rates are universal, due to the fact that they reflect properties of the transformation $\mathfrak{R}$. In particular, we expect $\mathfrak{R}^\ell$ to be hyperbolic at P_*, with a two-dimensional local unstable manifold. This manifold is given by the family of pairs $\beta \mapsto \mathtt{P}_*(\beta)$ whose factors are limit functions B_* and A_* in (1.6). The transversality condition mentioned in Conjecture 1.1 requires that the given family be transversal to the stable manifold of $\mathfrak{R}^\ell$. And β_* is the value of the parameter β where a given family intersects the stable manifold.

Our goal here is to verify this renormalization picture in a setup that is restricted but includes most of the essential aspects. After finding an appropriate transformation $\mathfrak{R}$, the first step in any renormalization group (RG) analysis is to prove the existence of a small invariant set, such as a periodic orbit. For the type of skew-product maps considered here, a fixed point of $\mathfrak{R}^3$ was obtained in [50] for $\varrho \in \{{}^1\!/_2, 0\}$. A theorem concerning the existence of a fixed point of $\mathfrak{R}^6$ associated with $\varrho = {}^1\!/_4$ was announced in [53]. A proof of this theorem will be given in Section 4.

Constructing a fixed point P_* for $\mathfrak{R}^\ell$ is a local analysis (near an approximate fixed point). By contrast, proving that the AM family and others are attracted to the unstable manifold of $\mathfrak{R}^\ell$ at P_* is a global analysis and much harder. A simplified version of this problem was considered in [55], in a situation where the Hofstadter Hamiltonian reduces to skew-product maps with factors that take values in the circle $\mathbb{R} \cup \{\infty\}$. Here we prove convergence (1.6) with $\mathrm{SL}(2, \mathbb{R})$ factors, including the AM factors, but only in a restricted one-parameter setup.

Before describing our main results, we would like to mention a peculiarity of these skew-product maps. High accuracy computations suggest [53] that the eigenvalue μ_1 associated with $\varrho \in \{{}^1\!/_2, 0\}$ is a zero of the polynomial $\mathcal{P}_6(z) = z^4 - 196z^3 - 58z^2 - 4z + 1$. And the eigenvalue μ_1 associated with $\varrho = {}^1\!/_4$ is a zero of $\mathcal{P}_3(z) = z^4 - 30z^3 - 24z^2 - 10z - 1$.

In both cases, the product of the two real roots of $\mathcal{P}_\ell$ is $(-\alpha)^{-\ell}$. Furthermore, the value of μ_2 appears to be $\alpha^{-\ell}$. In fact, computations that were carried out in the context of the present paper suggest that the eigenvalues of $D\mathfrak{R}^\ell(P_*)$ can all be written down in algebraic form. It is unusual that universal constants associated with critical phenomena are algebraically related to basic system parameters, such as the flux parameter α here. Known exceptions are statistical mechanics models in 2 dimensions, where the large scale asymptotic is governed by a conformal symmetry. The class of skew-product maps that includes the AM maps seems to be governed by symmetries as well, but it is not clear how the symmetries of the Hofstadter model [23,27,28] generate the algebraic eigenvalues that are observed here.

2. Main results

As is common in the renormalization of maps that include a circle rotation, we first generalize the notion of periodicity by considering commuting pairs of maps. Consider the map $F = (1, \mathbf{1})$ on $\mathbb{R} \times \mathbb{R}^2$, defined by $F(x, y) = (x + 1, y)$. A skew-product map $G = (\alpha, A)$ with a factor $A : \mathbb{R} \to \mathrm{SL}(2, \mathbb{R})$ represents a map on on the cylinder $\mathbb{T} \times \mathbb{R}^2$ if and only if G commutes with F. A more general skew-product map $F = (1, B)$ on $\mathbb{R} \times \mathbb{R}^2$ can be viewed as defining a cylinder $\mathbb{T}_F \times \mathbb{R}^2$ embedded in $\mathbb{R} \times \mathbb{R}^2$, by identifying points on the orbit of F. If G commutes with F, then G defines a map on this cylinder $\mathbb{T}_F \times \mathbb{R}^2$.

Consider now pairs (F, G) of maps $F = (1, B)$ and $G = (\alpha, A)$ on $\mathbb{R} \times \mathbb{R}^2$ that commute. Here, α can be an arbitrary irrational number between 0 and 1. Then the renormalized pair is defined by the equation

$$\mathfrak{R}((F, G)) = \bigl(\check{F}, \check{G}\bigr), \qquad \check{F} = \Lambda^{-1} G \Lambda, \quad \check{G} = \Lambda^{-1} F G^{-c} \Lambda, \tag{2.1}$$

where c is the integer part of α^{-1}, and where $\Lambda(x, y) = \bigl(\alpha x, L_1 y\bigr)$. Here, L_1 is a suitable nonsingular 2×2 matrix that can chosen to depend on the pair (F, G). By construction, the first component of $\check{F}$ is again 1, while the first component of $\check{G}$ is $\check{\alpha} = \alpha^{-1} - c$. We note that $\alpha \mapsto \check{\alpha}$ is the Gauss map that appears in the continued fraction expansion of α.

In what follows, α is assumed to be the inverse golden mean. Its continued fraction expansion is $\alpha = 1/(1 + 1/(1 + \ldots))$, so α is a fixed point of the Gauss map, and $c = 1$ in the equation (2.1). As mentioned earlier, we expect to find a period of $\mathfrak{R}$ that is a multiple of 3. This leads us to consider orbits of the third iterate of $\mathfrak{R}$, which is of the form

$$\mathfrak{R}^3(P) = \bigl(\Lambda_3^{-1} G^2 F^{-1} \Lambda_3 \,, \Lambda_3 F G^{-1} F G^{-2} \Lambda_3\bigr), \qquad P = (F, G), \tag{2.2}$$

with $\Lambda_3(x, y) = \bigl(\alpha^3 x, L_3 y\bigr)$ for some suitable nonsingular 2×2 matrix L_3.

What plays an important role in our analysis are symmetry properties. A 2×2 matrix Σ will be called a reflection, if $\Sigma^2 = \mathbf{1}$ and $\det(\Sigma) = -1$. An invertible map H on $\mathbb{R} \times \mathbb{R}^2$ is said to be reversible with respect to Σ, if

$$H^{-1} = \mathcal{S} H \mathcal{S}, \qquad \mathcal{S}(x, y) = (-x, \Sigma y). \tag{2.3}$$

For a skew-product map $H = (\gamma, C)$, reversibility with respect to Σ is equivalent to the property

$$C_\circ(x)^{-1} = \Sigma C_\circ(-x) \Sigma, \qquad C_\circ(x) \stackrel{\text{def}}{=} C\bigl(x - \tfrac{\gamma}{2}\bigr). \tag{2.4}$$

The matrix-valued function $C_\circ$ defined by (2.4) will be referred to as the symmetric factor of H, even if H is not reversible. A pair $P = (F, G)$ will be called reversible if both F and G are reversible with respect to the same reflection Σ. Since the matrix Σ depends on a choice of coordinates, we will specify it only when necessary.

The following result was announced in [52]. A proof will be given in Subsection 4.3.

Theorem 2.1. *Let α be the inverse golden mean. Then $\mathfrak{R}^6$ has a reversible fixed point $P_\star = (F_\star, G_\star)$ with $F_\star = (1, B_\star)$ and $G_\star = (\alpha, A_\star)$ commuting. The factors $B_\star$ and $A_\star$ are non-constant entire functions with values in* $\mathrm{SL}(2, \mathbb{R})$. *The scaling L_6 at $P_\star$ has real eigenvalues $\mathcal{V}$ and $\mathcal{V}^{-1}$ whose sum is $2\alpha^{-3}$, up to an error less that 10^{-429},*

To be more precise, the scaling matrix L_6 mentioned in this theorem is the product of the matrix $L_3(P_\star)$ appearing in the transformation $P_\star \mapsto P = \mathfrak{R}^3(P_\star)$, and the matrix $L_3(P)$ appearing in the transformation $P \mapsto P_\star = \mathfrak{R}^3(P)$. The exact form of L_6 depends on the chosen coordinates.

As a by-product of our (computer-assisted) proof of this theorem, we have accurate bounds on the various quantities involved, as well as other numerical data. These data include approximate values for the two expanding eigenvalue μ_1 and μ_2 of $D\mathfrak{R}^6(P_\star)$.

As will be described in Section 4, the AM family for $\xi = {}^\alpha\!/_2$ is reversible, due to the fact that $x \mapsto E - \lambda\cos\bigl(2\pi x\bigr)$ is an even function. By choosing the y-scaling L_3 appropriately, reversibility (for a fixed Σ) is preserved under renormalization. So we expect Conjecture 1.1 to hold within a class of reversible pairs. The fixed point $P_\star$ described in Theorem 2.1 is associated with the fibered rotation number $\varrho = {}^1\!/_4$. In the AM family, this corresponds to the energy $E_* = 0$.

Our main goal in this paper is to prove Conjecture 1.1 in a simplified setting where the analysis can be restricted to one-parameter families. This lead us to consider maps that are anti-reversible. To be more precise, define $-(\gamma, C) = (\gamma, -C)$. Using the same notation as in (2.3), we say that H is anti-reversible with respect to Σ, if $H^{-1} = -\mathcal{S}H\mathcal{S}$. A pair (F, G) is said to be anti-reversible, if F is reversible and G anti-reversible with respect to the same reflection Σ

If we choose $\xi = {}^\alpha\!/_2 - {}^1\!/_4$, then the AM map G is anti-reversible, but only for $E = 0$, due to the fact that $x \mapsto E - \lambda\sin\bigl(2\pi x\bigr)$ is an odd function precisely when $E = 0$. So the idea is to restrict our analysis to anti-reversible pairs. By choosing the y-scaling L_3 appropriately, anti-reversibility (for a fixed Σ) is preserved under renormalization. So we expect Conjecture 1.1 to hold for one-parameter families in this restricted class, except that the RG transformation has only a single expanding direction.

Based on Theorem 2.1, we expect to find a fundamental period 6 in this case. Somewhat unexpectedly, we find a period 3.

Theorem 2.2. *Let α be the inverse golden mean. Then $\mathfrak{R}^3$ has an anti-reversible fixed point $P_* = (F_*, G_*)$ with $F_* = (1, B_*)$ and $G_* = (\alpha, A_*)$ commuting. The factors B_* and A_* are non-constant entire functions with values in* $\mathrm{SL}(2, \mathbb{R})$. *The scaling L_3 at P_* is an orthogonal reflection in $\mathbb{R}^2$ about some line (that depends on the choice of coordinates). An extension of $\mathfrak{R}_3$ to pairs that need not commute is hyperbolic, with a single expanding direction with eigenvalue $\mu_2 \geq \alpha^{-3}$.*

A proof of this theorem will be given in Subsection 5.1.

Remark 1. In the anti-reversible case, our RG transformations $\mathfrak{R}^3$ and $\mathfrak{R}_3$ include an extra step $(B,A)\mapsto(-B,-A)$. We will ignore this step here, e.g. by identifying pairs of factors up to a sign.

For the pair P_* and the eigenvalue μ_2 described in Theorem 2.2 we also have the following. Here, and in the remaining part of this paper, α always denotes the inverse golden mean, unless specified otherwise.

Theorem 2.3. *Consider the* AM *factors (1.4) with $\xi = {}^{\alpha}\!/_{2} - {}^{1}\!/_{4}$ and energy $E=0$. Denote by p_n the n-th Fibonacci number and let $q_n = p_{n+1}$. Then there exists an open disk $D\subset\mathbb{C}$ centered at the origin, such that the limits*

$$
\begin{aligned}
B_*(s,x) &= \lim_{n\to\infty} L_3^{-n} A^{*p_{3n}}\bigl(\mu_2^{-3n}s,\alpha^{3n}x\bigr)L_3^n\,,\\
A_*(s,x) &= \lim_{n\to\infty} L_3^{-n} A^{*q_{3n}}\bigl(\mu_2^{-3n}s,\alpha^{3n}x\bigr)L_3^n\,,
\end{aligned}
\tag{2.5}
$$

exist for all $s\in D$ and all $x\in\mathbb{C}$. Here, L_3 is some orthogonal reflection in the plane. The functions $(s,x)\mapsto A_*(s,x)$ and $(s,x)\mapsto B_*(s,x)$ are analytic on $D\times\mathbb{C}$, and the convergence in (2.5) is uniform on compact subsets of this domain. The family of pairs $s\mapsto \mathtt{P}_*(s)$ associated with the limit factors (2.5) is a parametrization of the local unstable manifold of $\mathfrak{R}_3$ at P_*. Furthermore, the same holds for any real-analytic family $\mathtt{P}$ *in some open neighborhood (in a suitable topology) of the* AM *family, after an initial affine re-parametrization $s\mapsto s_*+cs$ with $c\neq 0$.*

A proof of this theorem will be given in Subsection 5.2. To be more precise, our proof of Theorems 2.1, 2.2, and 2.3 is computer-assisted. This means that some estimates have been verified (rigorously) with the aid of a computer. The main steps and ideas are described in Section 8. For details we refer to the source code of our programs [56].

The main part of our analysis is carried out in a space $\mathcal{F}_\rho$ of pairs of maps $F=(1,B)$ and $G=(\alpha,A)$ whose symmetric factors $A_\circ$ and $B_\circ$ are analytic in a bounded domain $|x|<\rho_F$ and $|x|<\rho_G$, respectively. The "suitable topology" mentioned in Theorem 2.3 only compares factors on this bounded domain; so in particular, the factors need not be periodic. Entire analyticity of A_* and B_* is obtained a-posteriori from the fact that the transformation $\mathfrak{R}^3$ is analyticity-improving. (And it is not hard to see that these factors are of finite exponential type.)

To be more specific, we consider the fixed point problem for $\mathfrak{R}_3$ instead of $\mathfrak{R}^3$, where

$$
\mathfrak{R}_3(P) = \bigl(\Lambda_3^{-1}GF^{-1}G\Lambda_3\,,\Lambda_3G^{-1}FG^{-1}FG^{-1}\Lambda_3\bigr)\,. \tag{2.6}
$$

This makes no difference for commuting pairs. But for non-commuting pairs, which need to be included in our analysis, the transformation $\mathfrak{R}^3$ does not in general preserve (anti)reversibility, while $\mathfrak{R}_3$ does. After constructing a fixed point (F_*,G_*) for the transformation $\mathfrak{R}_3$, we can use (2.5) to conclude that F_* and G_* commute. Our extension of $\mathfrak{R}_3$ to nearly-commuting pairs also includes a "commutator correction" which makes this transformation contracting in the direction of non-commuting perturbations.

Hyperbolicity of $\mathfrak{R}_3$ is proved via estimates on the derivative $D\mathfrak{R}_3$ on some cylinder C_1' centered at P_*. In order to prove Theorem 2.3, we show that some RG iterate of the anti-reversible AM family defines a curve that is properly aligned with a cylinder $C_0' \subset C_1'$. This constitutes the global part of our analysis. What remains is again a purely local problem.

One of the claims in Theorem 2.3 is that the parameter value s_* for which the AM pair is attracted to the fixed point P_* is zero. This is specific to the AM family and has to be proved separately. Similarly, our guess that $\mu_2 = \alpha^{-3}$ is based on special properties of the AM family. A proof is again outside the scope of renormalization.

In our proof that $s_* = 0$, we use the fact that the Lyapunov exponent of the AM map G for a spectral energy is $L(G) = \max\{0, \log\lambda\}$. The general idea is that $q \mapsto G^q$ tends to infinity if $L(G)$ is positive. An argument along these lines shows that $s_* \le 0$. Proving that $s_* \ge 0$ turns out to be significantly harder.

A useful tool in our proof of Theorem 2.3 is a Lyapunov exponent for pairs $P = (F, G)$. This exponent $L(P)$ is defined in such a way that it agrees with $L(G)$, if $F = (1, \mathbf{1})$ and $G = (\alpha, A)$, with α the inverse golden mean. It also has the property that

$$L\bigl(\mathfrak{R}(P)\bigr) = \alpha^{-1} L(P) . \tag{2.7}$$

This shows e.g. that $L(P_*) = 0$. The equation (2.7) also suggests that $\mu_2 = \alpha^{-3}$. Unfortunately, we can only prove that $\mu_2 \ge \alpha^{-3}$.

The problem of proving $\mu_2 \le \alpha^{-3}$ is related to the question of whether L takes a positive value on the local unstable manifold of $\mathfrak{R}_3$ at P_*. Our pursuit of this question has led to some interesting observations that we shall now describe.

Given that $L(P)$ is an asymptotic quantity, it is necessary to consider the unstable manifold $\mathcal{W}^u$ globally, at least on the side where we expect $L(P)$ to be positive. Based on numerical experiments, our conjecture is that $\mathcal{W}^u$ gets attracted to a "supercritical" fixed point $P_\diamond$. In fact, all AM pairs with $\lambda > 1$ and $E = 0$ appear to be get attracted to this fixed point.

To be more specific, we have to describe an extension of $\mathfrak{R}_3$ to pairs of skew-product maps whose factors need not have determinant 1. Let $H = (\gamma, C)$. If $\det(C)$ is the constant function $x \mapsto 1$, then the inverse H^{-1} of H agrees with the quasi-inverse

$$H^\dagger = \bigl(-\alpha, C^\dagger(\,.\,-\alpha)\bigr) , \quad \text{where} \quad C^\dagger = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \quad \text{if} \quad C = \begin{bmatrix} a & b \\ c & d \end{bmatrix} . \tag{2.8}$$

Thus, we can extend the domain of $\mathfrak{R}_3$ by replacing the inverse maps in our definition (2.6) by their quasi-inverses. But we assume that the determinants are nonnegative. In our proof of Theorems 2.1 and 2.2, we use such an extension of $\mathfrak{R}_3$ for pairs whose factors have determinants close to $x \mapsto 1$. The extended transformation $\mathfrak{R}_3$ includes (as its last step) a normalization that divides each factors by the square root of its determinant. So any fixed point of $\mathfrak{R}_3$ or $\mathfrak{R}_3^2$ has factors that take values in $\mathrm{SL}(2, \mathbb{R})$.

In what follows, we allow factors that (are nonzero but) can have arbitrary nonnegative constant determinants. But our RG transformation now includes a normalization step that divides each factor by its norm. This is useful in cases where the norms would otherwise tend to infinity under iteration of $\mathfrak{R}_3$. For such an extension we find the following.

Theorem 2.4. *There exists entire functions $b_\diamond$ and $a_\diamond$ of order 1, with $x \mapsto b_\diamond(x - 1/2)$ even and $x \mapsto a_\diamond(x - \alpha/2)$ odd, such that the pair $P_\diamond = ((1, B_\diamond), (\alpha, A_\diamond))$ with*

$$B_\diamond(x) = b_\diamond(x)\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \qquad A_\diamond(x) = a_\diamond(x)\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \tag{2.9}$$

is a fixed point of $\mathfrak{R}_3$ with $L_3 = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. The zeros of $b_\diamond$ are all simple and define a non-periodic bi-infinite sequence of real numbers whose gaps take exactly three distinct values: $1/2$, α^{-1}, and $\alpha^{-1} + 1/2$. The zeros of $a_\diamond$ have an analogous property, except that the gaps only take two distinct values: $1/2$ and $\alpha^{-1} - 1/2$.

Numerically, we find that $P_\diamond$ attracts supercritical AM pairs, as well as pairs $\mathtt{P}(s)$ with $s > 0$ on the unstable manifold of P_*. Our computations covered several values of λ between $1 + 2^{-32}$ and 2, both for $\xi = \alpha/2 - 1/4$ and $\xi = \alpha/2$. To be more precise, the y-scaling has to include a rotation; otherwise the limit can be a rotated version of $P_\diamond$. We would expect similar behavior for other energies in the spectrum of H_λ^α, as well as for other quadratic irrationals α that have a periodic continued fraction expansion.

For the inverse golden mean, it should be possible to prove that the (anti)reversible AM pair with λ sufficiently large is attracted to $P_\diamond$ under iteration of $\mathfrak{R}_3$, but such an analysis would go beyond the scope of this paper. A strong-coupling fixed point for an approximate renormalization scheme has been constructed in [34].

What we will prove here is the following.

Theorem 2.5. *Let P_0 be an anti-reversible* AM *pair with coupling constant $\lambda > 1$. Consider the set of accumulation points of the sequence $n \mapsto \mathfrak{R}^{3n}(P_0)$ in the space $\mathcal{F}_\rho$ mentioned earlier. This set K_* is compact and invariant under $\mathfrak{R}_3$. Let $P = ((1, B), (\alpha, A))$ be any pair in K_*. Then A and B extend to non-constant entire functions, with $b_\circ = \operatorname{tr}(B_\circ)$ even and $a_\circ = \operatorname{tr}(A_\circ)$ odd. Furthermore, $B(x) = 0$ wherever $b_\diamond(x) = 0$, and $A(x) = 0$ wherever $a_\diamond(x) = 0$.*

A proof of Theorems 2.4 and 2.5 will be given in Section 7.

We have not investigated the asymptotic behavior of subcritical pairs, like the AM pairs for $\lambda < 1$. Results on almost-reducibility [44] suggest that such pairs converge to some $\mathfrak{R}$-invariant set that consists of pairs whose factors are constant. (The y-scaling L_1 could be different from what we use here.) The action of $\mathfrak{R}$ on pairs with constant factors is trivial. In particular, it is easy to find periodic orbits for any rational fibered rotation number ϱ. Whether or not AM pairs with non-small positive coupling constant $\lambda < 1$ and rational fibered rotation number ϱ converge to such a "subcritical" fixed point (of $\mathfrak{R}^\ell$ for some ℓ) is a global question and not easy to answer.

3. The RG transformation for anti-reversible pairs

The main goal in this section is to properly formulate the fixed point problem considered in Theorem 2.2. Our RG analysis of anti-reversible pairs will be continued in Section 5, after having covered the reversible case in Section 4.

3.1. Some basic facts and identities

Let $P=(F,G)$ be a pair of skew-product maps $F=(1,B)$ and $G=(\alpha,A)$ whose factors B and A take values in $\mathrm{GL}(2,\mathbb{R})$ and have positive determinants. Then the renormalized pair $\tilde{P}=\mathfrak{R}_3(P)$ is given by

$$\tilde{P}=\bigl(\tilde{F},\tilde{G}\bigr)\,,\qquad \tilde{F}=\Lambda_3^{-1}\hat{F}\Lambda_3\,,\qquad \tilde{G}=\Lambda_3^{-1}\hat{G}\Lambda_3\,,\tag{3.1}$$

where

$$\hat{F}=GF^\dagger G\,,\qquad \hat{G}=G^\dagger FG^\dagger FG^\dagger\,.\tag{3.2}$$

Here, $F^\dagger$ and $G^\dagger$ denote the quasi-inverses of F and G, respectively, as defined in (2.8). The first component of $\hat{F}$ is $2\alpha-1=\alpha^3$. So after scaling by α^3, the first component of $\tilde{F}$ is again 1. Similarly, the first component of $\hat{G}$ is $2-3\alpha=\alpha^4$. So after scaling by α^3, the first component of $\tilde{G}$ is again α. The symmetric factor $\hat{B}$ of $\hat{F}$ is given by

$$\hat{B}_\circ(x)=A_\circ\bigl(\tfrac{\alpha-1}{2}+x\bigr)B_\circ(x)^\dagger A_\circ\bigl(\tfrac{1-\alpha}{2}+x\bigr)\,,\tag{3.3}$$

and for the symmetric factor $\hat{A}$ of $\hat{G}$ we obtain

$$\hat{A}_\circ(x)=A_\circ\bigl((1-\alpha)+x\bigr)^\dagger B_\circ\bigl(\tfrac{1-\alpha}{2}+x\bigr)A_\circ(x)^\dagger B_\circ\bigl(\tfrac{\alpha-1}{2}+x\bigr)A_\circ\bigl((\alpha-1)+x\bigr)^\dagger\,.\tag{3.4}$$

The symmetric factors associated with $\tilde{P}=\mathfrak{R}_3(P)$ are now obtained via scaling:

$$\tilde{B}_\circ(x)=L_3^{-1}\hat{B}_\circ\bigl(\alpha^3x\bigr)L_3\,,\qquad \tilde{A}_\circ(x)=L_3^{-1}\hat{A}_\circ\bigl(\alpha^3x\bigr)L_3\,.\tag{3.5}$$

Notice that such a relation is obvious for the regular factors. But it holds for the symmetric factors as well, as a short computation shows. Our choice for the y-scaling matrices L_3 will be described in Subsection 3.2.

Next, let us consider some consequences of (anti)reversibility. To this end, and for reference later on, define

$$J=\begin{bmatrix}0&1\\-1&0\end{bmatrix},\qquad S=\begin{bmatrix}1&0\\0&-1\end{bmatrix},\qquad M=2^{-1/2}\begin{bmatrix}1&1\\1&-1\end{bmatrix}.\tag{3.6}$$

With the exception of Section 4, (anti)reversibility in this paper is defined with respect to the reflection $\Sigma=iJ$. Notice that conjugacy by iJ keeps real matrices real.

Assume now that $F=(1,B)$ is reversible and $G=(\alpha,A)$ anti-reversible, both with respect to $\Sigma=iJ$. A short computation shows that this condition is equivalent to

$$B_\circ(x)^\top=B_\circ(-x)\,,\qquad A_\circ(x)^\top=-A_\circ(-x)\,.\tag{3.7}$$

Here, $C^\top$ denotes the transpose of a matrix C.

What makes skew-products over irrational rotations difficult to deal with is that products $A^{*q}(x)$ with large q can vary vastly in size, as a function of x. If the Lyapunov exponent $L=L(G)$ is positive, then $A^{*q}(x)$ grows asymptotically like e^{qL} for typical values of x. Particularly large factors can be obtained via the identity

$$(-1)^m A_\circ^{*2m}(iy)=U(iy)^*U(iy)\,,\tag{3.8}$$

where $U(x) = A_\circ((m - {}^1\!/_2)\alpha + x) \cdots A_\circ({}^\alpha\!/_2 + x)$, and where $U^* = \overline{U}^\top$ denotes the adjoint of U. So in particular, $(-1)^m A_\circ^{*2m}(iy)$ is a positive matrix for $y \in \mathbb{R}$. This fact will be used in several of our proofs.

On the other hand, products with many factors can be of order 1 in size. Consider the case where $\det(A) = 1$. Using that $A_\circ(-x) = -J^{-1}A_\circ(x)^{-1}J$, we have

$$(-1)^m A_\circ^{*(2m+1)}(x) = V(x)A_\circ(x)J^{-1}V(-x)^{-1}J\,, \tag{3.9}$$

where $V(x) = A_\circ(m\alpha + x) \cdots A_\circ(\alpha + x)$. If $A_\circ(0) = -J$, then this implies that

$$(-1)^m A_\circ^{*(2m+1)}(0) = -J\,. \tag{3.10}$$

This applies e.g. to the AM map with $\xi = {}^\alpha\!/_2 - {}^1\!/_4$ and $E = 0$. And it is independent of the value of λ. So for $\lambda > 1$, sub-products that appear in A^q that are of the form (3.10) are much smaller than sub-products of the form (3.8) for $y = 0$. This is the mechanism that produces the zeros described in Theorem 2.5.

In the remaining part of this paper, we consider the AM maps with respect to the basis defined by the column vectors of the matrix M given in (3.6). In this representation, the symmetric factor of the anti-reversible AM map is given by

$$A_\circ = \begin{bmatrix} t_\circ & t_\circ + 1 \\ t_\circ - 1 & t_\circ \end{bmatrix}, \qquad t_\circ(x) = \lambda \sin(2\pi x)\,. \tag{3.11}$$

3.2. Scaling and normalization

As mentioned earlier, we choose L_3 to be a reflection matrix. In the coordinates considered here,

$$L_3 = L(\vartheta) \stackrel{\text{def}}{=} \begin{bmatrix} \cos(\vartheta + \pi/4) & -\sin(\vartheta + \pi/4) \\ -\sin(\vartheta + \pi/4) & -\cos(\vartheta + \pi/4) \end{bmatrix}. \tag{3.12}$$

Notice that $L(\vartheta) = L(0)e^{-\vartheta J}$, where $e^{-\vartheta J}$ is a rotation by ϑ. Notice also that $L_3^2 = \mathbf{1}$. So L_3 drops out in a fixed point equation for $\mathfrak{R}_3^2$. But if $P = (F, G)$ is a fixed point of $\mathfrak{R}_3^2$, then a conjugacy by any rotation yields another fixed point.

The goal is to get uniqueness by choosing $\vartheta = \vartheta(P)$ in such a way that $\tilde{P} = \mathfrak{R}_3(P)$ satisfies a suitable normalization condition. With a y-scaling L_3 of the form (3.12), the symmetric factor of $\tilde{F} = \Lambda_3^{-1}\hat{F}\Lambda_3$ is given by

$$\tilde{B}_\circ(0) = e^{\vartheta J}\begin{bmatrix} a & u \\ v & d \end{bmatrix}e^{-\vartheta J}, \qquad \begin{bmatrix} a & u \\ v & d \end{bmatrix} \stackrel{\text{def}}{=} -L(0)\hat{B}_\circ(0)L(0)\,, \tag{3.13}$$

where $\hat{B}_\circ$ is as described in (3.3). The negative sign on the right hand side of this equation is due to the step $(B, A) \mapsto (-B, -A)$ mentioned in Remark 1. As a normalization condition, we impose that the two entries on the main diagonal of $\tilde{B}_\circ(0)$ agree. A straightforward computation shows that this determines $\mathfrak{c} = \cos(2\vartheta)$ and $\mathfrak{s} = \sin(2\vartheta)$ as follows:

$$\mathfrak{s} = \frac{a - d}{q}\,, \qquad \mathfrak{c} = -\frac{u + v}{q}\,, \qquad q = \sqrt{(u + v)^2 + (a - d)^2}\,. \tag{3.14}$$

As tedious as such computations may be, explicit expressions like (3.14) are needed in a computer-assisted proof that is by nature highly constructive. In order to get explicit expressions for the derivative $D\mathfrak{R}_3(P)\dot{P}$, it is convenient to consider a family of pairs P that depend differentiably on a parameter. Then the quantities a, u, v, d defined by (3.13) depend differentiably on the parameter as well. Using the "dot notation" for derivatives with respect to the parameter, the derivatives of $\mathfrak{s}$ and $\mathfrak{c}$ are given by

$$\dot{\mathfrak{s}} = \frac{\mathfrak{c}}{q}\bigl[\mathfrak{c}\bigl(\dot{a}-\dot{d}\bigr)+\mathfrak{s}(\dot{u}+\dot{v})\bigr]\,, \qquad \dot{\mathfrak{c}} = -\frac{\mathfrak{s}}{q}\bigl[\mathfrak{c}\bigl(\dot{a}-\dot{d}\bigr)+\mathfrak{s}(\dot{u}+\dot{v})\bigr]\,. \tag{3.15}$$

At this point we have defined the "basic" version of our RG transformation $\mathfrak{R}_3$ for skew-product pairs $P = (F, G)$. We are not assuming that $F = (1, B)$ and $G = (\alpha, A)$ commute, nor that A and B have determinant 1. But the transformation does not behave as desired for non-commuting pairs or for factors that have determinants $\neq 1$. Denote this basic version by $\mathcal{R}_3$. Our extended version of $\mathfrak{R}_3$ is defined as

$$\mathfrak{R}_3 = \mathfrak{N}\circ\mathcal{R}_3\circ\mathfrak{C}\,, \tag{3.16}$$

where $\mathfrak{C}$ is a "commutator correction" that will be defined later, and where $\mathfrak{N}$ performs a re-normalization of determinants.

To normalize determinants, we simply choose

$$\mathfrak{N}\bigl((\gamma, C)\bigr) = \bigl(\gamma, \mathcal{N}(C)\bigr)\,, \qquad \mathcal{N}(C) = [\det(C)]^{-1/2}C\,. \tag{3.17}$$

If the determinant of C is close to 1, then (3.17) is well-defined and $\mathcal{N}(C)$ has determinant 1. We note that, if $H = (\gamma, C)$ is (anti)reversible, then $\det(C)$ is an even function, so $\mathfrak{N}(H)$ is still (anti)reversible. For a pair $P = (F, G)$ we define $\mathfrak{N}$ component-wise.

For estimates of derivatives $D\mathfrak{R}_3(P)\dot{P}$, we use that the derivative of $\mathcal{N}$ at $C = \left[\begin{smallmatrix} a & u \\ v & d\end{smallmatrix}\right]$ is given by

$$D\mathcal{N}(C)\dot{C} = \det(C)^{-1/2}\dot{C} - \tfrac{1}{2}\det(C)^{-3/2}\bigl[a\dot{d}+d\dot{a}-u\dot{v}-v\dot{u}\bigr]C\,. \tag{3.18}$$

3.3. Commutators

The linearization of the basic transformation $\mathcal{R}_3$ at the fixed point P_* can have non-contracting directions that are associated with non-commuting perturbations of P_*. Formally, one can see that $D\mathcal{R}_3(P_*)$ must have an eigenvalue -1. And numerically, another eigenvalue is the number $\mathcal{V} = 8.3524100320\ldots$ that appears in Theorem 2.1. The goal is to eliminate these two eigenvalues. This will be done in the next section.

First we need some generalities. Consider the commutator $\Theta = FG(GF)^{-1}$ for a pair $P = (F, G)$. A straightforward computation shows that the commutator for the renormalized pair $\tilde{P} = \bigl(\tilde{F}, \tilde{G}\bigr)$ is given by

$$\tilde{\Theta} = (G\Lambda_3)^{-1}\Theta^{-1}(G\Lambda_3)\,. \tag{3.19}$$

If we write $\Theta = (0, C)$ and $\tilde{\Theta} = \bigl(0, \tilde{C}\bigr)$, then

$$\tilde{C}(x) = L(\vartheta)^{-1} A\bigl(\alpha^3 x\bigr)^{-1} C\bigl(\alpha^3 x + \alpha\bigr)^{-1} A\bigl(\alpha^3 x\bigr) L(\vartheta)\,. \tag{3.20}$$

Consider the change of variables $x = \frac{1+\alpha}{2} + z$ and define

$$\mathcal{C}(P, z) = C\bigl(\tfrac{1+\alpha}{2} + z\bigr)\,, \qquad \mathcal{A}(P, z) = A_0\bigl(\tfrac{1}{2} + \alpha^3 z\bigr) L(\vartheta_P)\,. \tag{3.21}$$

Then the equation (3.20) becomes

$$\mathcal{C}\bigl(\tilde{P}, z\bigr) = \mathcal{A}(P, z)^{-1} \mathcal{C}\bigl(P, \alpha^3 z\bigr)^{-1} \mathcal{A}(P, z)\,. \tag{3.22}$$

From this equation one can see that the eigenvalues of $\Theta \mapsto \tilde{\Theta}$ at $\Theta = \mathrm{I}$ are determined by the behavior of $\mathcal{C}(P, z)$ near $z = 0$.

Thus, consider $\mathcal{C}(P) = \mathcal{C}(P, 0) = C\bigl(\frac{1+\alpha}{2}\bigr)$. An explicit computation shows that

$$\mathcal{C}(P) = XY^{-1}\,, \qquad X = B_0\bigl(\tfrac{\alpha}{2}\bigr) A_0\bigl(-\tfrac{1}{2}\bigr)\,, \qquad Y = A_0\bigl(\tfrac{1}{2}\bigr) B_0\bigl(-\tfrac{\alpha}{2}\bigr)\,. \tag{3.23}$$

Assume now that G is anti-reversible with respect to $\Sigma = iJ$. Then $JXJ = Y^{-1}$. So X, Y, and XY^{-1} are of the form

$$X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad Y = -\begin{bmatrix} a & c \\ b & d \end{bmatrix}, \qquad XY^{-1} = \begin{bmatrix} -1 + b(b-c) & -a(b-c) \\ d(b-c) & -1 - c(b-c) \end{bmatrix}, \tag{3.24}$$

with $ad - bc = 1$. If XY^{-1} is the identity matrix, then $X = Y = J$. So in our applications, the matrix elements a and d are close to zero.

3.4. Commutator corrections

The commutator correction map $\mathfrak{C}$ is the first step in our RG transformation (3.16). The goal is for $P' = \mathfrak{C}(P)$ to satisfy $\mathcal{C}(P') = \mathbf{1}$. We define $\mathfrak{C}$ as a composition of three maps. To simplify notation, each of these maps will be denoted by $P \mapsto P'$.

Step 1. Here we replace the symmetric factor $A_\circ$ of G by

$$A'_\circ = RA_\circ R\,, \qquad R = \begin{bmatrix} \rho & r \\ r & \rho \end{bmatrix}, \qquad \rho^2 = 1 + r^2\,, \tag{3.25}$$

while keeping $B'_\circ = B_\circ$. The goal is to choose r in such a way that $\operatorname{tr}(X') = 0$. Notice that R is reversible, in the sense that $J^{-1}RJ = R^{-1}$. Since G is anti-reversible, this guarantees that the map G' is anti-reversible as well. Write

$$A_\circ\bigl(-\tfrac{1}{2}\bigr) = \begin{bmatrix} t_{\scriptscriptstyle A} + s_{\scriptscriptstyle A} & u_{\scriptscriptstyle A} \\ v_{\scriptscriptstyle A} & t_{\scriptscriptstyle A} - s_{\scriptscriptstyle A} \end{bmatrix}, \qquad B_\circ\bigl(\tfrac{\alpha}{2}\bigr) = \begin{bmatrix} t_{\scriptscriptstyle B} + s_{\scriptscriptstyle B} & u_{\scriptscriptstyle B} \\ v_{\scriptscriptstyle B} & t_{\scriptscriptstyle B} - s_{\scriptscriptstyle B} \end{bmatrix}, \tag{3.26}$$

and define

$$\begin{aligned} \varepsilon &= \operatorname{tr}(X_\circ)\,, \\ \tau &= 4t_{\scriptscriptstyle A} t_{\scriptscriptstyle B} + (u_{\scriptscriptstyle A} + v_{\scriptscriptstyle A})(u_{\scriptscriptstyle B} + v_{\scriptscriptstyle B})\,, \\ \sigma &= (u_{\scriptscriptstyle A} + v_{\scriptscriptstyle A}) 2t_{\scriptscriptstyle B} + 2t_{\scriptscriptstyle A}(u_{\scriptscriptstyle B} + v_{\scriptscriptstyle B})\,. \end{aligned} \tag{3.27}$$

A tedious but trivial computation shows that $\mathrm{tr}(X')=0$, if we choose

$$r=\frac{-\varepsilon}{\sqrt{\frac{1}{2}\bigl(\sigma^2-2\varepsilon\tau\bigr)+\frac{1}{2}\sqrt{\bigl(\sigma^2-2\varepsilon\tau\bigr)^2-4\bigl(\tau^2-\sigma^2\bigr)\varepsilon^2}}}\,. \tag{3.28}$$

And for the derivative with respect to a parameter, we obtain

$$\dot r=-\frac{\dot\varepsilon+r(\rho\dot\sigma+r\dot\tau)}{\varphi}\,,\qquad \varphi=\sigma\rho+r\bigl(2\tau+r\sigma\rho^{-1}\bigr)\,. \tag{3.29}$$

<u>Step 2</u>. Assume now that $\mathrm{tr}(X)=0$. The second correction $P\mapsto P'$ is defined via a transformation

$$A'_\circ=\mathcal{K}A_\circ\mathcal{K}\,,\qquad B'_\circ=\mathcal{K}^{-1}B_\circ\mathcal{K}^{-1}\,,\qquad \mathcal{K}=\begin{bmatrix}\kappa^{1/2}&0\\0&\kappa^{-1/2}\end{bmatrix}\,, \tag{3.30}$$

and the goal is to have

$$X'=\begin{bmatrix}s&w\\-w&-s\end{bmatrix}\qquad\text{if}\quad X=\begin{bmatrix}s&b\\c&-s\end{bmatrix}. \tag{3.31}$$

Recall that $X\approx J$ in our applications, so that $s\approx 0$, $b\approx 1$, and $c\approx -1$. Clearly $X'=\mathcal{K}^{-1}X\mathcal{K}$ is of the desired form, if we choose

$$\kappa=\sqrt{-b/c}\,. \tag{3.32}$$

Then $w=\sqrt{-bc}$. The derivative with respect to a parameter is trivial, so we will not give it here.

<u>Step 3</u>. Assume now that $X=\left[\begin{smallmatrix}s&w\\-w&-s\end{smallmatrix}\right]$. The third correction $P\mapsto P'$ is of the form

$$A'_\circ=RA_\circ R\,,\qquad B'_\circ=R^{-1}B_\circ R^{-1}\,,\qquad R=\begin{bmatrix}\rho&r\\r&\rho\end{bmatrix}\,, \tag{3.33}$$

with $\rho^2=1+r^2$. The goal is to determine r in such a way that $X'=R^{-1}XR$ is equal to J. An explicit computation show that this is achieved with

$$r=\frac{-s}{\sqrt{2\bigl(w^2-s^2\bigr)+2w\sqrt{w^2-s^2}}}\,. \tag{3.34}$$

For the derivative with respect to a parameter, we find that

$$\dot r=-\frac{\frac{1}{2}\dot s+r(\dot w\rho+\dot s r)}{\psi}\,,\qquad \psi=w\rho+r\bigl(2s+w\rho^{-1}r\bigr)\,. \tag{3.35}$$

3.5. The fixed point problem

Consider now the transformation $\mathfrak{R}_3$ defined by (3.16). Our first goal is to prove that $\mathfrak{R}_3$ has a fixed point P_* that has potentially the properties described in Theorem 2.2. As is common in many computer-assisted proofs, we associate with the given transformation $\mathfrak{R}_3$ a quasi-Newton map $\mathfrak{M}$ that we hope to be a contraction near some approximate fixed point $\bar{P}$. Picking an approximate inverse $\mathrm{I}-M$ of $\mathrm{I}-D\mathfrak{R}_3\bigl(\bar{P}\bigr)$, we define

$$\mathfrak{M}(p)=\mathfrak{R}_3\bigl(\bar{P}+(\mathrm{I}-M)p\bigr)-\bar{P}+Mp\,. \tag{3.36}$$

Here, the sum of map-pairs is defined component-wise, and $c_1(\gamma,C_1)+c_2(\gamma,C_2)$ is defined as $(\gamma,c_1C_1+c_2C_2)$. Notice that, if p is a fixed point of $\mathfrak{M}$, then $P=\bar{P}+(\mathrm{I}-M)p$ is a fixed point of $\mathfrak{R}_3$.

The following function spaces have already been used in [50]. Given $\rho>0$, denote by $\mathcal{G}_\rho$ the space of all real analytic functions g on $(-\rho,\rho)$ that have a finite norm

$$\|g\|_\rho=\sum_{n=0}^{\infty}|g_n|\rho^n\,,\qquad g(x)=\sum_{n=0}^{\infty}g_nx^n\,. \tag{3.37}$$

Notice that every function $g\in\mathcal{G}_\rho$ extends analytically to the complex disk $|x|<\rho$. Furthermore, $\mathcal{G}_\rho$ is a Banach algebra under the pointwise product of functions.

The space of matrix functions

$$C_\circ=\begin{bmatrix} t_\circ+s_\circ & u_\circ\\ v_\circ & t_\circ-s_\circ\end{bmatrix}, \tag{3.38}$$

with $t_\circ$, $u_\circ$, $v_\circ$, and $s_\circ$ belonging to $\mathcal{G}_\rho$ will be denoted by $\mathcal{G}_\rho^4$. The norm of $C_\circ\in\mathcal{G}_\rho^4$ is defined as $\|C_\circ\|_\rho=\|t_\circ\|_\rho+\|u_\circ\|_\rho+\|v_\circ\|_\rho+\|s_\circ\|_\rho$.

Given a pair $\rho=(\rho_{\mathrm{F}},\rho_{\mathrm{G}})$ of positive real numbers, we define $\mathcal{F}_\rho$ to be the vector space of all pairs $\mathcal{P}=(B_\circ,A_\circ)$ in $\mathcal{G}_{\rho_{\mathrm{F}}}^4\times\mathcal{G}_{\rho_{\mathrm{G}}}^4$, equipped with the norm $\|\mathcal{P}\|_\rho=\|B_\circ\|_{\rho_{\mathrm{F}}}+\|A_\circ\|_{\rho_{\mathrm{G}}}$. The subspace of pairs $\mathcal{P}\in\mathcal{F}_\rho$ that satisfy the (anti)reversibility conditions (3.7) will be denoted by $\mathcal{F}_\rho^r$.

For simplicity, and when no confusion can arise, we will identify a skew-product map $H=(\gamma,C)$ with its symmetric factor $C_\circ$. Referring to the representation (3.38), we note that H is reversible with respect to iJ, if and only if the functions $t_\circ$ and $s_\circ$ are even, while $v_\circ(-x)=u_\circ(x)$. Or $C_\circ$ is anti-reversible, precisely if $t_\circ$ and $s_\circ$ are odd, while $v_\circ(x)=-u_\circ(-x)$.

In our applications, we always choose $\varrho_{\mathrm{F}}\le\varrho_{\mathrm{G}}$. Under these conditions, (3.3) and (3.4) show that $\mathfrak{R}_3$ is well-defined on $\mathcal{F}_\rho$ if

$$\tfrac{1}{2}<\rho_{\mathrm{G}}\le\rho_{\mathrm{F}}<\alpha^{-3}\rho_{\mathrm{G}}-\tfrac{1}{2}\alpha^{-1}\,. \tag{3.39}$$

To be more precise, the conditions needed in the normalization step $\mathfrak{N}$ and for the commutator correction $\mathfrak{C}$ (all of which represent ad-hoc choices) also require some mild non-degeneracy properties. We note that the domain conditions for the transformation $\mathfrak{R}$ are more restrictive than the conditions (3.39) for $\mathfrak{R}_3$. But both are satisfied with comfortable margins in the case $\rho_{\rm F}=2$ and $\rho_{\rm G}={}^{11}\!/\!_8$ considered below.

For reference later on, we note that the transformation $\mathfrak{R}_3$ is compact, due to the analyticity-improving property of $\mathfrak{R}_3$. To be more precise, the transformation $(B_\circ,A_\circ)\mapsto\bigl(\tilde B_\circ,\tilde A_\circ\bigr)$ defined by the equations (3.3), (3.4), and (3.5) maps bounded sets in $\mathcal{F}_\rho$ to bounded sets in $\mathcal{F}_{\rho'}$, for some choice of $\rho'_{\rm F}>\rho_{\rm F}$ and $\rho'_{\rm G}>\rho_{\rm G}$. And the inclusion map from $\mathcal{F}_{\rho'}$ into $\mathcal{F}_\rho$ is compact.

Lemma 3.1. *Let $\rho=(2,{}^{11}\!/\!_8)$. Then there exist a pair $\bar P$ in $\mathcal{F}_\rho^r$, a bounded linear operator M on $\mathcal{F}_\rho^r$, and positive constants ε,K,δ satisfying $\varepsilon+K\delta<\delta$, such that the transformation $\mathfrak{M}$ defined by (3.36) is analytic in B_δ and satisfies*

$$\|\mathfrak{M}(0)\|_\rho\le\varepsilon\,,\qquad \|D\mathfrak{M}(p)\|_\rho\le K\,,\qquad p\in B_\delta\,, \tag{3.40}$$

where B_δ denotes the open ball of radius δ in $\mathcal{F}_\rho^r$, centered at the origin. Every pair $p\in B_\delta$ has the following properties. The matrix components of $P=\bar P+(\mathrm{I}-M)p$ are non-constant and satisfy the bound $\bigl\|P-\bar P\bigr\|_\rho<10^{-450}$. Furthermore, the angle $\vartheta=\vartheta(P)$ satisfies $\sin(2\vartheta)=-0.01760801\ldots$

Our proof of this lemma is computer-assisted and will be described in Section 8. These estimates will be used in Subsection 5.1 to give a proof of Theorem 2.2.

Remark 2. The angle ϑ mentioned in Lemma 3.1 depends on the choice of coordinates. So it seems to say something about the AM model, but it is not clear what. In this context, we note that the change of coordinates M which yields AM factors of the form (3.11) achieves nothing useful in the case $\xi={}^{\alpha}\!/\!_2-{}^{1}\!/\!_4$ considered here. It was chosen since it diagonalizes the y-scaling in the reversible case $\xi={}^{\alpha}\!/\!_2$.

4. The RG transformation for reversible pairs

The goal here is to reduce the proof of Theorem 2.1 to technical estimates similar to those in Lemma 3.1. An extra step is necessary to prove that the pair $P_\star$ commutes.

4.1. Scaling and normalization

We work in a basis where the AM factor $A_\circ$ for $\xi={}^{\alpha}\!/\!_2$ and $E=0$ takes the form (3.11), with the sine replaced by a cosine. The corresponding AM map G is reversible with respect to $\Sigma=S$, with S as defined in (3.6). So throughout this section, we restrict to pairs that are reversible with respect to $\Sigma=S$. Referring to (3.38), reversibility of $H=(\gamma,C)$ is equivalent to the functions $t_\circ,u_\circ,v_\circ$ being even and $s_\circ$ odd.

The matrix L_3 that enters the definition $\Lambda_3(x,y)=\bigl(\alpha^3x,L_3y\bigr)$ of the scaling used for $\mathfrak{R}^3$ and $\mathfrak{R}_3$ is taken to be of the form

$$L_3=Se^{\sigma_3S}=\begin{bmatrix}e^{\sigma_3}&0\\0&-e^{-\sigma_3}\end{bmatrix}, \tag{4.1}$$

with $\sigma_3=\sigma_3(P)$ depending on the pair P being renormalized. Notice that L_3 commutes with S, so conjugacy by Λ_3 preserves reversibility.

Instead of $\mathfrak{R}^6$, we first consider the transformation

$$\mathfrak{R}_6 = \mathfrak{R}_3^2 \circ \mathfrak{C}\,, \qquad \mathfrak{R}_3 = \mathfrak{N} \circ \mathcal{R}_3\,, \tag{4.2}$$

where $\mathcal{R}_3$ is the "basic" RG transformation $P \mapsto \tilde{P}$ defined by the equations (3.1) and (3.2). The transformation $\mathfrak{N}$ re-normalizes determinants, as described after (3.17). The transformation $\mathfrak{C}$ is a commutator correction that will be described below.

With a y-scaling L_3 of the form (4.1), the symmetric factor of $\tilde{F} = \Lambda_3^{-1}\hat{F}\Lambda_3$ is given by

$$\tilde{B}_\circ(0) = \Lambda_3^{-1}\hat{B}_\circ(0)\Lambda_3 = \begin{bmatrix} a & -e^{-2\sigma_3}u \\ -e^{2\sigma_3}v & d \end{bmatrix}\,, \qquad \begin{bmatrix} a & u \\ v & d \end{bmatrix} \stackrel{\text{def}}{=} \hat{B}_\circ(0)\,, \tag{4.3}$$

where $\hat{B}_\circ$ is as described in (3.3). We determine $\sigma_3 = \sigma_3(P)$ is such a way that the off-diagonal elements of $\tilde{B}_\circ(0)$ are equal in modulus. In other words, $e^{-2\sigma_3}|u| = e^{2\sigma_3}|v|$. Unless $uv = 0$, which does not occur in the cases considered, this trivially determines the scaling exponent $\sigma_3(P)$.

4.2. Commutator correction

Unlike in the anti-reversible case, the largest eigenvalue of $\mathfrak{R}_3^2(P_\star)$ in the non-commuting direction appears to be 1. Our goal here is to eliminate this eigenvalue. One reason is that an eigenvalue 1 makes a quasi-Newton map ill-defined. Another reason is that the correction will be needed to prove that the pair $P_\star$ is in fact commuting.

The commutator for $P = (F, G)$ at $x = \frac{1+\alpha}{2}$ is again given by the equation (3.23). By reversibility, we have $SXS = Y^{-1}$. So X, Y, and XY^{-1} are of the form

$$X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\,, \qquad Y = \begin{bmatrix} d & b \\ c & a \end{bmatrix}\,, \qquad XY^{-1} = \mathbf{1} + (a-d)\begin{bmatrix} a & -b \\ c & -d \end{bmatrix}\,, \tag{4.4}$$

with $ad - bc = 1$. In particular $\operatorname{tr}\bigl(XY^{-1}\bigr) = 1 + (a-d)^2$. So reversibility implies that the trace of the commutator does not change to first order. This motivates the following.

Consider a commutator correction $\mathfrak{C} : P \mapsto P'$ of the form

$$A'_\circ = RA_\circ R\,, \qquad B'_\circ = R^{-1}B_0R^{-1}\,, \qquad R = \begin{bmatrix} \rho & r \\ r & \rho \end{bmatrix}\,, \tag{4.5}$$

with $\rho^2 = 1 + r^2$. Then $\mathfrak{C}(P') = X'Y'^{-1}$, with $X' = R^{-1}XR$ and $Y' = RYR^{-1}$. Ideally, we can find r in such away that $X' = Y'$, or equivalently, that

$$Y = R^{-2}XR^2\,. \tag{4.6}$$

As it tuns out, this can be achieved not only to first order, but exactly, by choosing

$$r = \frac{q}{2}\left(\frac{2}{(1-q^2)^{1/2} + 1 - q^2}\right)^{1/2}\,, \qquad q = -\frac{a-d}{c-b}\,. \tag{4.7}$$

This completes the definition (4.2) of the transformation $\mathfrak{R}_6$. For estimates of the derivative $D\mathfrak{R}_6(P)\dot{P}$ we use that

$$\dot{r} = \frac{\dot{q}}{2}\cdot\frac{\rho}{1-q^2}\,, \qquad \dot{q} = \frac{(\dot{a}-\dot{d}) - q(\dot{c}-\dot{b})}{c-b}\,. \tag{4.8}$$

4.3. Proof of Theorem 2.1

In this subsection we give a proof of Theorem 2.1 based on estimates that have been verified with the aid of a computer. We start by solving the fixed point equation for $\mathfrak{R}_6$. To this end, we use again a quasi-Newton map of the type (3.36), namely

$$\mathfrak{M}(p) = \mathfrak{R}_6\bigl(\bar{P} + (\mathrm{I} - M)p\bigr) - \bar{P} + Mp\,, \tag{4.9}$$

where $\bar{P}$ is an approximate fixed point of $\mathfrak{R}_6$, and where $\mathrm{I} - M$ is an approximation for the inverse of $\mathrm{I} - D\mathfrak{R}_3\bigl(\bar{P}\bigr)$. The relevant function spaces are the spaces $\mathcal{G}_\rho$ and $\mathcal{F}_\rho$ defined in Subsection 3.5. But $\mathcal{F}_\rho^r$ now denotes the subspace of $\mathcal{F}_\rho$ of pairs that are reversible with respect to S.

Lemma 4.1. *Let $\rho = (3, 2)$. Then there exist a pair $\bar{P}$ in $\mathcal{F}_\rho^r$, a bounded linear operator M on $\mathcal{F}_\rho^r$, and positive constants ε, K, δ satisfying $\varepsilon + K\delta < \delta$, such that the transformation $\mathfrak{M}$ defined by (4.9) is analytic in B_δ and satisfies*

$$\|\mathfrak{M}(0)\|_\rho \le \varepsilon\,, \qquad \|D\mathfrak{M}(p)\|_\rho \le K\,, \qquad p \in B_\delta\,, \tag{4.10}$$

where B_δ denotes the open ball of radius δ in $\mathcal{F}_\rho^r$, centered at the origin. Every pair $p \in B_\delta$ has the following properties. The matrix components of $P = \bar{P} + (\mathrm{I} - M)p$ are non-constant and satisfy the bound $\bigl\|P - \bar{P}\bigr\|_\rho < 10^{-439}$. Furthermore, the six-step scaling factor $\mathcal{V} = e^{\sigma_6(P)}$ satisfies the bound described in Theorem 2.1.

Our proof of this lemma is computer-assisted and will be described in Section 8.

By the contraction mapping theorem, Lemma 4.1 guarantees the existence of a fixed point $p_\star \in B_\delta$ for $\mathfrak{M}$ and thus a fixed point $P_\star = \bar{P} + (\mathrm{I} - M)p_\star$ for $\mathfrak{R}_6$. The symmetric factors for $F_\star$ and $G_\star$ are analytic in the disks $|x| < \rho_{\mathrm{F}}$ and $|x| < \rho_{\mathrm{G}}$, respectively. A trivial computation, using the expressions (3.3) and (3.4) for the symmetric factors of $\hat{F} = GF^\dagger G$ and $\hat{G} = G^\dagger FG^\dagger FG^\dagger$, respectively, shows that radii of the domains of analyticity increase with each iteration of $\mathfrak{R}_3$ by a factor larger than 1. The factor approaches α^{-3} as the number of iterations increases. This shows that $B_\star$ and $A_\star$ extend to entire functions.

What remains to be proved is that the components $F_\star$ and $G_\star$ of the pair $P_\star$ commute. To this end, consider the commutator factor $\mathcal{C}(P, z)$ defined by (3.21). It admits a representation (3.22), with

$$\mathcal{A}(P, z) = A_0\bigl(\tfrac{1}{2} + \alpha^3 z\bigr)Se^{\sigma_3(P)S}\,. \tag{4.11}$$

Let $P_3 = \mathfrak{R}_3(P)$ and $P_6 = \mathfrak{R}_3(P_3)$. Applying the identity (3.22) twice, we obtain

$$\mathcal{C}(P_6, z) = \mathcal{A}_2(P, z)^{-1}\mathcal{C}\bigl(P, \alpha^6 z\bigr)\mathcal{A}_2(P, z)\,, \tag{4.12}$$

where

$$\mathcal{A}_2(P, z) = \mathcal{A}\bigl(P, \alpha^3 z\bigr)\mathcal{A}(P_3, z)\,. \tag{4.13}$$

Let $\mathcal{C}(P) = \mathcal{C}(P, 0)$.

Consider now the pair $P = \mathfrak{C}(P_\star)$. Then $\mathcal{C}(P)$ is the identity matrix. This follows from our definition of the commutator correction $\mathfrak{C}$. Given that $P_\star$ is a fixed point of $\mathfrak{R}_6 = \mathfrak{R}_3^2 \circ \mathfrak{C}$ and thus $P_6 = P_\star$, we see from (4.12) that $\mathcal{C}(P_\star)$ is the identity matrix as well. This implies in particular that $P = P_\star$, so that $P_\star$ is a fixed point of $\mathfrak{R}_3^2$.

In the case $P = P_\star$, the equation (4.12) is a linear fixed point equation for the function $z \mapsto \mathcal{C}(P_\star, z)$. We already know that $\mathcal{C}(P_\star, z)$ is the identity matrix for $z = 0$. Whether or not the same holds for $z \neq 0$ depends on the eigenvalues of the matrix $\mathcal{A}_2(P_\star) = \mathcal{A}_2(P_\star, 0)$.

Lemma 4.2. *The eigenvalues of $\mathcal{A}_2(P_\star)$ are $\nu = 2.8900536382\ldots$ and ν^{-1}.*

Our proof of this lemma is computer-assisted, as will be described in Section 8. It also verifies that the origin $z = 0$ belongs to the domain of analyticity of the function that appear in (4.12). But this could easily be checked by hand as well.

We note that ν appears to satisfy the equation $\nu + \nu^{-1} = 2\alpha^{-1}$. If this is the case, and if the scaling factor $\mathcal{V}$ in Theorem 2.1 satisfies $\mathcal{V} + \mathcal{V}^{-1} = 2\alpha^{-3}$, then $\nu^2 = \mathcal{V}$.

In some open neighborhood of the origin in $\mathbb{C}$, we have either $\mathfrak{C}(P_\star, z) = \mathbf{1}$ for all z, or else

$$\mathcal{C}(P_\star, z) = \mathbf{1} + z^n \bigl[\mathcal{C}_n + \mathcal{O}(1)\bigr]\,, \tag{4.14}$$

for some nonzero matrix $\mathcal{C}_n$ and some integer $n \geq 1$. Substituting this expression for $\mathcal{C}(P_\star, z)$ into (4.12) yields the identity

$$\mathcal{C}_n = \alpha^{6n} \mathcal{A}_2(P_\star)^{-1} \mathcal{C}_n \mathcal{A}_2(P_\star)\,. \tag{4.15}$$

The eigenvalues of $\mathcal{C}_n \mapsto \alpha^{6n} \mathcal{A}_2(P_\star)^{-1} \mathcal{C}_n \mathcal{A}_2(P_\star)$ are α^{6n} and $\alpha^{6n}\nu^{\pm 2}$. They are all less than 1, so the equation (4.15) cannot have a solution $\mathcal{C}_n \neq 0$. This shows that the commutator of $F_\star$ and $G_\star$ is constant and equal to the identity in some open neighborhood of $x = \frac{1+\alpha}{2}$. Given that $B_\star$ and $A_\star$ are entire analytic, this implies that $F_\star$ and $G_\star$ commute.

At this point, the proof of Theorem 2.1 is reduced to the task of verifying the bounds in Lemmas 4.1 and 4.2.

5. Hyperbolicity

Here we consider again the anti-reversible case and the AM maps (3.11).

5.1. Proof of Theorem 2.2

Our goal here is to prove Theorem 2.2, with the exception of the inequality $\mu_2 \geq \alpha^{-3}$, based on estimates that can be (and have been) verified with the aid of a computer. The inequality $\mu_2 \geq \alpha^{-3}$ will be proved in Section 6.

By the contraction mapping theorem, Lemma 3.1 guarantees the existence of a fixed point $p_* \in B_\delta$ for $\mathfrak{M}$ and thus a fixed point $P_* = \bar{P} + (\mathrm{I} - M)p_*$ for $\mathfrak{R}_3$. For the same reasons as in the reversible case, the factors B_* and A_* associated with P_* extend to entire functions.

In order to prove hyperbolicity and related properties, we consider the transformation $\mathfrak{T}$ defined by

$$\mathfrak{T}(p) = \mathcal{L}^{-1}\bigl[\mathfrak{R}_3^2\bigl(P_* + \mathcal{L}p\bigr) - P_*\,\bigr]\,, \tag{5.1}$$

where $\mathcal{L}$ is a suitable linear isomorphism of $\mathcal{F}_\rho^r$. Clearly $p_* = 0$ is a fixed point of $\mathfrak{T}$. We expect the derivative $D\mathfrak{T}(0)$ to have an eigenvalue α^{-6} and no other spectrum outside the open unit disk. Thus, we consider a decomposition $\mathcal{F}_\rho^r = \mathcal{U} \oplus \mathcal{W}$, where $\mathcal{U}$ is a convenient one-dimensional subspace of $\mathcal{F}_\rho^r$. We will refer to $\mathcal{U}$ and $\mathcal{W}$ as the vertical and horizontal subspaces, respectively. Now the isomorphism $\mathcal{L}$ is chosen in such a way that the expected expanding direction of $D\mathfrak{T}(0)$ is roughly vertical. Writing an element $q \in \mathcal{F}_\rho^r$ as $q = \left[\begin{smallmatrix} u \\ w \end{smallmatrix}\right]$, with $u \in \mathcal{U}$ and $w \in \mathcal{W}$, we obtain a representation

$$D\mathfrak{T}(p)q = \begin{bmatrix} M_{uu}(p) & M_{uw}(p) \\ M_{wu}(p) & M_{ww}(p) \end{bmatrix} \begin{bmatrix} u \\ w \end{bmatrix}, \qquad q = \begin{bmatrix} u \\ w \end{bmatrix}. \tag{5.2}$$

By choosing $\mathcal{L}$ properly, the operators $M_{uw}(p) : \mathcal{W} \to \mathcal{U}$ and $M_{wu}(p) : \mathcal{U} \to \mathcal{W}$ can be made small for all p near $p_* = 0$. And $M_{uu}(p)$ should be close to $\alpha^{-6} \simeq 18$. In order to simplify notation, we identify $\mathcal{U}$ with $\mathbb{R}$ by choosing a unit vector $u_0 \in \mathcal{U}$ and identifying the vector tu_0 with the coefficient t.

Specific estimates are obtained in terms of an enclosure

$$N_{uu}^- \leq M_{uu}(p) \leq N_{uu}^+ \tag{5.3}$$

and upper bounds

$$\|M_{uw}(p)\| \leq N_{uw}\,, \qquad \|M_{wu}(p)\| \leq N_{wu}\,, \qquad \|M_{ww}(p)\| \leq N_{ww}\,, \tag{5.4}$$

that hold for all pairs p in a suitable cylinder C_1. Here, and in what follows, $\|.\|$ denotes the norm in $\mathcal{F}_\rho$. To be more precise, we determine two cylinders C_0 and C_1, such that

$$C_0 \subset C_1\,, \qquad C_j = [-h_j, h_j] \times \{w \in \mathcal{W} : \|w\| < r\}\,, \tag{5.5}$$

with $0 < r < h_0 < h_1$. Notice that both cylinders are centered at $p_* = 0$. The goal is to show that $\mathfrak{T}$ maps C_0 into C_1 and $C_1 \setminus C_0$ into the complement of C_0. To this end, it suffices to prove that

$$\begin{aligned} &N_{uu}^+ h_0 + N_{uw} r < h_1\,, \qquad N_{wu} h_0 + N_{ww} r < r\,, \\ &N_{uu}^- h_0 - N_{uw} r > h_0\,. \end{aligned} \tag{5.6}$$

Here, we have used that $\mathfrak{T}(p+q) - \mathfrak{T}(p) = \int_0^1 D\mathfrak{T}(p+sq)q\,ds$ whenever p and $p+q$ both belong to C_1.

Lemma 5.1. *There exists a linear isomorphism $\mathcal{L}$ of $\mathcal{F}_\rho^r$, as well as positive real numbers $r < h_0 < h_1, N_{uu}^- < N_{uu}^+, N_{uw}, N_{wu}, N_{ww}$ that satisfy (5.6), such that the derivative of the transformation $\mathfrak{T}$ defined by (5.1) satisfies the bounds (5.3) and (5.4) for every $p \in C_1$. So $\mathfrak{T}$ maps C_0 into C_1 and $C_1 \setminus C_0$ to the complement of C_0, with room to spare for taking interiors and/or closures. Let*

$$a_\pm = N_{uu}^\pm \pm N_{uw}\,, \quad b = N_{wu} + N_{ww}\,, \quad c = N_{uu}^- - N_{uw} - N_{wu} - N_{ww}\,. \tag{5.7}$$

Then $a_+ < 19$, $b < \frac{1}{4}$, *and* $c > 17$.

Our proof of this lemma is computer-assisted, as will be described in Section 8.

One of the consequences of the "uniform hyperbolicity" described in Lemma 5.1 is the following.

Corollary 5.2. *Let* $p \in C_0$. *Then either* $\mathfrak{T}^n(p)$ *belongs to* $C_1 \setminus C_0$ *for some* $n > 0$, *or else* $\mathfrak{T}^n(p) \to 0$ *as* $n \to \infty$.

Proof. Let $p \in C_0$. Since the orbit of p can exit C_0 only via the set $C_1 \setminus C_0$, it suffices to consider the case where $p_n = \mathfrak{R}^n(p)$ belongs to C_0 for all $n \geq 0$.

Write $p_n = \left[\begin{smallmatrix} u_n \\ w_n \end{smallmatrix}\right]$ with $u_n \in \mathcal{U}$ and $w_n \in \mathcal{W}$. From (5.3) and (5.4) we see that

$$|u_{n+1}| \geq N_{uu}^{-}|u_n| - N^{uw}\|w_n\|\,, \qquad \|w_{n+1}\| \leq N_{wu}|u_n| + N_{ww}\|w_n\|\,, \tag{5.8}$$

for all $n \geq 0$. Assume for contradiction that $|u_m| - \|w_m\| > 0$ for some $m \geq 0$. Then

$$|u_{m+1}| - \|w_{m+1}\| \geq N_{uu}^{-}|u_m| - N_{uw}\|w_m\| - N_{wu}|u_m| - N_{ww}\|w_m\| > c\|w_m\|\,, \tag{5.9}$$

with $c > 0$ as defined in (5.7). So we have $|u_n| - \|w_n\| > 0$ for all $n \geq m$. Combining this with the first inequality in (5.8), we find that

$$|u_{n+1}| > a|u_n|\,, \tag{5.10}$$

for all $n \geq m$, with $a = a_-$ as defined in (5.7). Given that $a > 1$, this leads to a contradiction. So we must have $|u_n| \leq \|w_n\|$ for all $n \geq 0$. By the second inequality in (5.8), this implies that

$$\|w_{n+1}\| \leq N_{wu}|u_n| + N_{ww}\|w_n\| \leq b\|w_n\|\,, \tag{5.11}$$

for all $n \geq 0$, with b as defined in (5.7). Given that $b < 1$, we find that $w_n \to 0$ as $n \to \infty$. But $|u_n| \leq \|w_n\|$ for all $n \geq 0$, so $u_n \to \infty$ as well. Thus, $p_n \to 0$ as claimed. **QED**

The bounds $a_- > 17$ and $b < 1/4$ from Lemma 5.1 yield information about the spectrum of $D\mathfrak{T}(0)$, using e.g. the theorem below. We note that the operator $D\mathfrak{T}(0)$ is compact, for the reasons described before Lemma 3.1.

Theorem 5.3. ([45]) *Let A be a compact linear operator on a real Banach space $\mathbb{R} \times W$. For $u \in \mathbb{R}$ and $w \in W$ write $A(u + w) = u' + w'$ with $u' \in \mathbb{R}$ and $w' \in W$. Assume that there exist positive real numbers $b < a$ such that $\|w'\| \leq b \max\{|u|, \|w\|\}$, and such that $|u'| \geq a|u|$ whenever $|u| \geq \|w\|$. Then A has a simple eigenvalue of modulus $\geq a$ and no other eigenvalue of modulus $> b$.*

Here, in the real setting, a non-real number $\xi + i\eta$ is said to be an eigenvalue of A if there exists nonzero vectors x and y such that $Ax = \xi x - \eta y$ and $Ay = \xi y + \eta x$.

As a consequence of Lemma 5.1 and Theorem 5.3 we have the following.

Corollary 5.4. *The derivative $D\mathfrak{T}(p_*)$ at $p_* = 0$ has a real eigenvalue $\lambda \geq a_-$ and no other spectrum outside the disk $|z| \leq b$. The local unstable manifold of $\mathfrak{T}$ at the fixed point*

$p_*=0$ is the graph of a real analytic function $\mathtt{p}_*$ from an open neighborhood of the origin in $\mathcal{U}$ to $\mathcal{W}$. Furthermore, $\mathtt{p}_*$ extends to a real analytic function on an open neighborhood of $[-h_0,h_0]$, taking values in $\{w\in\mathcal{W}:\|w\|<r\}$.

The existence and real analyticity of the local unstable manifold near $p_*=0$ follows from standard theorems on invariant manifolds. Its extension is obtained by iterating $\mathfrak{T}$ and using Corollary 5.2.

Clearly Corollary 5.4 translates trivially to an analogous result for the transformation $\mathfrak{R}_3^2$. In fact, an analogous result holds for $\mathfrak{R}_3$ as well, since P_* is a fixed point of $\mathfrak{R}_3$. Our reason for considering the second iterate of $\mathfrak{R}_3$ in this section is that it was easier to find a good isomorphism $\mathcal{L}$ in this case.

5.2. Proof of Theorem 2.3, Part I

Our goal here is to prove Theorem 2.3, based on on estimates that can be (and have been) verified with the aid of a computer. A transversality condition that is needed, and the claim that $s_*=0$, will be proved in Section 6.

Here we consider the unstable manifold of $\mathfrak{R}_3$ at P_* to be a curve in the cylinder $C_0'=P_*+\mathcal{L}C_0$ rather than a graph. A possible parametrization of this curve is given by $\mathtt{P}_*(t)=P_*+\mathcal{L}(tu_0+\mathtt{p}_*(t))$, with t ranging in $[-h_0,h_0]$. Here $u_0\in\mathcal{U}$ is the unit vector mentioned earlier. The projection of $\mathtt{P}_*$ onto $\mathcal{L}\mathcal{U}$ is a strictly increasing function; and as t increases from $-h_0$ to h_0, the curve $\mathtt{P}$ connects the bottom of C_0' to the top.

Given a real number $c>0$, consider the extension of $\mathfrak{R}_3$ to one-parameter families of pairs $s\mapsto\mathtt{P}(s)$, defined by the equation

$$\mathfrak{F}_c(\mathtt{P})(s)=\mathfrak{R}_3(\mathtt{P}(cs))\,. \tag{5.12}$$

Lemma 5.5. *Consider the* AM *family* $\mathtt{P}$ *for* $\lambda=e^s$. *Then there exist real numbers* $\sigma,\varepsilon>0$ *and an integer* $m>0$*, such that the following holds. Consider the curve* $\mathtt{P}_0=\mathfrak{F}_{\alpha^3}^{2m}(\mathtt{P})$. *Then* $\mathtt{P}_0(s)$ *lies in the interior of* C_0' *for* $-\sigma<s<\sigma$. *As* s *is increased from* $-\sigma-\varepsilon$ *to* $\sigma+\varepsilon$*, the curve* $\mathtt{P}_0$ *enters the cylinder* C_0' *through the bottom (corresponding to* $u=-h_0$ *for* C_0*) at* $s=-\sigma$ *and leaves it through the top (corresponding to* $u=-h_0$ *for* C_0*), at* $s=\sigma$. *An analogous statement holds if* $\mathtt{P}$ *is replaced by* $\mathfrak{F}_{\alpha^3}(\mathtt{P})$.

Our (computer-assisted) proof of this lemma implements the transformation $\mathfrak{F}_{\alpha^3}$ on a space of curves $s\mapsto\mathtt{P}(s)$ in $\mathcal{F}_\rho$ that are analytic in a disk $|s|<\delta$ of radius $\delta=2^{-96}$. In this space, we determine bounds on the curve $\mathtt{P}_0=\mathfrak{F}_{\alpha^3}^{2m}(\mathtt{P})$ for $m=66$ that imply the claims of Theorem 2.3 via strict inequalities. For further details we refer to Section 8.

Some immediate consequences of Lemma 5.5 are the following. There exists an increasing sequence $n\mapsto s_n^-$ and a decreasing sequence $n\mapsto s_n^+$, with $s_n^-<s_n^+$ for all n, such that the curve $\mathtt{P}_n=\mathfrak{F}_1^{2n}(\mathtt{P}_0)$ enters the bottom of the cylinder C_0' at the parameter value s_n^- and leaves the top of the cylinder for the first time at a parameter value s_n^+.

Pick a parameter value s_∞ that belongs to $[s_n^-,s_n^+]$ for every n. Then the orbit $n\mapsto\mathfrak{R}_3^{2n}(\mathtt{P}_0(s_\infty))$ converges to P_* by Corollary 5.2. If n is sufficiently large, then the pair $\mathtt{P}_n(s_\infty)$ is close enough to P_* for perturbative arguments to apply. In particular, if $\mathtt{P}_n$

intersects the local stable manifold transversally, then we can use the graph transform [6] to characterize convergence.

The graph transform $\mathfrak{F}$ associated with $\mathfrak{R}_3^2$ takes the form $\mathfrak{F}(\mathtt{P}) = \mathfrak{R}_3^2 \circ \mathtt{P} \circ R$. Here, $R = R(\mathtt{P})$ is a real analytic function defined near s_∞. Its dependence on $\mathtt{P}$ is real analytic and can be chosen in such a way that $\mathfrak{F}$ has an attracting fixed point. By construction, this fixed point is the (canonically parametrized) local unstable manifold of $\mathfrak{R}_3$ at P_*. If $\mathtt{P}$ is any curve in the domain of $\mathfrak{F}$, then the sequence $k \mapsto \mathfrak{F}^k(\mathtt{P})$ converges to the fixed point of $\mathfrak{F}$, and $k \mapsto R\bigl(\mathfrak{F}^k(\mathtt{P})\bigr)$ converges to the function $s \mapsto \mu_2^{-1}s$. In fact, R can be chosen affine, at the expense of possibly weakening the rate of convergence.

Assume now that s_∞ must have the value 0, and that the curves $\mathtt{P}_n$, for n sufficiently large, are transversal to the local stable manifold of $\mathfrak{R}_3$ at P_*. These properties will be proved in Section 6.

In this case, the re-parametrization function R can be chosen linear. Since convergence of $k \mapsto R\bigl(\mathfrak{F}^k(\mathtt{P}_n)\bigr)$ to the function $s \mapsto \mu_2^{-2}s$ is exponential, we can in fact choose the fixed re-parametrization $s \mapsto \mu_2^{-2}s$ at each step. This show that, if $\mathtt{P}$ is the AM family with parameter $\lambda = e^s$, then the sequence $n \mapsto \mathfrak{F}^n_{\mu_2^{-2}}(\mathtt{P}_0)$ converges to $\mathtt{P}_*$, modulo a one-time linear re-parametrization. Here, we assume that the local unstable manifold $\mathtt{P}_*$ has been parametrized in such a way that it is a fixed point of $\mathfrak{F}_{\mu_2^{-2}}$.

The same holds if $\mathtt{P}$ is replaced by $\mathfrak{F}_{\alpha^3}(\mathtt{P})$. So the above arguments can be repeated for the graph transform associated with $\mathfrak{R}_3$. We note that the pairs $\mathtt{P}_*(t)$ are limits of renormalized AM pairs, so they are commuting.

Since transversality to the local stable manifold is stable under small perturbations, we can repeat the same arguments for a sufficiently small perturbation of the AM curve. The only difference is that a one-time affine re-parametrization is needed.

6. The Lyapunov exponent

The Lyapunov exponent will be used to establish a connection between observable quantities and local properties of $\mathfrak{R}_3$ near the fixed point P_*.

6.1. The critical coupling

Let $G = (\alpha, A)$ be the anti-reversible AM map with coupling constant $\lambda = e^s$. Let λ_* be a value of the parameter λ for which the pair $P = (F, G)$ with $F = (1, \mathbf{1})$ gets attracted to P_* under the iteration of $\mathfrak{R}_3^2$. The goal here is to show that $\lambda_* = 1$.

First, we claim that $\lambda_* \le 1$. To see why, consider $\lambda > 1$. Then the Lyapunov exponent $L(G) = \log\lambda$ is positive. Thus, by Proposition 6.4 in Subsection 6.2, the sequence of functions $n \mapsto \|A^{*q_n}(\alpha^n\boldsymbol{.})\|$ cannot stay bounded on $[-1, 1]$ as $n \to \infty$. So we cannot have $\mathfrak{R}^n(P) \to P_*$ as $n \to \infty$ along multiples of 3.

Our next goal is to exclude the possibility $\lambda_* < 1$. The following is Theorem 3.4 in [40]. It applies to the AM map $G = (\alpha, A)$, for any irrational α whose continued fraction denominators q_n satisfy $\lim_n q_n^{-1} \log q_{n+1} = 0$.

Theorem 6.1. ([40]) *Let $0 < \lambda < 1$ and assume that E belongs to the spectrum of H_λ^α. There exists constants $a, b, c > 0$ such that for all $q > 0$,*

$$\|A^{*q}(z)\| \le bq^a\,, \qquad |\operatorname{Im} z| \le c\,. \tag{6.1}$$

An analogous result for Diophantine α was probably proved earlier. Corollary 4.5 in [44] comes close, but it considers only the real domain.

In what follows, if $G=(\alpha,A)$ is an arbitrary skew-product map, we will write $A^{\alpha * q}$ instead of A^{*q} for the product (1.1), in order to emphasize the dependence on α.

Let G be an anti-reversible AM map for energy zero and α the inverse golden mean. Consider the corresponding pair $P=(F,G)$ with $F=(1,\mathbf{1})$, and its RG iterates $P_n=\mathfrak{R}_3^{2n}$. We choose here even powers of $\mathfrak{R}_3$, so that the reflections $L(0)$ that are part of the scaling $L_3=L(0)e^{-\vartheta J}$ cancel. And for the fixed point P_*, the rotations cancel as well, since $L(0)e^{-\vartheta J}L(0)=e^{-\vartheta J}$. Thus, in order to simplify notation, consider $\mathfrak{R}_3$ with $L_3=\mathbf{1}$. Then we can perform an initial rotation $A\mapsto e^{\vartheta J}Ae^{-\vartheta J}$ in such a way that $P_n\to P_*$ with $L_3=\mathbf{1}$ fixed. Then $P_n=(F_n,G_n)$ with $F_n=(1,B_n)$ and $G_n=(\alpha,A_n)$, where

$$A_n(x)=A^{\alpha * q_{6n}}\bigl(\alpha^{6n}x\bigr)\,,\qquad B_n(x)=A^{\alpha * q_{6n-1}}\bigl(\alpha^{6n}x\bigr)\,. \tag{6.2}$$

As described earlier, the factors A_* and B_* of the fixed point P_* are entire, due to the analyticity-improving property of $\mathfrak{R}_3$. For the same reason, we have convergence $A_n\to A_*$ and $B_n\to B_*$, uniformly on compact subsets of $\mathbb{C}$.

Let now u and v be fixed but arbitrary nonnegative integers, not both zero. Then

$$A_n^{\alpha * u}(\,.+v)B_n^{1*v}\to A_*^{\alpha * u}(\,.+v)B_*^{1*v}\,, \tag{6.3}$$

uniformly on compact subsets of $\mathbb{C}$.

Proposition 6.2. *Assume that $\lambda_*<1$. Then $x\mapsto A_\infty^{\alpha * u}(x+v)B_\infty^{1*v}(x)$ is a polynomial whose degree cannot be larger than the constant a in Theorem 6.1.*

Proof. Let M be some fixed 2×2 matrix and define

$$f_n=\operatorname{tr}\bigl(MA_n^{\alpha * u}(\,.+v)B_n^{1*v}\bigr)\,,\quad f_*=\operatorname{tr}\bigl(MA_*^{\alpha * u}(\,.+v)B_*^{1*v}\bigr)\,. \tag{6.4}$$

By Theorem 6.1 we have a bound

$$|f_n(z)|\le 2b(uq_{6n-1}+vq_{6n})^a\le C(\alpha u+v)^a\alpha^{-6na}\,,\qquad |\operatorname{Im} z|\le c\alpha^{-6n}\,, \tag{6.5}$$

for some fixed constant $C>0$. Now restrict to a disk $|z|\le r$ of radius $r>0$. If n is sufficiently large, then

$$\partial_z^k f_n(z)=\frac{k!}{2\pi i}\int_{\Gamma_n}\frac{f_n(\zeta)}{(\zeta-z)^{k+1}}\,d\zeta\,, \tag{6.6}$$

where Γ_n is the path along the two circles in $\mathbb{C}/\bigl(\alpha^{-6n}\mathbb{Z}\bigr)$ at $\operatorname{Im}\zeta=\pm c\alpha^{-6n}$. Using the bound (6.5) and the fact that $|\Gamma_n|=2\alpha^{-6n}$, we have

$$\begin{aligned}\bigl|\partial_z^k f_n(z)\bigr|&\le\frac{k!}{2\pi}2\alpha^{-6n}\bigl(c\alpha^{-6n}-r\bigr)^{-k-1}C(\alpha u+v)^a\alpha^{-6na}\\&\le C_k(\alpha u+v)^a\alpha^{6n(k-a)}\,,\end{aligned} \tag{6.7}$$

for some constant $C_k > 0$. Thus, if $k > a$, then the derivative $\partial_z^k f_*(z) = \lim_n \partial_z^k f_n(z)$ vanishes on the disk $|z| < r$.

This shows that f_* is a polynomial of degree $\lfloor a \rfloor$ or less. Since M was arbitrary, we conclude that $x \mapsto A_\infty^{\alpha * u}(x+v) B_\infty^{1*v}(x)$ is a polynomial of degree $\lfloor a \rfloor$ or less. **QED**

Theorem 6.3. $\lambda_* = 1$.

Proof. We have already established that $\lambda_* \leq 1$. Assume for contradiction that $\lambda_* < 1$. In what follows, x and y denote real numbers. Given any $m > 0$, denote by d_m the polynomial degree of A_*^{*m}, meaning the maximal degree of any of the components of A_*^{*m}. Clearly, $x \mapsto A_*^{\alpha * m}(\omega x + z)$ has degree d_m as well, for any complex numbers $\omega \neq 0$ and z.

Consider now the identity (3.8), which holds whenever (α, A) is anti-reversible. By taking the trace of $(-1)^m A_\circ^{*2m}(iy)$, we obtain the square of the Hilbert-Schmidt norm of $U(iy) = A_\circ((m - {}^1\!/_2)\alpha + iy) \cdots A_\circ({}^\alpha\!/_2 + iy)$. Applying this to $A = A_*$, we see that the function $y \mapsto A_*^{*2m}(iy)$ must have degree $2d_m$. Iterating this argument shows that $A_*^{*2^j}$ has degree $2^j d_1$. But the degree of $A_*^{*2^j m}$ cannot exceed a, by Proposition 6.2. Thus, we must have $d_1 = 0$.

An analogous argument shows that B_* has degree 0 as well. But we know that neither A_* nor B_* are constant. So $\lambda_* = 1$. **QED**

6.2. A Lyapunov exponent for pairs

Let α be an irrational number between 0 and 1. To a pair $P = (F, G)$ with $F = (1, B)$ and $G = (\alpha, A)$, we associate a renormalized pair as in (2.1) by setting

$$\mathfrak{R}(P) = \left(\Lambda^{-1} G \Lambda, \Lambda^{-1} F G^{-c} \Lambda\right). \tag{6.8}$$

For simplicity, we restrict here to the trivial scaling $\Lambda = \Lambda(P)$, given by $\Lambda(x, y) = (\alpha x, y)$. Consider now the iterates $P_n = \mathfrak{R}^n(P)$. The components of P_n are of the form $F_n = (1, B_n)$ and $G_n = (\alpha_n, A_n)$. After choosing a suitable norm for the factors A_n, a Lyapunov-type exponent for pairs can be defined by setting

$$\ell(P) = \limsup_{n \to \infty} q_n^{-1} \log \|A_n\|. \tag{6.9}$$

where q_n is the n-th continued fraction denominator for α.

Consider first the case where $F = (1, \mathbf{1})$. Then the functions A_n are scaled versions of A^{q_n}. To be more precise, $A_n(x) = A^{*q_n}(\bar{\alpha}_n x)$, where $\bar{\alpha}_n = \alpha_0 \alpha_1 \cdots \alpha_{n-1}$. Taking the sup-norm in (6.9) on a domain $|x| < \varepsilon$, we have $\ell(P) = \ell_\varepsilon(G)$, where

$$\ell_\epsilon(G) = \limsup_{n \to \infty} q_n^{-1} \log \sup_{|x| \leq \epsilon} \|A^{*q_n}(\bar{\alpha}_n x)\|. \tag{6.10}$$

Here, and in what follows, we assume that A is a continuous 1-periodic function on $\mathbb{R}$, taking values in $\mathrm{SL}(2, \mathbb{R})$.

Proposition 6.4. *Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ be of finite type, in the sense that the sequence $n \mapsto \alpha_n$ is bounded away from zero. Then the* $\limsup$ *in (6.10) exists as a limit, and $\ell_\epsilon(G)$ agrees with the Lyapunov exponent $L(G)$.*

Proof. We may assume that $L(G) > 0$. Then, by Oseledets' theorem [41], we have

$$\lim_q \frac{1}{q} \log \frac{\|A^{*q}(x_0)v_0\|}{\|v_0\|} = L(G)\,, \tag{6.11}$$

for almost every $x_0 \in \mathbb{R}$, and for all vectors v_0 outside some one-dimensional subspace of $\mathbb{R}^2$ that can depend on x_0. Now fix such an x_0 and v_0.

Using the three-gap theorem [3,2,4], we can find an integer $k > 0$ and sequences $n \mapsto t_n$ and $n \mapsto s_n$ of positive integers, such that

$$q_n \le t_n \le q_{n+k}\,, \qquad |x_0 + t_n\alpha - s_n| \le \epsilon\bar{\alpha}_n\,. \tag{6.12}$$

Setting $x_n = x_0 + t_n\alpha - s_n$ and $v_n = A^{*t_n}(x_0)v_0$, we have

$$\begin{aligned} \frac{1}{q_n}\log\frac{\|A^{*q_n}(x_n)v_n\|}{\|v_n\|} &= \left(1+\frac{t_n}{q_n}\right)\frac{1}{q_n+t_n}\log\frac{\|A^{*(q_n+t_n)}(x_0)v_0\|}{\|v_0\|} \\ &\quad - \frac{t_n}{q_n}\,\frac{1}{t_n}\log\frac{\|A^{*t_n}(x_0)v_0\|}{\|v_0\|}\,. \end{aligned} \tag{6.13}$$

Notice that $1 \le t_n/q_n \le C$ for some fixed constant C. Thus, the right hand side of (6.13) converges to $L(G)$ as $n \to \infty$. This yields a lower bound $\ell_\epsilon(G) \ge L(G)$. The upper bound $\ell_\epsilon(G) \le L(G)$ follows from Furman's theorem [29]. **QED**

In what follows, we assume that α is the inverse golden mean. Then $c = 1$ in the equation (6.8), and $\alpha_n = \alpha$ for all n. In addition, we restrict to pairs in the set $\mathcal{F}'_\rho$ defined below, where $\rho = (\rho_{\mathrm{F}}, \rho_{\mathrm{G}})$ is assumed to satisfy

$$\tfrac{1}{2}\alpha^{-2} < \rho_G\,, \qquad \tfrac{1}{2}\alpha + \alpha\rho_G \le \rho_F \le \alpha^{-1}\rho_G\,. \tag{6.14}$$

Definition 6.5. *Define $\mathcal{F}'_\rho$ to be the set of pairs $P = (F, G)$ in $\mathcal{F}^r_\rho$ with the property that the maps F and G commute, and that their factors have determinant* 1.

The condition (6.14) guarantees that $\mathfrak{R}$ defines a dynamical system on $\mathcal{F}'_\rho$. This condition is satisfied e.g. for the values $\rho = (2, {}^{11}\!/_8)$ and $\rho = (3, 2)$ that are used in Lemma 3.1 and Lemma 4.1, respectively.

Using that $q_n^{-1} = \bigl(1+\alpha^2\bigr)\alpha^n + \mathcal{O}\bigl(\alpha^{3n}\bigr)$, we can rewrite (6.9) as

$$\ell(P) = \limsup_{n\to\infty} L_n(P)\,, \tag{6.15}$$

where

$$L_n(P) = \bigl(1+\alpha^2\bigr)\alpha^n \log\|A_{n,\circ}\|_{\rho_G}\,, \qquad A_{n,\circ}(x) = A_\circ^{*q_n}\bigl(\alpha^n x\bigr)\,. \tag{6.16}$$

Here, we have used that scaling and "symmetrizing" commute, as mentioned after (3.5).

Remark 3. If we restrict n in (6.16) to multiples of 3 and replace $\mathfrak{R}^3$ by $\mathfrak{R}_3$, then the weaker domain condition (3.39) is sufficient. In addition, the restriction to commuting pairs can be omitted in this case. We will not do this here, simply to avoid complicating the notation.

Our main reason for considering a Lyapunov exponent based on the functionals L_n is that it transforms conveniently under renormalization:

$$L_n\bigl(\mathfrak{R}(P)\bigr) = \alpha^{-1} L_{n-1}(P)\,, \qquad n = 1, 2, \ldots \tag{6.17}$$

So we have $\ell\bigl(\mathfrak{R}(P)\bigr) = \alpha^{-1}\ell(P)$. Clearly, ℓ vanishes on any periodic orbit of $\mathfrak{R}$.

Proposition 6.6. *Let $G = (\alpha, A)$ be an anti-reversible* AM *map with α the inverse golden mean. Then the limit $\lim\limits_{n\to\infty} L_n(P)$ exists and is equal to $L(G)$.*

Proof. Let $r = \rho_{\scriptscriptstyle G}$. Consider the sup norm in (6.10) for small $\epsilon > 0$. Since the evaluation map $g \mapsto g(x)$ is continuous on $\mathcal{G}_r$ for $|x| \le r$, this norm is bounded from above by $C\|A_{n,\circ}\|_r$ for some fixed constant $C > 0$. So we have $\liminf\limits_{n\to\infty} L_n(P) \ge L(G)$.

Next, consider the AM map G_δ with factor $A_\delta(x) = A(x+i\delta)$. Using the (well known) fact that $\log\bigl\|A^{*q}(x+i\delta)\bigr\|$ is a convex function of δ, we have

$$L(P) \le \limsup_{n\to\infty} q_n^{-1} \sup_{|x|\le\epsilon} \log\bigl\|A_\circ^{*q_n}\bigl(\alpha^n x \pm i\delta\bigr)\bigr\| \le \ell_\epsilon(G_\delta) = L(G_\delta)\,, \tag{6.18}$$

for $\epsilon > 0$ sufficiently large (depending only on r) and every $\delta > 0$. The "$\pm$" in the equation (6.18) includes a maximum over the two signs. But by anti-reversibility, the supremum over $|x| \le \epsilon$ does not depend on the sign.

Now we can use the fact [48] that $L(G_\delta) = \max\bigl\{0, \log\lambda + 2\pi\delta\bigr\}$. Thus, taking $\delta \to 0$ in (6.18) yields $\limsup\limits_{n\to\infty} L_n(P) \le L(G)$. This proves the claim in Proposition 6.6. **QED**

For more general pairs in $\mathcal{F}'_\rho$, the $\limsup$ in (6.15) may not be a limit. An easy way to cure this problem is to define a slightly different Lyapunov-type exponent as follows:

$$L(P) = \limsup_{n\to\infty} \frac{1}{1+\alpha^2}\Bigl[L_n(P) + \alpha^2 L_{n-1}(P)\Bigr]\,. \tag{6.19}$$

Theorem 6.7. *Assume that $P \in \mathcal{F}'_\rho$ for some choice of $\rho = (\rho_{\scriptscriptstyle F}, \rho_{\scriptscriptstyle G})$ that satisfies (6.14). Then the sequence $n \mapsto L_n(P) + \alpha^2 L_{n-1}(P)$ is decreasing. As a consequence, the* $\limsup$ *in (6.19) is achieved as a limit, and L is upper semi-continuous on $\mathcal{F}'_\rho$. Furthermore, the value of $L(P)$ does not depend on the choice of ρ.*

Proof. For $n \ge 0$ define

$$b_n = b_n(\rho) = \log\|B_{n,\circ}\|_{\rho_F}\,, \qquad a_n = a_n(\rho) = \log\|A_{n,\circ}\|_{\rho_G}\,. \tag{6.20}$$

Assume now that $n \geq 1$. Then the domain conditions (6.14) guarantee that

$$b_n \leq a_{n-1}\,, \qquad a_n \leq b_{n-1} + a_{n-1}\,. \tag{6.21}$$

Using that $1+\alpha = \alpha^{-1}$, this yields the bound

$$\begin{aligned} L_n(P) + \alpha^2 L_{n-1}(P) &= \bigl(1+\alpha^2\bigr)\bigl[\alpha^n a_n + \alpha^{n+1} a_{n-1}\bigr] \\ &\leq \bigl(1+\alpha^2\bigr)\bigl[\alpha^n b_{n-1} + \alpha^n a_{n-1} + \alpha^{n+1} a_{n-1}\bigr] \\ &\leq \bigl(1+\alpha^2\bigr)\bigl[\alpha^n a_{n-2} + \alpha^n(1+\alpha) a_{n-1}\bigr] \\ &= \alpha^2 L_{n-2}(P) + L_{n-1}(P)\,, \end{aligned} \tag{6.22}$$

for $n \geq 2$. This shows that the sequence $n \mapsto L_n(P) + \alpha^2 L_{n-1}(P)$ is decreasing and thus has a limit. Since each L_k is continuous, the limit is upper semi-continuous.

Next consider another domain parameter ϱ satisfying (6.14). Let us write $L_n(\varrho)$ instead of $L_n(P)$ when the domain parameter is ϱ, and $L_n(\rho)$ when the domain parameter is ρ. Using that $\mathfrak{R}$ is analyticity-improving, there exists $k > 0$ and a constant $c_k > 0$ such that

$$\|B_{n+k}\|_{\varrho_F} \leq e^{c_k} \|B_n\|_{\rho_F}^{q_{n-2}} \|A_n\|_{\rho_G}^{q_{n-1}}\,, \quad \|A_{n+k}\|_{\varrho_F} \leq e^{c_k} \|B_n\|_{\rho_F}^{q_{n-1}} \|A_n\|_{\rho_G}^{q_n}\,, \tag{6.23}$$

for every $n \geq 0$. Taking logarithms and using the identity

$$\begin{bmatrix} q_{k-2} & q_{k-1} \\ q_{k-1} & q_k \end{bmatrix} \begin{bmatrix} \alpha \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^k \begin{bmatrix} \alpha \\ 1 \end{bmatrix} = \alpha^{-k} \begin{bmatrix} \alpha \\ 1 \end{bmatrix}, \tag{6.24}$$

we find that

$$\alpha^2 L_{n+k-1}(\varrho) + L_{n+k}(\varrho) \leq C_k \alpha^n + \alpha^2 L_{n-1}(\rho) + L_n(\rho)\,, \tag{6.25}$$

with $C_k = \bigl(1+\alpha^2\bigr)^2 \alpha^k c_k$. An analogous inequality holds if ϱ and ρ are exchanged. This shows that $L(P)$ is independent of the choice of ρ. **QED**

Using upper semicontinuity and (6.17), together with the fact that the pair P_* commutes, we immediately obtain the following.

Corollary 6.8. *Consider a pair $P \in \mathcal{F}'_\rho$. Then $L(\mathfrak{R}(P)) = \alpha^{-1} L(P)$. So in particular, $L(P_*) = 0$. Furthermore, if $\mathfrak{R}^{3n}(P) \to P_*$ as $n \to \infty$, then $L(P) = 0$.*

6.3. Transversal intersection

At this point we can complete the proof of 2.3. We already know from Subsection 6.1 that $s_* = 1$. What remains to be proved is that some RG iterate of the AM family intersects the local stable manifold $\mathcal{W}^s$ of $\mathfrak{R}_3$ transversally, and that $\mu_2 \geq \alpha^{-3}$.

To this end, consider the splitting $\mathcal{F}_\rho = W^u \oplus W^s$, where W^u is the unstable subspace for the operator $D\mathfrak{R}_3(P_*)$ and W^s the stable subspace. Then $P_* + W^s$ is tangent to $\mathcal{W}^s$ at P_*. And $\mathcal{W}^s$ is the graph of a real analytic function from W^s to $P_* + W^u$. To be more precise, this holds locally, near P_*. So we restrict our analysis to a suitable open ball B

in $\mathcal{F}_\rho$ that is centered at P_*. Using that $\mathcal{W}^s$ is a graph, we can define the height of a pair $P \in B$ relative to $\mathcal{W}^s$ in the direction of W^u.

Consider the AM family $s \mapsto \mathtt{P}(s)$ associated with the factor (3.11) for $\lambda = e^s$. Define $\mathtt{P}_n = \mathfrak{F}_1^n(\mathtt{P})$, where $\mathfrak{F}_1$ denotes the pointwise version of $\mathfrak{R}_3$ defined by (5.12). Notice that $\mathfrak{R}_3$ may be replaced by $\mathfrak{R}^3$, since the AM pairs commute. Recall that Lyapunov exponent of $\mathtt{P}(s)$ is $\max\{0,s\}$. So by (2.7), the Lyapunov exponent of $\mathtt{P}_n(s)$ is $\alpha^{-3n}s$. And recall from Subsection 6.1 that 0 is the unique value of s for which $\mathtt{P}(s)$ is attracted to P_* under the iteration of $\mathfrak{R}_3$. If n is sufficiently large, so that the pair $\mathtt{P}_n(0)$ lies on $\mathcal{W}^s$ in B, then the pairs $\mathtt{P}_n(s)$ for $s > 0$ cannot lie on $\mathcal{W}^s$. Here, we have Corollary 6.8. In what follows, k denotes some (large) value of n for which this holds.

Now choose $t > 0$ such that $\mathtt{P}_*(t)$ belongs to B. Let Σ_0 be a codimension 1 subspace of $\mathcal{F}_\rho$ that passes through $\mathtt{P}_*(t)$ and is transversal to the unstable manifold $\mathtt{P}_*$. We may assume also that Σ_0 does not intersect $\mathcal{W}^s$. For $n = 1, 2, \ldots$, define Σ_n to be the inverse image of Σ_{n-1} under $\mathfrak{R}_3$, restricted to B. If B has been chosen sufficiently small, then the λ-Lemma [14] guarantees the following. The sets Σ_n are real analytic manifolds. Furthermore, the sequence $n \mapsto \Sigma_n$ accumulates at the stable manifold $\mathcal{W}^s$ of $\mathfrak{R}_3$ at P_* asymptotically like μ_2^{-n}. To be more precise, if $\mathtt{Q}$ is any (real analytic) curve in B that crosses $\mathcal{W}^s$ transversally at the point $\mathtt{Q}(0)$, then near this point, and for sufficiently large k, the curve $\mathtt{Q}$ intersects Σ_n for a unique parameter value s_n. Furthermore, the sequence $n \mapsto \mu_2^n s_n$ converges to a nonzero constant.

Consider now the curve $\mathtt{Q} = \mathtt{P}_k$. Denote by $h(s)$ the height of $\mathtt{Q}(s)$ above $\mathcal{W}^s$ in the direction of W^u. By analyticity, we have $h(s) = as^m + \mathcal{O}\big(s^{m+1}\big)$ for some $m \geq 1$, with $a \neq 0$. Thus, for sufficiently large n, the curve $\mathtt{Q}$ intersects Σ_n at some parameter value $s_n > 0$. We may choose the smallest such value. By construction, the point $\mathfrak{R}^{3n}(\mathtt{Q}(s_n))$ lies on Σ_0. So we must have $\mu_2^n s_n^m \geq b$ for sufficiently large n, where b is some positive constant. Using that the Lyapunov exponent is bounded on B, and that $\mathfrak{R}^{3n}(\mathtt{Q}(s_n))$ has Lyapunov exponent $\alpha^{-3(n+k)}s_n$, we also have $\alpha^{-3n}s_n \leq c$ for some $c > 0$. Thus, $\mu_2 \geq \alpha^{-3m}$. We know from Lemma 5.1 that $\mu_2 \leq \sqrt{19}$. And for $m \geq 2$ we have $\alpha^{-3m} > 17$. So we must have $m = 1$. This implies that $\mathtt{Q} = \mathfrak{F}_1^k(\mathtt{P})$ intersects $\mathcal{W}^s$ transversally, and that $\mu_2 \geq \alpha^{-3}$. This completes the proof of Theorem 2.3.

Remark 4. The parameter values s_n depend on the value of k defining the curve $\mathtt{Q} = \mathtt{P}_k$. Writing $s_n = s_{k,n}$ we obtain $\mathfrak{R}^{3(k+n)}(\mathtt{P}(s_{k,n})) \to \mathtt{P}_*(t)$ in the limit $k \to \infty$. If we assume that $\mu_2 = \alpha^{-3}$, then this implies that $\mathtt{P}_*(t)$ has a positive Lyapunov exponent.

7. Supercritical maps

The main goal in this section is to prove Theorems 2.4 and 2.5.

7.1. Limiting zeros

Here we consider the factor-normalization that was mentioned before Theorem 2.4. Recall from (3.16) that $\mathfrak{R}_3 = \mathfrak{N} \circ \mathcal{R}_3 \circ \mathfrak{C}$, where $\mathfrak{C}$ is a commutator correction, and where $\mathfrak{N}$ normalizes determinants to 1. The factor-normalization of a pair $P = ((1,B),(\alpha,A))$ is component-wise, $\mathfrak{N}(P) = ((1,\mathcal{M}(B),(\alpha,\mathcal{N}(A))$, with $\mathcal{M}$ and $\mathcal{N}$ of the form

$$\mathcal{M}(B) = M(B)^{-1}B\,, \qquad \mathcal{N}(A) = N(A)^{-1}A\,. \tag{7.1}$$

Up to now we have used $M(B)=\det(B)^{1/2}$ and $N(A)=\det(A)^{1/2}$.

Here, we are interested only in pairs that commute and whose factors have constant nonnegative determinants. So we omit the commutator correction $\mathfrak{C}$ and choose for M and N the norms in $\mathcal{F}_\rho$. Other choices would work equally well, as long as they guarantee that the orbits of $\mathfrak{R}_3$ remain bounded without tending to zero. In order to simplify the description, we also choose a trivial y-scaling $L_3=\mathbf{1}$.

Let K_0 be a set of pairs $P\in\mathcal{F}_\rho$ whose norm is bounded by some fixed constant, and that satisfy $L(P)\ge\varepsilon$ for some fixed $\varepsilon>0$. Recall that the transformation $\mathfrak{R}^3$ is compact, as described before Lemma 3.1. Here, and in what follows, we assume that the domain parameter ρ satisfies the condition (3.39). Consequently, the sets $K_n=\mathfrak{R}^{3n}(K_0)$ for $n>0$ all have compact closures. Denote by K_* the set of all accumulation points from the sequence $n\mapsto K_n$. This set is compact and invariant under $\mathfrak{R}_3$. By taking K_0 invariant under conjugacies by a rotation, the limit set K_* has the same property. Notice that the pairs in K_* belong to $\mathcal{F}_\rho$, so their factors are analytic.

Let $P\in K_0$ and define $P_n=\mathfrak{R}_3^n(P)$ for $n\ge 0$. By (3.3), (3.4), and (3.5), the symmetric factors associated with the pairs P_n are related via

$$N_nB_{n,\circ}(x)=A_{n-1,\circ}\bigl(\alpha^3\bigl(x-\tfrac{\alpha^{-1}}{2}\bigr)\bigr)B_{n-1,\circ}\bigl(\alpha^3x\bigr)^\dagger A_{n-1,\circ}\bigl(\alpha^3\bigl(x+\tfrac{\alpha^{-1}}{2}\bigr)\bigr)\,, \tag{7.2}$$

and

$$\begin{aligned}M_nA_{n,\circ}(x)=A_{n-1,\circ}&\bigl(\alpha^3\bigl(x+\alpha^{-1}\bigr)\bigr)^\dagger B_{n-1,\circ}\bigl(\alpha^3\bigl(x+\tfrac{\alpha^{-1}}{2}\bigr)\bigr)\times\\ &\times A_{n-1,\circ}\bigl(\alpha^3x\bigr)^\dagger B_{n-1,\circ}\bigl(\alpha^3\bigl(x-\tfrac{\alpha^{-1}}{2}\bigr)\bigr)A_{n-1,\circ}\bigl(\alpha^3\bigl(x-\alpha^{-1}\bigr)\bigr)^\dagger\,.\end{aligned} \tag{7.3}$$

Here, M_n and N_n are the normalization factors that appear in the definition of $P_n=\mathfrak{R}_3(P_{n-1})$. Assume now that K_0 is a set of pairs $P=(F,G)$ with $F=(1,\mathbf{1})$ and $G=(\alpha,A)$, and with the property that $A_\circ(0)=-J$. The latter condition is satisfied e.g. for the anti-reversible AM family. In this case, (3.10) implies that

$$A_{n,\circ}(0)=\pm\varepsilon_nJ\,,\qquad \varepsilon_n=\mathcal{O}\bigl(\alpha^{-3n}\bigr)\,. \tag{7.4}$$

Here, ε_n is a product of n factors N_m^{-1} or M_m^{-1} for $m=1,2,\ldots,n$. The given estimate on ε_n uses the fact that, under the iteration of $\mathcal{R}_3$, the norms grow at least as quickly as the Lyapunov exponents. Thus, as $n\to\infty$, the values at $x=0$ of the factors $A_{n,\circ}$ tend to zero, uniformly in our sequence $n\mapsto K_n$.

There are many other values of x where $A_{n,\circ}(x)$ tends to zero. An example is $x=\alpha^{-1}$. At this value of x, the last factor in (7.3) is of size $\mathcal{O}\bigl(\alpha^{-3n}\bigr)$ for large n. Thus, as $n\to\infty$, the values at $x=\alpha^{-1}$ of the factors $A_{n,\circ}$ tend to zero as well. The search for such zeros can be made more systematic as follows.

Let $P=((1,B),(\alpha,A))$ be a pair in K_*, and let $\tilde P=\bigl((1,\tilde B),(\alpha,\tilde A)\bigr)$ be the image of P under the transformation $\mathfrak{R}_3$. Denote by $\mathcal{A}$ and $\mathcal{B}$ be the set of zeros of $A_\circ$ and $B_\circ$, respectively. Then (7.2) shows that the set of zeros of $\tilde B_\circ$ includes the set

$$\tilde{\mathcal{B}}=\biggl[\alpha^{-3}\mathcal{B}\bigcup_{m=\pm1}\Bigl(\alpha^{-3}\mathcal{A}+\tfrac{m}{2}\alpha^{-1}\Bigr)\biggr]\,, \tag{7.5}$$

and (7.3) shows that the set of zeros of $\tilde{A}_\circ$ includes

$$\tilde{\mathcal{A}} = \biggl[\bigcup_{m=\pm 1}\Bigl(\alpha^{-3}\mathcal{B} + \tfrac{m}{2}\alpha^{-1}\Bigr)\bigcup_{m=0,\pm 1}\Bigl(\alpha^{-3}\mathcal{A} + m\alpha^{-1}\Bigr)\biggr]. \tag{7.6}$$

This defines a transformation that maps a pair of sets of complex numbers $\mathcal{P} = (\mathcal{B}, \mathcal{A})$ to a pair $\tilde{\mathcal{P}} = \bigl(\tilde{\mathcal{B}}, \tilde{\mathcal{A}}\bigr)$. Consider iterates $\mathcal{P}_1, \mathcal{P}_2, \ldots$ under this transformation $\mathcal{P} \mapsto \tilde{\mathcal{P}}$, starting with a pair $\mathcal{P}_0 = (\mathcal{B}_0, \mathcal{A}_0)$.

Notice that the imaginary parts of non-real points in $\mathcal{B}_0$ or $\mathcal{A}_0$ expand by a factor α^{-3} under the transformation $\mathcal{P} \mapsto \tilde{\mathcal{P}}$. Thus, the limit sets ($\limsup$ or $\liminf$) $\mathcal{B}_*$ and $\mathcal{A}_*$ are always subsets of $\mathbb{R}$.

In the case at hand, $\mathcal{A}_0$ includes 0, since $A_\circ(0) = 0$ for all pairs in K. So consider $\mathcal{B}_0 = \{\}$ and $\mathcal{A}_0 = \{0\}$. Then $\mathcal{A}_1$ includes 0 as well, due to the map $x \mapsto \alpha^{-3}x$ that appears in (7.6) for $m = 0$. So it is clear that $\mathcal{B}_{n+1} \supset \mathcal{B}_n$ and $\mathcal{A}_{n+1} \supset \mathcal{A}_n$ for all n. Thus, $\mathcal{B}_n \nearrow \mathcal{B}_\infty$ and $\mathcal{A}_n \nearrow \mathcal{A}_\infty$ for a pair of sets $\mathcal{P}_\infty = (\mathcal{B}_\infty, \mathcal{A}_\infty)$.

A zero that appears for the AM family but that is not included in the above set $\mathcal{A}_\infty$ is $x = {}^1\!/_2$. And $x = -{}^1\!/_2$ appears as well, due to anti-reversibility. To see how these zeros occur, denote by $C_n(x)$ the product of the last two factors in (7.3). An explicit computation shows that

$$C_n(x) = A_0^{*q_{3n-2}}\bigl(\alpha^{3n}x + \tfrac{q_{3n-1}}{2}\alpha\bigr). \tag{7.7}$$

Using that $q_k\alpha - q_{k-1} = (-1)^k\alpha^{k+1}$, one finds that $C_n({}^\sigma\!/_2) = J$ for $\sigma = (-1)^{3n-1}$. This implies that the factor $A_\circ$ for an anti-reversible AM pair $P \in K_*$ has a zero at $x = \pm{}^1\!/_2$. The same holds for other anti-reversible pairs $P = ((1, \mathbf{1}), (\alpha, A))$ with A of Schrödinger type (1.4).

To see how the zeros at $\pm{}^1\!/_2$ propagate under iteration of $\mathcal{P} \mapsto \tilde{\mathcal{P}}$, consider $\mathcal{B}_0 = \{\}$ and $\mathcal{A}_0 = \bigl\{-{}^1\!/_2, {}^1\!/_2\bigr\}$. Then $\mathcal{A}_1$ includes $\pm{}^1\!/_2$ as well, since $\pm{}^1\!/_2$ is a fixed point of the map $x \mapsto \alpha^{-3}x \mp \alpha^{-1}$ that appears in (7.6). Again we have $\mathcal{B}_n \nearrow \mathcal{B}_\infty$ and $\mathcal{A}_n \nearrow \mathcal{A}_\infty$ for a pair of sets $\mathcal{P}_\infty = (\mathcal{B}_\infty, \mathcal{A}_\infty)$.

7.2. The supercritical fixed point

The recursion relations (7.5) and (7.6) can be obtained more easily from pairs of commuting skew-product maps with factors that take values in $\mathrm{GL}(1, \mathbb{R})$ instead of $\mathrm{GL}(2, \mathbb{R})$. A map $g : (x, y) \mapsto (x + \alpha, a(x)y)$ of this type will again be written as $g = (\alpha, a)$. Given a pair $p = (f, g)$ of such maps $f = (-1, b)$ and $g = (\alpha, a)$, the renormalized pair $\mathcal{R}(p)$ is given by

$$\mathcal{R}(p) = \bigl(\lambda^{-1}g\lambda\,, \lambda^{-1}fg^c\lambda\bigr), \qquad \lambda(x, y) = (-\alpha x, y), \tag{7.8}$$

with the same positive integer c as in (2.1). The analogue of anti-reversibility here is the requirement that $b_\circ$ be even and $a_\circ$ odd. The analogue of a Schrödinger pair is a pair $p = (f, g)$ with components $f = (-1, 1)$ and $g = (\alpha, a)$, where a is periodic with period 1.

Let us restrict now to the case where α is the inverse golden mean. Then $c = 1$ at every RG step. If p is anti-reversible, then the relation between the zeros for $p = (f, g)$ and the zeros for $\tilde{p} = \mathcal{R}^3(p)$ is trivially given by (7.5) and (7.6).

Consider again skew product maps with factors in $\mathrm{GL}(2,\mathbb{R})$. Let P_0 be an anti-reversible AM pair with $\lambda > 1$. Consider the pairs $P_n = \mathfrak{R}_3^n(P_0)$ for $n \geq 1$ and the associated symmetric factors $B_{n,\circ}$ and $A_{n,\circ}$. These factors can be obtained iteratively via (7.2) and (7.3). The following are numerical observations.

Observation 5. As n increases, the factors $B_{n,\circ}$ and $A_{n,\circ}$ approach symmetric matrices with constant null spaces.

In other words, if we start with $K_0 = \{P\}$, then the limit set K_* appears to consist of pairs whose symmetric factors are of the form

$$B_\circ(x) = b_\circ(x) W W^\top\,, \qquad A_\circ(x) = a_\circ(x) V V^\top\,, \tag{7.9}$$

where W and V are constant unit vectors in $\mathbb{R}^2$. Notice that $b_\circ = \mathrm{tr}(B_\circ)$ has to be even and $a_\circ = \mathrm{tr}(A_\circ)$ odd. This follows from the (anti)reversibility property (7.9). The part of Observation 5 that we cannot prove is that $A_\circ$ is symmetric, and that V and W are constant. It is possible to give some formal arguments, but we will not do this here. If V and W are assumed to be constant, then the fact that F has to commute with both G and $G^\dagger$ implies that $W = V^\dagger$.

Observation 6. The sequence of pairs $n \mapsto P_n$ converges to a fixed point of $\mathfrak{R}_3$.

To be more precise, if we use the trivial y-scaling $L_2 = \mathbf{1}$, as we we have done so far in this section, then convergence is to a period 2 of $\mathfrak{R}_3$. The traces $b_\circ$ and $a_\circ$ reproduce after one step of $\mathfrak{R}_3$, but the directions V and $W = V^\dagger$ only repeat after two steps. Using instead $L_3 = L(\vartheta)$ of the form (3.12), with $\vartheta = \vartheta(P)$ chosen appropriately, the sequence $n \mapsto P_n$ converges numerically to a fixed point of $\mathfrak{R}_3$, with the properties described in Theorem 2.4.

Proof of Theorem 2.4. We give only a sketch here, since the proof follows closely the steps used in [55].

First, notice that the fixed point equation for a pair of the form described in Theorem 2.4 reduces to a fixed point equation for the functions $b_\diamond$ and $a_\diamond$. This is essentially the fixed point equation for the third power $\mathcal{R}^3$ of the operator $\mathcal{R}$ defined by (7.8), with $c = 1$, except for a constant re-normalization of the factors b and a.

For simplicity, let us re-define $b_\diamond$ and $a_\diamond$ to be the symmetric factors of the desired fixed point. Then we have good guess where their zeros are: starting with $\mathcal{B}_0 = \{\}$ and $\mathcal{A}_0 = \{-1/2, 0, 1/2\}$, an iteration of the map $\mathcal{P} \mapsto \tilde{\mathcal{P}}$ yields a limit pair $\mathcal{P}_\infty = (\mathcal{B}_\infty, \mathcal{A}_\infty)$ that should be the set of zeros associated with the symmetric factors of the fixed point.

For the actual construction, it is more convenient to start with $\mathcal{B}_0 = \{\}$ and $\mathcal{A}_0 = \frac{1}{2}\mathbb{Z}$. This corresponds to starting with the pair P whose symmetric factors are

$$B_\circ(x) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \qquad A_\circ(x) = 2\sin(2\pi x)\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \tag{7.10}$$

and then iterating $\mathfrak{R}^3$ with $L_3 = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. Notice that the symmetric factor $A_\circ$ in this equation is the limit as $\lambda \to \infty$ of the anti-symmetric AM factor (in its standard form), after dividing it by λ.

The symmetric factors that are being generated by iterating the map $(B_\circ, A_\circ) \mapsto (\hat{B}_\circ, \hat{A}_\circ)$ defined by (3.3) and (3.4) have their zeros modulo $1/2$ on the orbit of the point $x = 0$ under the translation $x \mapsto x + \alpha$ on the circle $\mathbb{T}/2 = \mathbb{R}/(2\mathbb{Z})$. Thus, by the three-gap theorem [3,2,4], the gaps between adjacent zeros on $\mathbb{T}/2$ take at most 3 distinct values. To be more specific, there are exactly three distinct gaps, except when the orbit has length q_{3n}, at which point the largest gap gets closed. So after n iterations, the resulting factor $\hat{B}_\circ$ has q_{3n-1} zeros on $\mathbb{T}/2$ with 3 distinct gaps, while the factor $\hat{A}_\circ$ has q_{3n} zeros on $\mathbb{T}/2$ with 2 distinct gaps.

The same applies to the iteration $(B_\circ, A_\circ) \mapsto (\tilde{B}_\circ, \tilde{A}_\circ)$, except that the circle gets enlarged by a factor α^{-3} at each step. Furthermore, the zeros that lie within some fixed positive angle of the origin reproduce after each step [55]. This yields the limiting sequences of zeros described in Theorem 2.4.

The symmetric factors that are being generated during this iteration have a Weierstrass representation as products, determined (up to a constant factor) by their zeros. Here, we use the fact that $b_\circ$ is even and $a_\circ$ odd, so it is possible to work with products of order $1/2$. By controlling the zero sets $\mathcal{B}_n$ and $\mathcal{A}_n$, one finds uniform convergence on compact sets to a pair of limit functions $b_\diamond$ and $a_\diamond$. For details we refer to [55], where RG fixed points have been constructed for skew-products with meromorphic factors.

The gap sizes listed in (2.9) were obtained by iterating the map $\mathcal{P} \mapsto \tilde{\mathcal{P}}$ a few times, starting with the sets $\mathcal{B}_0 = \{\}$ and $\mathcal{A}_0 = \{-1/2, 0, 1/2\}$. **QED**

The following will be used in the proof of Theorem 2.5 below.

Proposition 7.1. *Let $\mathcal{B}_0 = \{\}$. Let $X = \{0\}$ or $X = \{-1/2, 1/2\}$ or $X = \{-1/2, 0, 1/2\}$. Assume that $X \subset \mathcal{A}_1$ whenever $X \subset \mathcal{A}_0$. Then $\mathcal{A}_0 = X$ lead to the same limit pair $\mathcal{P}_\infty$ as $\mathcal{A}_0 = X + \mathbb{Z}$.*

Proof. For $n \ge 0$ define $\mathbb{T}_n = \mathbb{R}/(\alpha^{3n}\mathbb{Z})$. Consider first the case $\mathcal{A}_0 = X + \mathbb{Z}$. By construction, the points in $\mathcal{B}_n$ constitute $|X|$ orbits of length q_{3n-1} for the translation $x \mapsto x + \alpha^{3n+1}$ on the circle $\mathbb{T}_n$, and the points in $\mathcal{A}_n$ constitute $|X|$ orbits of length q_{3n}. Furthermore, the orbits are symmetric with respect to $x \mapsto -x$.

Consider now $\mathcal{A}_0 = X$ instead of $\mathcal{A}_0 = X + \mathbb{Z}$. Then we must get the same orbits on $\mathbb{T}_n$, simply by counting points. For $m > 0$, denote by $a(n, m)$ and $b(n, m)$ the number of points in $\mathcal{A}_n$ and $\mathcal{B}_n$, respectively, that lie within a distance m from the origin in $\mathbb{T}_n$.

Next, construct $\mathcal{B}_n$ and $\mathcal{A}_n$ as subsets of $\mathbb{R}$ instead of $\mathbb{T}_n$. Now the number of points in $\mathcal{B}_n$ or $\mathcal{A}_n$ that lie within a distance m of the origin can be smaller than $b(n, m)$ or $a(n, m)$, respectively. However, this does not happen if n is sufficiently large. The reason is that map $\mathcal{P} \mapsto \tilde{\mathcal{P}}$ is "expanding" by a factor α^{-3}. To be more precise, let $c > \alpha^3$. Then for m larger than some constant that only depends on c, the points in $\tilde{\mathcal{B}}_n \cap [-m, m]$ and $\tilde{\mathcal{A}}_n \cap [-m, m]$ are determined via $\mathcal{P} \mapsto \tilde{\mathcal{P}}$ from the points in $\mathcal{B}_n \cap [-cm, cm]$ and $\mathcal{A}_n \cap [-cm, cm]$, if n is sufficiently large. This shows that $\mathcal{A}_0 = X$ leads to the same limit pair $\mathcal{P}_\infty$ as $\mathcal{A}_0 = X + \mathbb{Z}$. **QED**

Notice that $\liminf_n \mathcal{B}_n = \limsup_n \mathcal{B}_n$ and $\liminf_n \mathcal{A}_n = \limsup_n \mathcal{A}_n$ in the above.

Proof of Theorem 2.5. Consider the choice $M(B) = \|B_\circ\|$ and $N(A) = \|A_\circ\|$ in the definition (7.1) of our factor-normalization. Let $P \in K_*$. Given that P belongs to $\mathcal{F}_\rho$

and lies in the range of $\mathfrak{R}_3^n$ for every n, the factors B and A extend analytically to all of $\mathbb{C}$. Furthermore, $b_\circ = \mathrm{tr}(B_\circ)$ is even and $a_\circ = \mathrm{tr}(A_\circ)$ odd, due to the anti-reversibility property (7.9).

By construction, the symmetric factors $B_\circ$ and $A_\circ$ have norm 1 and determinant 0.

The claim in Theorem 2.5 concerning the zeros of B and A follows from our discussion in the previous subsection, together with the characterization of the set of zeros for the functions $b_\diamond$ and $a_\diamond$ given in the proof of Theorem 2.4, as well as Proposition 7.1. **QED**

Remark 7. If one assumes that some point P on the unstable manifold of $\mathfrak{R}_3$ at P_* converges to $P_\diamond$ under iteration of $\mathfrak{R}_3$, then it is possible to show that $\mu_2 = \alpha^{-3}$. The reason is that the asymptotic behavior under renormalization becomes trivial in this case.

8. Computer estimates

What remains to be done is to verify the estimates in Lemmas 3.1, 4.1, 4.2, 5.1, and 5.5. This is carried out with the aid of a computer. This part of the proof is written in the programming language Ada [57] and can be found in [56]. The following is meant to be a rough guide for the reader who wishes to check the correctness of our programs.

8.1. Enclosures and data types

Bounds on a vector x in a space $\mathcal{X}$, also referred to as enclosures for x, are given here by sets $X \subset \mathcal{X}$ that include x and are representable as data on a computer. Data of type `Ball` are pairs $\texttt{B} = (\texttt{B.C}, \texttt{B.R})$, where `B.C` and `B.R` are representable numbers, with $\texttt{B.R} \ge 0$. In a Banach algebra $\mathcal{X}$ with unit $\mathbf{1}$, the enclosure associated with a `Ball` `B` is the ball $\texttt{B}_{\mathcal{X}} = \{x \in \mathcal{X} : \|x - (\texttt{B.C})\mathbf{1}\| \le \texttt{B.R}\}$. Our other enclosures are closed convex subsets of $\mathcal{X}$ that admit a canonical decomposition

$$S = \sum_n x_n \texttt{B(n)}_{\mathcal{X}} \,, \tag{8.1}$$

where each x_n is a representable element in $\mathcal{X}$, and where each `B(n)` is a `Ball` with center $\mathbf{0}$ or $\mathbf{1}$. Notice that a `Ball` can have radius zero.

Consider now a disk $D = \{z \in \mathbb{C} : |z| < \rho\}$ with representable radius $\rho > 0$. An analytic function $g : D \to \mathcal{X}$ admits a Taylor series representation $g(z) = \sum_{n=0}^\infty g_n z^n$ with coefficients $g_n \in \mathcal{X}$. Denote by $\mathcal{G}$ the space of all such functions that have a finite norm $\|g\| = \sum_{n=0}^\infty \|g_n\|\rho^n$. Assume that $\mathcal{X}$ carries a type of enclosures named `Scalar`. To each such enclosure $\texttt{S}_{\mathcal{X}} \subset \mathcal{X}$ we can associate a set $\texttt{S}_{\mathcal{G}} \subset \mathcal{G}$ by replacing each ball $\texttt{B(n)}_{\mathcal{X}}$ in the decomposition (8.1) of $S = \texttt{S}_{\mathcal{X}}$ by the ball $\texttt{B(n)}_{\mathcal{G}}$.

Given an integer $D > 0$, our enclosures for functions in $\mathcal{G}$ are specified by data of type `Taylor1`. A `Taylor1` is in essence a pair `P=(P.F,P.C)`, where $\texttt{P.F} \le D$ is a nonnegative integer, and where `P.C` is an `array(0 .. D) of Scalar`. The corresponding enclosure is the set

$$\texttt{P}_{\mathcal{G}} = \sum_{n=0}^{m-1} \texttt{P.C(n)}_{\mathcal{X}} \mathcal{P}_n + \sum_{n=m}^{D} \texttt{P.C(n)}_{\mathcal{G}} \mathcal{P}_n \,, \qquad \mathcal{P}_n(z) = z^n \,, \tag{8.2}$$

where $m =$ `P.F`. Notice that the first sum in (8.2) is a polynomial, while each term in the second sum is in general a non-polynomial. This allows for efficient estimates in the problem considered here. For precise definitions we refer to the Ada package `Taylors1`.

Quadruples of `Taylor1` define a type `TMat2` that is used for enclosures of 2×2 matrices A with entries in $\mathcal{G}$. By including another component $\alpha \in \mathbb{Z}\bigl[\frac{1}{2}\sqrt{5} - \frac{1}{2}\bigr]$ we obtain enclosures for skew-product maps $G = (\alpha, A)$. The corresponding data type is named `Skew`. Enclosures on pairs $P = (F, G)$ are described by data of type `Skew2`. For details we refer to the child package `Taylors1.Skews2`.

In most of our packages, the type `Scalar` is generic, meaning unspecified, except for a list of available operations. When instantiated with `Scalar => Ball`, the packages `Taylors1` and `Taylors1.Skews2` define enclosures for the spaces $\mathcal{G}_\rho$, $\mathcal{G}_\rho^4$, and $\mathcal{F}_\rho$ described in Subsection 3.5.

A particular instantiation of `Taylors1`, with `Scalar => Ball`, is named `S_T` in the package `FamRG`. The resulting type `Taylor1` is named `TScalar`. A second instantiation of `Taylors1`, with `Scalar => TScalar`, is named named `T_T`. The type of `Taylor1` defined by `T_T` describes analytic functions from a disk $|s| < \delta$ to $\mathcal{G}_\rho$. Now it suffices to instantiate the child package `T_T.Skews2`, to obtain `Skew2`-type enclosures on analytic curves in the space $\mathcal{F}_\rho$. This covers all major data types used in our programs.

8.2. Bounds and procedures

After having defined enclosures for (elements in) the various spaces that are needed in our analysis, we need to implement bounds on maps between these space. In this context, a bound on a map $f : \mathcal{X} \to \mathcal{Y}$ is a function F that assigns to a set $X \subset \mathcal{X}$ of a given type (`Xtype`) a set $Y \subset \mathcal{Y}$ of a given type (`Ytype`), in such a way that $y = f(x)$ belongs to Y whenever $x \in X$. In Ada, such a bound F can be implemented by defining an appropriate `procedure F(X: in Xtype; Y: out Ytype)`. In practice, the domain of `F` is restricted. If `X` does not belong to the domain of `F`, the `F` raises an `Exception` which causes the program to abort.

Our type `Ball` is defined in the package `MPFR.Floats.Balls`, using centers `B.C` of type `MPFloat` and radii `B.R` of type `LLFloat`. Data of type `MPFloat` are high-precision floating point numbers, and the elementary operations for this type are implemented by using the open source MPFR library [60]. Data of type `LLFloat` are standard extended floating-point numbers [59] of the type commonly supported by hardware. Both types support controlled rounding. Bounds on the basic operations for this type `Ball` are defined and implemented in `MPFR.Floats.Balls`.

Using the definition (8.2) of a `Taylor1`-type enclosure, it is clearly possible to implement a bound `Prod` on the map $(f, g) \mapsto f * g$ from $\mathcal{G} \times \mathcal{G}$ to $\mathcal{G}$. This and other basic bounds that include the type `Taylor1` are defined in package `Taylors1`. Basic bounds that involve the types `Skew` and `Skew2` are defined in the package `Taylors1.Skew2`. This includes a bound `Normalize` on the map $\mathfrak{N}$ defined by (3.17). It also includes a bound `Inv` on the map $G \mapsto G^\dagger$ and a bound `Prod_GFG` on the product $(F, G) \mapsto GFG$. Combining the two yields bounds on the products that appear in the definition of the operator $\mathfrak{R}_3$.

Bounds on problem-specific maps such as $\mathfrak{R}_3$ are mostly defined in child packages of `Taylors1.Skew2`. Among the exceptions are the bounds named `Equalize` on the normalization transformations defined in Subsections 3.2 and 4.1. The package `Taylors1.Skew2.RG3r` implements bounds on operations such as $\mathfrak{R}_6$ that are used only for reversible pairs, while `Taylors1.Skew2.RG3a` implements bounds that are specific to anti-reversible pairs. This includes procedures named `Commutize` that provide bounds on the commutator-correction map $\mathfrak{C}$. A bound on the derivative of $\mathfrak{C}$ is named `DCommmutize`. The same naming convention is used for other derivative bounds.

Up to this level, the same bounds can be used for pairs and for families of pairs. To choose one or the other, is suffices to instantiate the package `Taylors1` and its children with the desired type of `Scalar`. But for derivative bounds on maps such as $\mathfrak{M}$, we need to be able to enumerate the degrees of freedom, so it matters whether `Scalar` encloses a number or a Taylor series.

8.3. Linear operators and modes

In the packages `MapR`, `FamRG`, and their children, degrees of freedom are associated with a type `Mode`. To simplify the discussion, consider first the space $\mathcal{G}_\rho$. In this case, a "coefficient mode" c_n represents a monomial $\mathcal{P}_n$ of degree n, and an "error mode" e_n represents the unit ball in the subspace of all functions $g = e\mathcal{P}_n$ with $e \in \mathcal{G}_\rho$. So `Taylor1`-type enclosure (8.2) in $\mathcal{G}_\rho$ is a finite linear combination of modes for $\mathcal{G}_\rho$. A proper collection of normalized modes $\{h_1, h_2, \ldots, h_m\}$ defines an analogue of a finite basis for $\mathcal{G}_\rho$. Due to our choice of norm in $\mathcal{G}_\rho$, the operator norm of a bounded linear $T : \mathcal{G}_\rho \to \mathcal{G}_\rho$ is simply $\|T\| = \max_n \|Th_n\|$. Given that each mode h_n admits a representation of type `Taylor1`, a bound on $\|T\|$ is easily obtained from a bound on T.

This generalizes readily to the space $\mathcal{F}_\rho^r$. The corresponding type `CMode` is defined in the package `MapRG`. For a "basis" of such modes we use the type `CModes`. In `MapRG` we also instantiate two generic packages `Linear` and `Linear.Contr` that implement bounds on a quasi-Newton map (and its derivative) of the type (3.36) in terms of bounds on the given map (and its derivative). Using these facilities, the child package `MapRG.RG3a` implements a bound `DContrNorm` on the norm of $D\mathfrak{M}$, where $\mathfrak{M}$ is the transformation defined by (3.36). This procedure is used to verify the bound on the operator norm of $D\mathfrak{M}(p)$ in Lemma 3.1.

The operator M that enters the definition (3.36) is a "matrix" on a subspace spanned by finitely many coefficient modes, and on the complementary subspace it is the zero operator. This matrix is included in [56] as a data file `Contr3aMat.132`. The matrix M that is used in the definition (4.9) is included in [56] as a file `Contr3rMat.134`. The operator $\mathcal{L}$ that enters the definition (5.1) is a "matrix" L on a subspace spanned by finitely many coefficient modes, and on the complementary subspace it is the identity operator. This matrix L is included in the file `Iso.132.28`. A bound on its inverse is obtained and saved by running the program `Invert_Iso`.

The child package `MapRG.RG3r` is an analogue of `MapRG.RG3a`, but for reversible pairs.

The package `FamRG` and its children define an analogue of the `MapRG` hierarchy, but for analytic curves $s \mapsto \mathtt{P}(s)$. The modes for this space are named `DModes`. However, these modes are not needed in our current proof. (An earlier version implemented the graph transform method that is described in Subsection 5.2.) The child package `FamRG.RG3a` implements the bounds that are used in our proof of Lemma 5.5. This includes a bound `Plain_FamRG3` on the transformation $\mathfrak{F}_c$ defined in (5.12), for the special value $c = \alpha^3$.

8.4. Organizing the bounds

Our proof of Lemma 5.5 is organized in the main program `Iter_AM_Fam_Sin6`. Starting with an enclosure for the anti-reversible AM family for λ near 1, it does little more than iterating the above-mentioned bound `Plain_FamRG3`. The bounds are in essence numerical computations, but, as should be clear by now, they include rigorous estimates of truncation errors and rounding errors.

The same is true for our proof of the remaining lemmas. The approximate solutions $\bar{P}$ referred to in Lemmas 3.1 and 4.1 have been computed numerically beforehand. First we used a numerical versions of `Iter_AM_Fam_Sin6` (or `Iter_AM_Fam_Cos6` in the reversible case) to obtain rough approximate fixed points for $\mathfrak{R}_3^2$. Then the approximations were improved via the procedures `IterContr` in the packages `MapRG.RG3a` and `MapRG.RG3r`, respectively. (Numerical versions of our programs are obtained simply by using for `Scalar` the type `Rep` instead of `Ball`.) The results are in the data files `approx-Fix3a.trunc` and `approx-Fix6r.trunc` in [56].

The main programs that are used to verify the estimates in Lemmas 3.1, 4.1, and 5.1 are `Check_RG3a_Fixpt`, `Check_RG6r_Fixpt`, and `CheckNorms_DRGN3`. They do little more than instantiating the required packages with the appropriate parameters, reading data files if needed, and then handing the task over to the proper procedure(s) in the instantiated packages.

Our programs were run successfully on a standard desktop machine, using a public version of the gcc/gnat compiler [58]. Instructions on how to compile and run these programs can be found in the file `README` that is included with the source code in [56].

References

[1] P.G. Harper, *Single band motion of conduction electrons in a uniform magnetic field*, Proc. Phys. Soc. Lond. A **68**, 874–892 (1955).

[2] V.T. Sós, *On the distribution* mod 1 *of the sequence* $n\alpha$, Ann. Univ. Sci. Budapest, Eötvös Sect. Math. **1**, 127–134 (1958).

[3] J. Surányi, *Über die Anordnung der Vielfachen einer reelen Zahl* mod 1, Ann. Univ. Sci. Budapest, Eötvös Sect. Math. **1**, 107–111 (1958).

[4] S. Świerczkowski, *On successive settings of an arc on the circumference of a circle*, Fundamenta Mathematicae **46**, 187–189 (1959).

[5] D.R. Hofstadter, *Energy levels and wave functions of Bloch electrons in rational and irrational magnetic fields*, Phys. Rev. B **14**, 2239–2249 (1976).

[6] M.W. Hirsch, C.C. Pugh, M. Shub, *Invariant Manifolds.* Lecture Notes in Mathematics, Volume **583**, Springer Verlag, Berlin · Heidelberg · New York (1977).

[7] M.J. Feigenbaum, *Quantitative universality for a class of non-linear transformations*, J. Stat. Phys. **19**, 25–52 (1978).

[8] P. Coullet, C. Tresser, *Itération d'endomorphismes et groupe de renormalisation*, J. Phys. Colloque C **539**, C5–25 (1978).

[9] L.P. Kadanoff, *Scaling for a critical Kolmogorov–Arnold–Moser trajectory.* Phys. Rev. Lett. **47**, 1641–1643 (1981).

[10] D.J. Thouless, M. Kohmoto, M.P. Nightingale, M. den Nijs, *Quantized Hall conductance in a two-dimensional periodic potential*, Phys. Rev. Lett. **49**, 405–408 (1982).

[11] R.S. MacKay, *Renormalisation in Area Preserving Maps.* Thesis, Princeton (1982). World Scientific, London (1993).

[12] J. Bellissard, B. Simon, *Cantor spectrum for the almost Mathieu equation*, J. Funct. Anal. **48**, 408–419 (1982).

[13] R. Johnson, J. Moser, *The rotation number for almost periodic potentials*, Commun. Math. Phys. **84**, 403–438 (1982).

[14] J. Palis, W. de Melo, *Geometric Theory of Dynamical Systems. An Introduction.* Springer-Verlag, Berlin · New York (1982).

[15] S. Ostlund, D. Rand, J. Sethna, E. Siggia, *Universal properties of the transition from quasiperiodicity to chaos in dissipative systems*, Physica, **8D**, 303-342 (1983).

[16] J. Avron, B. Simon, *Almost periodic Schrödinger operators. II. The integrated density of states*, Duke Math. J. **50**, 369–391 (1983).

[17] J.-P. Eckmann, H. Koch, P. Wittwer, *A computer-assisted proof of universality for area-preserving maps*, Mem. Amer. Math. Soc., Vol. **47**, No. 289, 1–121 (1984).

[18] J.-P. Eckmann, P. Wittwer, *A complete proof of the Feigenbaum conjectures*, J. Stat. Phys. **46**, 455–475 (1987).

[19] O.E. Lanford III, *Renormalization group method for critical circle mappings*, Nonlinear evolution and chaotic phenomena, G. Gallavotti and P.F. Zweifel (eds.), NATO adv. Sci. Inst. Ser. B **176**, Plenum, New York, 25–36 (1988).

[20] E. de Faria, *Proof of universality for critical circle mappings*, Thesis, CUNY (1992).

[21] L. Eliasson, *Floquet solutions for the 1-dimensional quasi-periodic Schrdinger equation*, Comm. Math. Phys. **146**, 447–482 (1992).

[22] D. Sullivan, *Bounds, quadratic differentials and renormalization conjectures*, AMS Centennial Publications 2: Mathematics into twenty-first century (1992).

[23] P.B. Wiegmann, A.V. Zabrodin, *Quantum group and magnetic translations. Bethe ansatz solution for the Harper's equation*, Modern Phys. Lett. B **8**, 311–318 (1994).

[24] Y. Last, *Zero measure spectrum for the almost Mathieu operator*, Comm. Math. Phys. **164**, 421–432 (1994).

[25] C. McMullen, *Renormalization and 3-manifolds which fiber over the circle*, Ann. Math. Studies **135**, Princeton University Press (1994).

[26] J. Bellissard, A. van Elst, H. Schulz-Baldes, *The Non-Commutative Geometry of the Quantum Hall Effect*, J. Math. Phys. **35**, 5373–5451 (1994).

[27] L.D. Faddeev, R.M. Kashaev, *Generalized Bethe ansatz equations for Hofstadter problem*, Commun. Math. Phys. **169**, 181–191 (1995).

[28] Y. Hatsugai, M. Kohmoto, Y.-S. Wu, *Quantum group, Bethe ansatz equations, and Bloch wave functions in magnetic fields*, Phys. Rev. B **53**, 9697–9712 (1996).

[29] A. Furman, *On the multiplicative ergodic theorem for uniquely ergodic systems*, Ann. Inst. Henri Poincaré **33**, 797–815 (1997).

[30] A.Y. Gordon, S. Jitomirskaya, Y. Last, B. Simon, *Duality and singular continuous spectrum in the almost Mathieu equation*, Acta Math. **178**, 169–183 (1997).

[31] A. Rüdinger, F. Piéchon, *Hofstadter rules and generalized dimensions of the spectrum of Harper's equation*, J. Phys. A **30**, 117–128 (1997).

[32] M. Lyubich, *Feigenbaum-Coullet-Tresser Universality and Milnor's Hairiness Conjecture*, Ann. Math. **149**, 319–420 (1999).

[33] S. Jitomirskaya, *Metal-insulator transition for the almost Mathieu operator*, Ann. of Math. **150**, 1159–1175 (1999).

[34] B.D. Mestel, A.H. Osbaldestin, B. Winn, *Golden mean renormalization for a generalized Harper equation: The strong coupling fixed point*, J. Math. Phys. *41* 8304–8330 (2000).

[35] D. Osadchy, J.E. Avron, *Hofstadter butterfly as quantum phase diagram*, J. Math. Phys. **42**, 5665–5671 (2001).

[36] M Yampolsky, *Hyperbolicity of renormalization of critical circle maps*, Publ. Math. Inst. Hautes Études Sci. **96**, 1–41 (2002).

[37] A. De Carvalho, M. Lyubich, M. Martens, *Renormalization in the Hénon Family, I: Universality But Non-Rigidity*, J. Stat. Phys. **121**, 611–669 (2005).

[38] D. Damanik, *The spectrum of the almost Mathieu operator*, Lecture series in the CRC 701 (2008), Preprint, 2009, arXiv:0908.1093.

[39] M. Goldstein and W. Schlag, *Fine properties of the integrated density of states and a quantitative separation property of the Dirichlet eigenvalues*, Geom. Funct. Anal. **18**, 755–869 (2008).

[40] A. Avila, *The absolutely continuous spectrum of the almost Mathieu operator*, Preprint 2008.

[41] V. Oseledets, *Oseledets theorem*, Scholarpedia **3**, 1846 (2008).

[42] D. Gaidashev, T. Johnson, *Dynamics of the universal area-preserving map associated with period doubling: hyperbolic sets*, Nonlinearity **22**, 2487–2520 (2009).

[43] A. Avila, S. Jitomirskaya, *The ten martini problem*, Ann. Math. **170**, 303–342 (2009).

[44] A. Avila, S. Jitomirskaya, *Almost localization and almost reducibility*, J. Eur. Math. Soc. **12**, 93–131 (2010).

[45] G. Arioli and H. Koch, *Integration of dissipative PDEs: a case study*, SIAM J. Appl. Dyn. Syst. **9**, 1119–1133 (2010).

[46] G. Arioli and H. Koch, *The critical renormalization fixed point for commuting pairs of area-preserving maps*, Comm Math. Phys. **295**, 415–429 (2010).

[47] A. Avila, R. Krikorian, *Monotonic cocycles*, Invent. Math. **202**, 271–331 (2015).

[48] A. Avila, *Global theory of one-frequency Schrödinger operators*, Acta Math. **215**, 1–54 (2015).

[49] I.I. Satija, *A tale of two fractals: the Hofstadter butterfly and the integral Apollonian gaskets*, Eur. Phys. J. Spec. Top. **225**, 2533–2547 (2016)

[50] H. Koch, *On hyperbolicity in the renormalization of near-critical area-preserving maps*, Discrete Contin. Dyn. Syst. **36**, 7029–7056 (2016).

[51] S. Jitomirskaya, C. Marx, *Dynamics and spectral theory of quasi-periodic Schrödinger-type operators*, Erg. Theor. Dyn. Syst. **37**, 2353–2393 (2017).

[52] H. Koch, *Golden mean renormalization for the almost Mathieu operator and related skew products*, J. Math. Phy. **62**, 1–12 (2021).

[53] H. Koch, S. Kocić, *Renormalization and universality of the Hofstadter spectrum*, Nonlinearity **33**, 4381–4389 (2020).

[54] I.I Satija, M. Wilkinson, *Nests and chains of Hofstadter butterflies*, J. Phys. A: Math. Theor. **53**, 085703 (2020).

[55] H. Koch, *On trigonometric skew-products over irrational circle-rotations*, Discrete Contin. Dyn. Syst. A. **41**, 5455–5471 (2021).

[56] H. Koch. The source code for our programs, and data files, are available at `web.ma.utexas.edu/users/koch/papers/skewunivers/`

[57] Ada Reference Manual, ISO/IEC 8652:2012(E), available e.g. at `www.ada-auth.org/arm.html`

[58] A free-software compiler for the Ada programming language, which is part of the GNU Compiler Collection; see `gnu.org/software/gnat/`

[59] *IEEE Standard for Floating-Point Arithmetic*, in IEEE Std 754-2019 (Revision of IEEE 754-2008), pp.1–84, 22 July 2019, doi: 10.1109/IEEESTD.2019.8766229

[60] The MPFR library for multiple-precision floating-point computations with correct rounding; see www.mpfr.org/