Periodic and quasiperiodic waves on the sphere

Gianni Arioli [1] and Hans Koch [2]

Abstract. We construct periodic and quasiperiodic solutions for the nonlinear plate equation on the unit sphere in $\mathbb{R}^3$. Periodic solutions for the nonlinear wave equation are constructed as well. The methods do not involve small parameters. In the case of a cubic nonlinear term, our approach is computer-assisted and thus yields detailed information about each solution. For nonlinearities that are in some sense close to linear, weak solutions are obtained by a variational method.

1. Introduction and main results

We consider the wave equation ($\nu = 1$) and plate equation ($\nu = 2$)

$$\partial_\tau^2 v + (-\Delta_{\mathbb{S}})^\nu v = f(v)\,, \tag{1.1}$$

for a function $v = v(x,\tau)$ on the product $\mathbb{S} \times \mathbb{R}$ of the sphere $\mathbb{S} = \bigl\{x \in \mathbb{R}^3 : |x| = 1\bigr\}$ with the real line. Here, $\Delta_{\mathbb{S}}$ denotes the Laplacean on $\mathbb{S}$, and f is a nonlinear function on $\mathbb{R}$. In this paper we focus on solutions that are periodic or quasiperiodic in time τ.

We parameterize the sphere via Euler angles $\vartheta \in [0,\pi]$ and $\varphi \in [-\pi,\pi]$, writing $v = v(\vartheta,\varphi,\tau)$. The corresponding point on the sphere is $x = (\sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi, \cos\vartheta)$.

Stationary solutions of the equation (1.1) are of the from $v(\vartheta,\varphi,\tau) = u(\vartheta,\varphi)$, for some function u on the sphere that satisfies

$$\mathcal{L}u = f(u)\,, \qquad \mathcal{L} = (-\Delta_{\mathbb{S}})^\nu\,. \tag{1.2}$$

The first three plots in Figure 1 depict solutions of this equation, with $\nu = 1$ and $f(u) = u^3$. Our plots use the sinusoidal projection, also known as Mercator equal-area projection. The coordinates used are $\xi = \varphi\sin\vartheta$ horizontally and $\eta = \pi/2 - \vartheta$ vertically. The value of u is indicated by colors and contour lines.

Some of the properties of these stationary solutions will be described below. For results on semilinear equations $-\Delta u = f(u)$ on planar domains we refer to [25,30] and references therein.

To find time-periodic solutions with period $T > 0$, we change variables to $t = \beta\tau$, with $\beta = 2\pi/T$, and consider functions that are 2π-periodic in t. In the simplest case, which we refer to as "periodic rotating waves", we have $v(\vartheta,\varphi,\tau) = u(\vartheta,\varphi+\beta\tau)$ for some function u on the sphere. The equation for $u = u(\vartheta,\varphi)$ is

$$\mathcal{L}u = f(u)\,, \qquad \mathcal{L} = \beta^2\partial_\varphi^2 + (-\Delta_{\mathbb{S}})^\nu\,. \tag{1.3}$$

The last plot in Figure 1 shows a solutions of this equation, with $\nu = 2$ and $\beta = \frac{8}{7}$. Some of its properties will be described below. The solutions in Figure 1 will be referred to as solutions 1,2,3, and 4. Here, and in what follows, we assume that $f(u) = u^3$, unless specified otherwise.

[1] Department of Mathematics, Politecnico di Milano, Piazza Leonardo da Vinci 32, 20133 Milano.

[2] Department of Mathematics, University of Texas at Austin, Austin, TX 78712

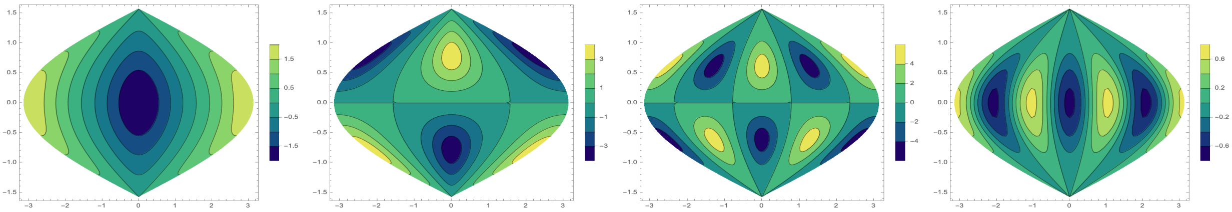

Figure 1. Solutions 1,2,3 of equation (1.2) and Solution 4 of equation (1.3).

For more general periodic solutions we make the ansatz $v(\vartheta, \varphi, \tau) = u(\vartheta, \varphi, \alpha\tau)$, where u is a function on $\mathbb{S} \times \mathbb{T}$, with $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$. The resulting equation for $u = u(\vartheta, \varphi, t)$ can be written as

$$\mathcal{L}u = f(u), \qquad \mathcal{L} = \alpha^2 \partial_t^2 + (-\Delta_{\mathbb{S}})^\nu. \tag{1.4}$$

Three solutions of this equation, referred to as solutions 5,6, and 7, are depicted in Figure 2. The corresponding values of ν and α are listed in Table 1. To be more precise, we only plot their values at time $t = 0$, since these solutions are close to (but different from) functions of the form $(x, t) \mapsto X(x)T(t)$. Additional data are given in Section 7. The solution numbers in Figure 2 and subsequent figures are links to short animations of the corresponding solutions.

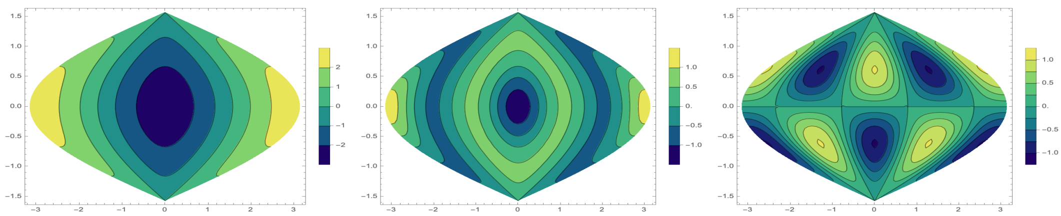

Figure 2. Solutions 5,6,7 of equation (1.4) at time $t = 0$.

One of our goals is to prove the existence of these periodic solutions. (Quasiperiodic solutions will be considered below.) Starting from an approximate numerical solution $\bar{u}$, we show that there exists a true solution u nearby. This is done with the aid of a computer. Similar techniques have been used in [20,22] for the nonlinear wave and beam equations on an interval. Computer-assisted proofs for the existence of periodic solutions of some other evolution-type PDEs are given e.g. in [19,21,27,28].

The existence of periodic solutions for some nonlinear parabolic equations can be proved also by KAM type methods (for small amplitudes) or variational methods. For specific results we refer to [14,15,16,17,18,24,26] or [2,3,4,5,6,7,8,9,10,11,12,13,16], respectively, and references therein.

In what follows, α is assumed to be a positive rational number. We consider solutions u in a space $\mathcal{B}_k$ of real analytic functions, imposing certain symmetries which guarantee that $\mathcal{L}$ has a bounded inverse on $\mathcal{B}_k$. The symmetries considered are of the following type.

Definition 1. We say that $u : \mathbb{S} \times \mathbb{T} \to \mathbb{C}$ has even time-parity, if $u \circ \mathcal{R} = u$ for the time reflection $\mathcal{R} : (\vartheta, \varphi, t) \mapsto (\vartheta, \varphi, -t)$. We say that u has even mixed-parity, if $u \circ \mathcal{R}' = u$ for the mixed reflection $\mathcal{R}' : (\vartheta, \varphi, t) \mapsto (\vartheta, -\varphi, -t)$. We say that u has even frequency-parity, if $u \circ \mathcal{R}'' = u$ for the time-translation $\mathcal{R}'' : (\vartheta, \varphi, t) \mapsto (\vartheta, \varphi, t + \pi)$. Odd parities are defined analogously. Furthermore, if k is a positive integer, we say that u is k-antisymmetric, if $u \circ R_k = -u$ for the rotation $R_k : (\vartheta, \varphi, t) \mapsto (\vartheta, \varphi + \pi/k, t)$.

We will restrict to nonlinearities f that are odd. Then $u \mapsto f(u)$ preserves time-parity, mixed-parity, frequency-parity, and k-antisymmetry.

For quasiperiodic solutions of (1.1), we make the ansatz $v(\vartheta, \varphi, \tau) = w(\vartheta, \varphi, \tau\omega)$, where $w = w(\vartheta, \varphi, y)$ is a function on $\mathbb{S} \times \mathbb{T}^2$, and where $\omega = (\beta, \alpha)$ with $\beta > 0$ an irrational number. To simplify the analysis, we restrict to solutions that represent rotating waves, meaning that $w(\vartheta, \varphi, (s,t)) = u(\vartheta, \varphi + s, t)$, for some function u on $\mathbb{S} \times \mathbb{T}$. The equation (1.1) reduces to

$$\mathcal{L}u = f(u)\,, \qquad \mathcal{L} = (\alpha\partial_t + \beta\partial_\varphi)^2 + (-\Delta_{\mathbb{S}})^\nu\,. \tag{1.5}$$

Notice that this equation is not invariant under time-reflection. Thus, in the quasiperiodic case, we only allow "mixed" as a possible parity.

We consider quasiperiodic solutions only for the plate equation, meaning $\nu = 2$. To avoid excessively small denominators when inverting $\mathcal{L}$, we restrict to irrationals β of bounded type. In fact, unless stated otherwise, $\beta = |i + \theta_j|$ for some integers i and $j > 0$, where $\theta_j > 0$ is defined by the equation $\theta_j = 1/(j + \theta_j)$. Notice that θ_1 is the inverse golden mean.

Figure 3 depicts three solutions of the equation (1.5) for $\nu = 2$. The corresponding values of α and β are listed in rows 8,9, and 10 of Table 1. To save space, only the values at time $t = 0$ are shown in Figure 3. But additional data are given Section 7.

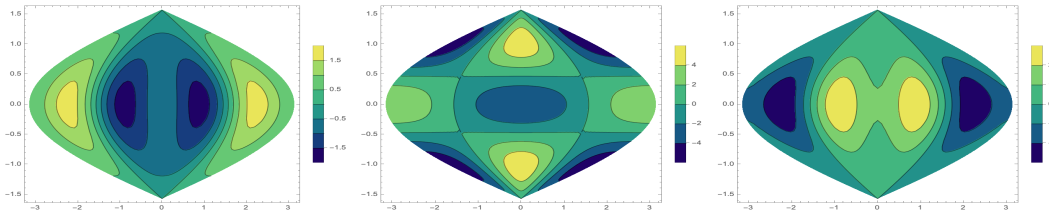

Figure 3. Solutions 8,9,10 of equation (1.5) at time $t = 0$.

Finding nontrivial quasiperiodic solutions turned out to be quite difficult. Solution 8 was found by chance, and the others are the result of extensive random searches. Most numerical solutions that we found were not accurate enough at a reasonable level of truncation. A likely cause for this are near-resonances at high frequencies. A quasiperiodic solution that seems particularly simple corresponds to row 11 of Table 1. Figure 4 shows snapshots of this solution at times $t_j = \pi j/4$ for $j = 0, 1, \ldots, 7$.

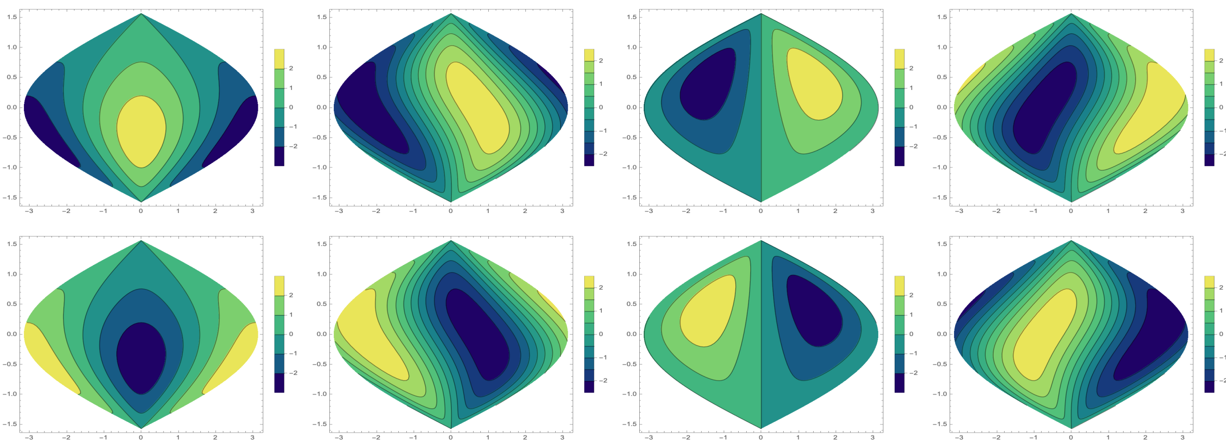

Figure 4. Solution 11 of equation (1.5) at times $t_j = \pi j/4$ for $j = 0, 1, \ldots, 7$.

In Section 2 we will define a Banach algebra $\mathcal{A}$ of analytic functions $u:\mathbb{S}\times\mathbb{T}\to\mathbb{C}$ that admit an expansion

$$u(\vartheta,\varphi,t)=\sum_{l,m,n}u_{l,m,n}\mathcal{Y}_l^m(\vartheta,\varphi)e^{int}\,,\qquad \mathcal{Y}_l^m(\vartheta,\varphi)=\mathcal{P}_l^m(\vartheta)e^{im\varphi}\,,\tag{1.6}$$

with exponentially decreasing coefficients $u_{l,m,n}$. The sum in this equation ranges over all integers l,m,n that satisfy $|m|\le l$. The functions $\mathcal{Y}_l^m$ are, modulo constant factors, the standard spherical harmonics, and the functions $\mathcal{P}_l^m$ are related to the associated Legendre functions. $\mathcal{Y}_l^m$ is an eigenfunction of the negative Laplacean $-\Delta_{\mathbb{S}}$ with eigenvalue $l(l+1)$.

The operator $\mathcal{L}$ is "diagonal" in the representation (1.6), with eigenvalues

$$\lambda_{l,m,n}=l^\nu(l+1)^\nu-(\alpha n+\beta m)^2\,,\qquad |m|\le l\,.\tag{1.7}$$

In order to avoid zero eigenvalues, we restrict our analysis to k-antisymmetric functions, as described in Definition 1. This is equivalent to the condition that $u_{l,m,n}=0$ unless $m\equiv k\pmod{2k}$.

The space of real-valued functions in $\mathcal{A}$ is denoted by $\mathcal{B}$. When no confusion can arise, we use the same symbols for the spaces that describe functions on $\mathbb{S}$ only. This corresponds to restricting the sum in (1.6) to $n=0$. The equations (1.2), (1.3), and (1.4) can be regarded as special cases of (1.5), if we set α and/or β equal to zero. In the same sense, the equation (1.7) describes the eigenvalues of $\mathcal{L}$ in all cases considered here.

Convention 2. All periodic (quasiperiodic) solutions considered in this paper have even time-parity (mixed-parity). And all non-stationary solutions have odd frequency-parity. In addition, we restrict to functions u that are odd under mixed-reflection (time-reflection) when considering quasiperiodic periodic (non-stationary periodic) solutions. The subspace of $\mathcal{B}$ characterized by these parities will be denoted by $\mathcal{B}_0$. The space of k-antisymmetric functions in $\mathcal{B}_0$ will be denoted by $\mathcal{B}_k$.

label	type	ν	α	β	k	$\updownarrow$	$\leftrightarrow$	norm
1	s	1			1	e	e	3.0979 ...
2	s	1			1	o	e	7.5742 ...
3	s	1			2	o	e	12.433 ...
4	r	2		8/7	3	e	e	1.2606 ...
5	p	2	3/5		1	e	e	5.9524 ...
6	p	1	17/5		1	e	e	2.9935 ...
7	p	1	17/5		2	o	e	2.1303 ...
8	q	2	2	$(\sqrt{5}-1)/2$	1	e		8.4941 ...
9	q	2	2	$\sqrt{2}$	1	e		22.955 ...
10	q	2	3/2	$\sqrt{2}$	1	e		16.275 ...
11	q	2	1	$2-\sqrt{2}$	1			12.301 ...

Table 1. Parameter values and properties of solutions.

The following theorem concerns the solutions depicted in Figures 1,2,3,4, that use the parameter values listed in Table 1 and $f(u)=u^3$. The label entries in this table links to a plot or animation of the given solution.

Theorem 1.1. *For each row in Table 1 there exists a real analytic solution w of the equation (1.1) with the properties indicated in the given row. Column 3 specifies the exponent ν in the equation (1.1). All solutions are k-antisymmetric, for the value of k that is specified in column 6. Column 2 describes the type of solution: an entry "s" indicates a stationary solution of the equation (1.1), given by a function $u \in \mathcal{B}_k$ satisfying (1.2); an entry "r" indicates a non-stationary rotating periodic solution, given by a function $u \in \mathcal{B}_k$ satisfying (1.3); an entry "p" indicates a non-stationary non-rotating periodic solution, given by a function $u \in \mathcal{B}_k$ satisfying (1.4); and an entry "q" indicates a nonperiodic quasi-periodic solution, given by a function $u \in \mathcal{B}_k$ satisfying (1.5). The values of α and/or β are given in Columns 4 and/or 5, respectively. The periodic (quasiperiodic) solutions all have even time-parity (mixed-parity), and all non-stationary solutions have odd frequency-parity. Columns 7 and 8 describe additional reflection symmetries, if present. Column 7 indicates an even (e) or odd (o) parity with respect to north-south reflection $\vartheta \mapsto \pi - \vartheta$. Column 8 indicates an even (e) or odd (o) parity with respect to east-west reflection $\varphi \mapsto -\varphi$. A bound on the norm of $u \in \mathcal{B}_k$ is given in column 9, and bounds on the leading coefficients in the (real version of) the expansion (1.6) are given in Section 7.*

The nonlinearity $f(u) = u^3$ has been chosen for its simplicity. What matters in our approach is that f is odd and can be estimated well as a function on the Banach algebra $\mathcal{B}$. So f need not be a polynomial. It suffices for $f : \mathbb{R} \to \mathbb{R}$ to be odd and extend analytically to a complex disk $|z| < r$ with sufficiently large radius r. In particular, Theorem 1.1 remains valid if f is a small perturbation of $u \mapsto u^3$ of type described above, with r larger than the norm in column 9. See Remark 5.

A proof of Theorem 1.1 is given in Section 3. Starting from an approximate numerical solution $\bar{u}$, we show that there exists a true solution u nearby. This is done with the aid of a computer. As a by-product we obtain accurate estimates [31] for each solution. This includes an explicit expression for $\bar{u}$, and a bound on the norm of $u - \bar{u}$ which is typically by a factor 10^{-12} smaller than the norm of $\bar{u}$. Here we take advantage of the fact that $\mathcal{L}$ has a compact inverse on $\mathcal{B}_k$. But the degree of compactness is relatively weak in the non-periodic case. In fact, compactness is not necessary for this type of proof, but $\mathcal{L}^{-1}$ has to be small on the complement of an appropriate finite-dimensional subspace.

We note that the equation (1.1) is Hamiltonian. Most known results on quasiperiodic solutions for such PDEs concern small amplitude solutions and are based on KAM type methods; see e.g. [14,15,16,17,18,24,26] and references therein. Nontrivial solutions can be obtained by a variational method as well. But the choice $f(u) = u^3$ for the equations (1.4) and (1.5) seems beyond the reach of existing techniques.

The following theorem is proved by using ideas developed in [3,11,16].

We say that a function $f : \mathbb{R} \to \mathbb{R}$ is near-linear, if f is increasing, has a weak derivative $f' \in \mathrm{L}^\infty(\mathbb{R})$ that is bounded away from 0, and satisfies

$$f(0) = 0 \,, \qquad \lim_{t \to 0} \frac{f(t)}{t} = b_0 \,, \qquad \frac{f(x)}{x} \le b_\infty \,, \tag{1.8}$$

for some positive constants $b_\infty < b_0$ and $|x|$ sufficiently large.

Theorem 1.2. *Fix $\nu \in \{1, 2\}$. Consider the equation (1.2), or (1.3) with β rational, or (1.4) with α rational, or (1.5) with α rational and β an irrational number of bounded type. Here $\alpha, \beta > 0$. Choose $k \ge 1$ and restrict to functions u that are k-antisymmetric. Consider the set Λ of all eigenvalues $\lambda_{l,m,n}$ of $\mathcal{L}$ that admit a k-antisymmetric eigenvector. Assume that $0 \notin \Lambda$. (This holds for all choices described in Table 1.) Let λ be the smallest positive* *eigenvalue in Λ. Then for every odd function f that is near-linear, with $b_\infty < \lambda < b_0$, the given equation has a nonzero solution u that belongs to the Sobolev space H^s for some $s > 1$. (Specific bounds on s are given in Section 4.) For periodic solutions, s is sufficiently large to guarantee that u is at least continuous.*

For periodic rotating waves, and either $\nu = 2$ or $\beta < 1$, the same results hold if β is chosen irrational.

The remaining part of this paper is organized as follows. In Section 2 we define spaces of analytic functions on $\mathbb{S}$ and $\mathbb{S} \times \mathbb{T}$. Proofs of Theorem 1.1 and Theorem 1.2 are given in Section 3 and Section 4, respectively. Section 5 is devoted to the linear operators $\mathcal{L}$ and its eigenvalues. Our proof of Theorem 1.1 is based on estimates given in Lemma 3.2. This lemma is proved with the aid of a computer, as described in Section 6. The full details for this part, including the source text of our programs and data files, are given in [31]. Section 7 contains bounds on the largest terms in the series (1.6) for the solutions described in Theorem 1.1.

2. Analytic functions on the sphere

2.1. Spherical harmonics

The standard (L^2 normalized) spherical harmonics Y_l^m and our un-normalized spherical harmonics $\mathcal{Y}_l^m$ are related to the Wigner matrices D^l and d^l via the identity

$$\mathcal{Y}_l^m(\vartheta,\varphi) = \sqrt{\frac{4\pi}{2l+1}}\, Y_l^m(\vartheta,\varphi) = D^l_{m,0}(-\varphi,\vartheta,0) = \mathcal{P}_l^m(\vartheta)e^{im\varphi}\,, \tag{2.1}$$

where $\mathcal{P}_l^m = d^l_{m,0}$. Here, m and l are integers that satisfy $|m| \le l$. From the definition of the Wigner d-matrix one readily sees that $\mathcal{P}_l^m$ is even, entire analytic, and takes real values for real arguments. We note that $\mathcal{P}_l^m$ agrees up to a constant factor with $\vartheta \mapsto P_l^m(\cos\vartheta)$, where P_l^m is the associated Legendre function.

The Wigner matrix D^l is a unitary matrix in an irreducible representation of the rotation group SO(3) or its double cover SU(2). Decomposing a product representation into irreducible representations yields the well-known product expansion

$$\mathcal{P}_{l_1}^{m_1}\mathcal{P}_{l_2}^{m_2} = \sum_{l_3} \Gamma^{m_1 m_2 m_3}_{l_1\, l_2\, l_3}\mathcal{P}_{l_3}^{m_3}\,, \qquad \Gamma^{m_1 m_2 m_3}_{l_1\, l_2\, l_3} \stackrel{\text{def}}{=} \langle l_1 m_1\, l_2 m_2 | l_3 m_3\rangle\langle l_1 0\, l_2 0 | l_3 0\rangle\,, \tag{2.2}$$

where $m_3 = m_1 + m_2$. The constants $\langle l_1 m_1\, l_2 m_2 | l_3 m_3\rangle$ are known as Clebsch-Gordan coefficients. They are zero unless $|l_1 - l_2| \le l_3 \le l_1 + l_2$. Furthermore, $\langle l_1 0\, l_2 0 | l_3 0\rangle$ vanishes unless $l_1 + l_2 + l_3$ is even. And $\sum_{l_3} |\langle l_1 m_1\, l_2 m_2 | l_3 m_3\rangle|^2 = 1$ by unitarity. Here, and in what follows, we assume that $|m_j| \le l_j$ for all j, and that $m_3 = m_1 + m_2$. For the coefficients $\Gamma^{m_1 m_2 m_3}_{l_1\, l_2\, l_3}$, the Cauchy-Schwarz inequality yields

$$\sum_{l_3} \bigl|\Gamma^{m_1 m_2 m_3}_{l_1\, l_2\, l_3}\bigr| \le \Bigl(\textstyle\sum_{l_3} \bigl|\langle l_1 m_1\, l_2 m_2 | l_3 m_3\rangle\bigr|^2\Bigr)^{1/2} \Bigl(\textstyle\sum_{l_3} \bigl|\langle l_1 0\, l_2 0 | l_3 0\rangle\bigr|^2\Bigr)^{1/2} = 1\,. \tag{2.3}$$

2.2. Spaces of analytic functions

Consider functions u on the sphere $\mathbb{S}$ that admit an expansion

$$u = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} u_{l,m} \mathcal{Y}_l^m \,. \tag{2.4}$$

For real-valued functions on $\mathbb{S}$ we have $u_{l,m} \in \mathbb{C}$. This applies e.g. to solutions of the equation (1.2) or (1.3). But when considering the equation (1.4) or (1.5) on $\mathbb{S} \times \mathbb{T}$, the coefficients $u_{l,m}$ are taken to be functions on $\mathbb{T}$. In this case, setting $u_{l,m}(t) = \sum_n u_{l,m,n} e^{int}$ with $u_{l,m,n} \in \mathbb{C}$, we obtain the expansion (1.6).

To cover both cases, let $\mathfrak{C}$ be a Banach algebra over $\mathbb{C}$. Given a pair $\rho = (\rho_{\rm L}, \rho_{\rm M})$ of real numbers ≥ 1, denote by $\mathcal{A}_\rho(\mathfrak{C})$ the vector space of all functions $u : \mathbb{S} \to \mathfrak{C}$ that admit an expansion (2.4) with coefficients $c_{l,m} \in \mathfrak{C}$, and that have a finite norm

$$\|u\|_\rho = \sum_{l,m} \|u_{l,m}\| \rho_{\rm L}^l \rho_{\rm M}^m \,. \tag{2.5}$$

The sum in this equation is an abbreviation for the double sum in (2.4).

In what follows, when the choice of $\mathfrak{C}$ does not matter, we write $\mathcal{A}_\rho$ in place of $\mathcal{A}_\rho(\mathfrak{C})$.

Lemma 2.1. *$\mathcal{A}_\rho$ is a Banach algebra under pointwise multiplication, in the sense that $\|uv\|_\rho \leq \|u\|_\rho \|v\|_\rho$ for every $u, v \in \mathcal{A}_\rho$. Furthermore, if $u \in \mathcal{A}_\rho$ with $\rho_{\rm L} > 1$, then $u = u(\vartheta, \varphi)$ extends analytically to a complex open neighborhood of $\mathbb{R}^2$.*

Proof. Consider first fixed integers m_1, m_2, and define $m_3 = m_1 + m_2$. By (2.2) we have

$$\mathcal{Y}_{l_1}^{m_1} \mathcal{Y}_{l_2}^{m_2} = \sum_{l_3} \Gamma_{l_1\,l_2\,l_3}^{m_1 m_2 m_3}\, \mathcal{Y}_{l_3}^{m_3} \,, \qquad l_1 \geq |m_1| \,, \quad l_2 \geq |m_2| \,. \tag{2.6}$$

The coefficients $\Gamma_{l_1\,l_2\,l_3}^{m_1 m_2 m_3}$ vanish unless $l_3 \leq l_1 + l_2$ and $|m_3| \leq |m_1| + |m_2|$. Thus, using (2.3), we have

$$\|\mathcal{Y}_{l_1}^{m_1} \mathcal{Y}_{l_2}^{m_2}\|_\rho \leq \sum_{l_3} \bigl|\Gamma_{l_1\,l_2\,l_3}^{m_1 m_2 m_3}\bigr| \rho_{\rm M}^{|m_3|} \rho_{\rm L}^{l_3} \leq \rho_{\rm L}^{l_1+l_2} \rho_{\rm M}^{|m_1|+|m_2|} \,. \tag{2.7}$$

Let now u and v be two functions in $\mathcal{A}_\rho$. To simplify notation, we define $u_{m,l} = 0$ and $v_{m,l} = 0$ whenever $l < |m|$. By using the bound (2.7), we immediately get

$$\begin{aligned} \|uv\|_\rho &\leq \sum_{m_1,l_1,m_2,l_2} \|u_{m_1,l_1} v_{m_2,l_2}\| \|\mathcal{Y}_{l_1}^{m_1} \mathcal{Y}_{l_2}^{m_2}\|_\rho \\ &\leq \sum_{m_1,l_1,m_2,l_2} \|u_{m_1,l_1}\| \|v_{m_2,l_2}\| \rho_{\rm L}^{l_1+l_2} \rho_{\rm M}^{|m_1|+|m_2|} = \|u\|_\rho \|v\|_\rho \,. \end{aligned} \tag{2.8}$$

This shows that $\mathcal{A}_\rho$ is a Banach algebra, as claimed.

As mentioned after (2.1), the functions $\mathcal{Y}_l^m$ are analytic on all of $\mathbb{C}^2$. To obtain explicit bounds, we can use that the Wigner matrix D^l defines a unitary representation of SO(3) on $\mathbb{C}^n$, with $n = 2l + 1$. In particular, the matrix element $d_{m,0}^l$ is an inner product $d_{m,0}^l(\vartheta) = \bigl\langle lm \bigl| e^{-i\vartheta J_2} \bigr| l0 \bigr\rangle$, where $|l0\rangle$ and $|lm\rangle$ are certain unit vectors in $\mathbb{C}^n$, and where J_2

is a Hermitian $n\times n$ matrix with eigenvalues in $\{-l,-l+1,\ldots,l-1,l\}$. This immediately yields the bound

$$\bigl|\mathcal{Y}_l^m(\vartheta,\varphi)\bigr|=\bigl|\langle lm|e^{-i\vartheta J_2}|l0\rangle e^{im\varphi}\bigr|\le e^{|l\,\mathrm{Im}\,\vartheta|}e^{|m\,\mathrm{Im}\,\varphi|}\,, \tag{2.9}$$

for all $\vartheta,\varphi\in\mathbb{C}$. Recall that $|m|\le l$. Thus, if $u\in\mathcal{A}_\rho$ with $\rho_{\rm L}>1$, then the series (2.4) for u converges uniformly on compact subsets of the domain $|\,\mathrm{Im}\,\vartheta|+|\,\mathrm{Im}\,\varphi|<\log\rho_{\rm L}$. Given that each term is analytic in this domain, the same holds for u. **QED**

Let $\mathfrak{R}$ be a Banach algebra over $\mathbb{R}$ of real-valued functions. Consider $\mathfrak{C}=\mathfrak{R}+i\mathfrak{R}$ equipped with the norm

$$\|x+iy\|=\|x\|+\|y\|\,,\qquad x,y\in\mathfrak{R}\,. \tag{2.10}$$

Then $\mathfrak{C}$ is a Banach algebra over $\mathbb{C}$ of complex-valued functions. Restricting to real-valued functions in $\mathcal{A}_\rho(\mathfrak{C})$ yields a Banach algebra $\mathcal{B}_\rho(\mathfrak{R})$ of real-valued functions. Every function u in this space admits a representation

$$u(\vartheta,\varphi)=\sum_{l\ge 0}a_{l,0}\mathcal{P}_l^0(\vartheta)+\sum_{m>0}\sum_{l\ge m}\mathcal{P}_l^m(\vartheta)\bigl[a_{l,m}\mathfrak{c}_m(\varphi)+b_{l,m}\mathfrak{s}_m(\varphi)\bigr]\,, \tag{2.11}$$

with $a_{l,m},b_{l,m}\in\mathfrak{R}$, where $\mathfrak{c}_m(\varphi)=\cos(m\varphi)$ and $\mathfrak{s}_m(\varphi)=\sin(m\varphi)$. Here, we have restricted to nonnegative values of m by using the symmetry property

$$\mathcal{P}_l^{-m}=(-1)^m\mathcal{P}_l^m\,. \tag{2.12}$$

A straightforward computation shows that the norm in $\mathcal{B}_\rho(\mathfrak{R})$ is given by

$$\|u\|_\rho=\sum_{l\ge 0}\|a_{l,0}\|\rho_{\rm L}^l+\sum_{m>0}\sum_{l\ge m}\bigl(\|a_{l,m}\|+\|b_{l,m}\|\bigr)\rho_{\rm L}^l\rho_{\rm M}^m\,. \tag{2.13}$$

Banach algebras of functions related to spherical harmonics have been used before in [25].

Convention 3. In what follows, we only consider $\rho_{\rm M}=1$ and write ρ in place of $(\rho,1)$.

For real analytic functions $h:\mathbb{T}\to\mathbb{R}$ we use a space $\mathcal{T}_\varrho$ with norm

$$\|h\|_\varrho=|a_0|+\sum_{n>0}\bigl(|a_n|+|b_n|\bigr)\varrho^n\,,\qquad h=a_0+\sum_{n>0}\bigl[a_n\mathfrak{c}_n+b_n\mathfrak{s}_n\bigr]\,, \tag{2.14}$$

where $\varrho>1$. Our solutions for (1.4) and (1.5) belong to $\mathcal{B}_\rho(\mathcal{T}_\varrho)$ for some $\rho,\varrho>1$.

Remark 4. The multiplication of two functions in $\mathcal{B}_\rho$ reduces to a large number of products of the form (2.6), and each such product involves a substantial number of Clebsch-Gordan coefficients. Computing these coefficients (accurately) each time they are needed is a prohibitive amount of work, so they have to be computed beforehand and stored. This in turn creates a nontrivial storage problem. Fortunately, the set of Clebsch-Gordan coefficients can be partitioned into a significantly smaller set of equivalence classes that share the same value up to a sign. We will not explain these issues here but refer to [25] for details and references.

3. The fixed point equation

Here we reduce the proof of Theorem 1.1 to specific estimates. Throughout this section, we consider a fixed but arbitrary row of Table 1. Let $\varrho = \frac{33}{32}$ and $\rho = \frac{65}{64}$.

The column entries in the chosen row specify the type of solution and the corresponding parameters. For the equations (1.2) and (1.3) that describe stationary solutions and periodic rotating waves, respectively, $\mathcal{B} = \mathcal{B}_\rho(\mathbb{R})$. For the equations (1.4) and (1.5) that describe ordinary periodic and quasiperiodic solutions, respectively, $\mathcal{B} = \mathcal{B}_\rho(\mathcal{T}_\varrho)$. The subspace $\mathcal{B}_k$ of $\mathcal{B}$ is defined by the chosen symmetries: the main symmetries as described in Convention 2, with the value of k given in Column 6, as well as the reflection symmetries described in Columns 7 and 8.

The following proposition gives a rough lower bound on the eigenvalues of $\mathcal{L}$ on the space $\mathcal{B}_k$. It implies that $\mathcal{L}$ has a compact inverse on $\mathcal{B}_k$. More accurate bounds will be proved in Section 5.

Proposition 3.1. *There exists constants $c_s, c_r, c_p, c_q > 0$ such that the eigenvalues of $\mathcal{L}$ on $\mathcal{B}_k$ satisfy the following bounds.*
(s) $|\lambda_{l,m}| \ge c_s l^\nu$ for $\mathcal{L}$ as defined by (1.2).
(r) $|\lambda_{l,m}| \ge c_r l^\nu$ for $\mathcal{L}$ as defined by (1.3).
(p) $|\lambda_{l,m,n}| \ge c_p(l^\nu + n)$ for $\mathcal{L}$ as defined by (1.4).
(q) $|\lambda_{l,m,n}| \ge c_q\bigl(l + n^{1/2}\bigr)$ for $\mathcal{L}$ as defined by (1.5) with $\nu = 2$.
(Here β can be any positive irrational number of bounded type.)

Recall that $\beta \in \mathbb{R} \setminus \mathbb{Q}$ is said to be of bounded type if the sequence $j \mapsto b_j$ in the continued fraction expansion $\beta = b_0 + 1/(b_1 + 1/(b_2 + 1/(b_3 + \ldots)))$ is bounded.

The goal is to find a solution $u \in \mathcal{B}_k$ of the equation $\mathcal{L}u = f(u)$. This is equivalent to solving the fixed point equation

$$u = G(u)\,, \qquad G(u) \stackrel{\text{def}}{=} \mathcal{L}^{-1} f(u)\,. \tag{3.1}$$

As is common in many computer-assisted proofs, we associate with G a quasi-Newton map $\mathcal{N}$ as follows. Given a function $\bar{u} \in \mathcal{B}_k$ and a bounded linear operator M on $\mathcal{B}_k$, define

$$\mathcal{N}(h) = G(\bar{u} + Ah) - \bar{u} + Mh\,, \qquad A = \mathrm{I} - M\,, \tag{3.2}$$

for every $h \in \mathcal{B}_k$. Clearly, if h is a fixed point of $\mathcal{N}$, then $\bar{u} + Ah$ is a fixed point of G.

Our goal is to apply the contraction mapping theorem to the map $\mathcal{N}$, acting on a ball $B_\delta = \{h \in \mathcal{B}_k : \|h\| < \delta\}$. Thus, $\bar{u}$ is chosen to be an approximate fixed point of G, and M is chosen in such a way that $A = \mathrm{I} - M$ is an approximate inverse of $\mathrm{I} - DG(\bar{u})$.

Lemma 3.2. *There exists a function $\bar{u} \in \mathcal{B}_k$ and a bounded linear operator M on $\mathcal{B}_k$, such that the following holds. The equation (3.2) defines a compact cubic map $\mathcal{N}$ on $\mathcal{B}_k$. Furthermore, there exist positive real numbers δ, ε, and K, satisfying $\varepsilon + K\delta < \delta$, such that*

$$\|\mathcal{N}(0)\| \le \varepsilon\,, \qquad \|D\mathcal{N}(h)\| < K\,, \qquad \forall h \in B_\delta\,. \tag{3.3}$$

For every $h \in B_\delta$, the function $u = \bar{u} + Ah$ has the symmetries described in Columns 6,7, and 8 of Table 1. In addition, u satisfies the norm bound given in Column 9 and the coefficient bounds given in Section 7. Furthermore, the associated function v is precisely of the type indicated in Column 2.

Our proof of this lemma is computer-assisted and will be described in Section 6.

Lemma 3.2 implies Theorem 1.1. Namely, the contraction mapping theorem guarantees that $\mathcal{N}$ has a fixed point $h \in \mathcal{B}_k$. The associated function $u = \bar{u} + Ah$ is a fixed point of G and thus a solution of the given equation $\mathcal{L}u = f(u)$. Clearly, u has the properties described in Lemma 3.2 after (3.3). In particular, the associated solution v of the equation (1.1) is precisely of the type indicated in Column 2. As described in Theorem 1.1, this means e.g. that our solutions labeled 8,9,10,11 are non-periodic.

Remark 5. Verifying the bounds in Lemma 3.2 is a finite computation with rigorous error estimates. The constant ε in (3.3) is obtained as an upper bound on the norm of $\mathcal{N}(0)$, while $K < 1$ is fixed beforehand. Given that $\mathcal{B}$ is a Banach algebra, it is clear that all necessary inequalities remain true under perturbations of the nonlinearity f by a small odd analytic function with the proper domain.

4. Weak solutions

In this section we give a proof of Theorem 1.2.

4.1. Existence for the dual equation

We consider the dual formulation that was used in [3,11,16] to find weak solutions of the nonlinear wave equation $\partial_t^2 u - \partial_x^2 u + f(u) = 0$ on an interval with zero boundary conditions or periodic boundary conditions. The functions f considered in [3,16] include $f(u) = u^3$, while [11] considers near-linear functions f.

We start by giving a purely formal description. Consider an equation $\mathcal{L}u = f(u)$ on some space U of functions $u : \Omega \to \mathbb{R}$. Denote by V the null space of the linear operator $\mathcal{L}$. Write $w = \mathcal{L}u$ and $\mathcal{L}^{-1}w = u + v$ for some $v \in V$. Then $\mathcal{L}u = f(u)$ if and only if $w = f(u)$. Assuming that f has an inverse $g = f^{-1}$, the equation for w and v is

$$\mathcal{L}^{-1}w = g(w) + v\,, \qquad \mathcal{L}v = 0\,. \tag{4.1}$$

This is the equation for a critical point of the functional Φ defined by

$$\Phi(w) = \int_\Omega \Bigl[-\tfrac{1}{2}w\mathcal{L}^{-1}w + G(w)\Bigr]\,, \qquad G' = g\,, \quad w \in U \cap V^\perp\,, \tag{4.2}$$

assuming that $\mathcal{L}^{-1}$ extends to a self-adjoint linear operator on $\mathrm{L}^2(\Omega)$.

In the cases considered here, $\mathcal{L}$ is defined on a space $\mathcal{B}$ of analytic functions and has a compact inverse $A = \mathcal{L}^{-1}$ on some subspace $\mathcal{B}_k$. The eigenfunctions of $A : \mathcal{B}_k \to \mathcal{B}_k$ constitute an orthogonal set in $\mathrm{L}^2 = \mathrm{L}^2(\Omega)$, where $\Omega = \mathbb{S}$ in equation (1.2) and (1.3), and $\Omega = \mathbb{S} \times \mathbb{T}$ in equation (1.4) and (1.5). Thus, A extends trivially to a compact self-adjoint linear operator on L^2, if we set $Av = 0$ for $v \in \mathcal{B}_k^\perp$. Using that both $\mathcal{B}_k$ and $\mathcal{B}_k^\perp$ are characterized by symmetries, we will show below that it suffices to solve the equation (4.1) with $v = 0$. This makes our setup similar to the one considered in [11].

Under the assumptions on f stated before Theorem 1.2, f has an inverse $g = f^{-1}$ that is increasing, globally Lipschitz, and satisfies

$$g(0) = 0\,, \qquad \lim_{t \to 0} \frac{g(t)}{t} = a_0\,, \qquad \frac{g(x)}{x} \ge a_\infty\,, \tag{4.3}$$

for $|x|$ sufficiently large, with $a_0 < a_\infty$. Notice that $a_0 = b_0^{-1}$ and $a_\infty = b_\infty^{-1}$.

In the following lemma, g can be any function with the above-mentioned properties.

Lemma 4.1. *Let (Ω,Σ,μ) be a measure space with $\mu(\Omega)$ finite and positive. Let A be a compact self-adjoint linear operator on $\mathrm{L}^2=\mathrm{L}^2(\Omega,\Sigma,\mu)$. Denote by κ be the largest eigenvalue of A, and assume that $a_0<\kappa<a_\infty$. Then the equation $Aw=g(w)$ has a nonzero solution $w\in\mathrm{L}^2$.*

Proof. Let G be the antiderivative of g satisfying $G(0)=0$. Consider the functional $\Phi:\mathrm{L}^2\to\mathbb{R}$ defined by

$$\Phi(w)=-\tfrac{1}{2}\langle w,Aw\rangle+\Psi(w)\,,\qquad \Psi(w)=\int_\Omega G(w)\,d\mu\,. \tag{4.4}$$

Notice that Φ is differentiable at every $w\in\mathrm{L}^2$. In fact, Φ is of class C^1 with Lipschitz derivative. Namely, if $u,v,h\in\mathrm{L}^2$, then

$$\begin{aligned}\bigl|[D\Psi(u)-D\Psi(v)]h\bigr| &\le \left|\int_\Omega [g(u)-g(v)]h\,d\mu\right|\\ &\le a\int_\Omega |u-v||h|\,d\mu\le a\|u-v\|\|h\|\,,\end{aligned} \tag{4.5}$$

where a is the Lipschitz constant for g. Furthermore, Φ is coercive in the following sense. By (4.3) there exists $C>0$ such that $G(x)\ge\frac{1}{2}a_\infty x^2-C$ for all x. So we have

$$\Phi(w)\ge\tfrac{1}{2}\bigl(-\kappa+a_\infty\bigr)\|w\|^2-C\mu(\Omega)\,. \tag{4.6}$$

Since $\kappa<a_\infty$ by assumption, this shows e.g. that $c=\inf\Phi$ exists in $\mathbb{R}$.

Let $n\mapsto w_n$ be a minimizing sequence in L^2, meaning that $\Phi(w_n)\to c$. By (4.6), this sequence is bounded. So some subsequence converges weakly in L^2 to some $w\in\mathrm{L}^2$. To simplify notation, denote this subsequence again by $n\mapsto w_n$. Given that A is compact, we have $Aw_n\to Aw$, and thus

$$\langle w_n,Aw_n\rangle=\langle w_n,Aw\rangle+\langle w_n,Aw_n-Aw\rangle\to\langle w,Aw\rangle+0\,. \tag{4.7}$$

Since the sequence $n\mapsto\Phi(w_n)$ converges, so does the sequence $n\mapsto\Psi(w_n)$.

Given that $G'=g$ is increasing, the function Ψ is convex. Thus $\liminf\Psi(w_n)\ge\Psi(w)$, implying that $c=\lim\Phi(w_n)\ge\Phi(w)$. But $\Phi(w)$ cannot be less that $\inf\Phi=c$. So w is a minimizer of Φ,

$$\Phi(w)=\inf\Phi\,. \tag{4.8}$$

As a critical point of Φ, the function w satisfies $-Aw+g(w)=\nabla\Phi(w)=0$, as claimed.

To see that $w\ne 0$, let h be a normalized eigenvector of A for the eigenvalue κ. Then

$$\Phi(th)\le\tfrac{1}{2}\bigl(-\kappa t^2+a_0t^2\bigr)+\mathcal{O}\bigl(t^2\bigr)\,, \tag{4.9}$$

for $t\ne 0$ near 0. By assumption, we have $\kappa>a_0$. So $\Phi(th)<0$ for $t\ne 0$ sufficiently close to 0. This shows that $\inf\Phi<0$, implying that $w\ne 0$. **QED**

4.2. Regularity

Assume now that $g = f^{-1}$, where f has the properties described before Theorem 1.2. Notice that $g(w)$ belongs to L^2 if and only if $w \in \mathrm{L}^2$. This follows from the fact that $|g(x)| \le a|x|$ and $|f(x)| \le b|x|$ for some $a, b > 0$.

In our application of Lemma 4.1, we have $\Omega = \mathbb{S}$ or $\Omega = \mathbb{S} \times \mathbb{T}$.

In what follows, $(\Omega, \mathfrak{g})$ can be any compact oriented smooth Riemannian manifold without boundary. So Ω carries a canonical volume form defining a finite measure μ. Denote by Δ_Ω the Laplace-Beltrami operator on Ω. It is well-known that the eigenfunctions $Y_0, Y_1, Y_2, \ldots$ of $-\Delta_\Omega$ are smooth and constitute a complete orthogonal basis for $\mathrm{L}^2 = \mathrm{L}^2(\Omega, \mu)$. The associated eigenvalues are real numbers $\mu_j \ge 0$ that grow asymptotically like $\mu_j \sim j^{2/n}$, where n is the dimension of Ω. Consider the normalization $\|Y_j\|_{\mathrm{L}^2} = 1$ for all j. As usual, we define the Sobolev space $\mathrm{H}^s = \mathrm{H}^s(\Omega, \mu)$ for $s \in \mathbb{R}$ to be the space of all distributions $u = \sum_j u_j Y_j$ in the dual of $\mathrm{C}^\infty(\Omega)$, with the property that the sum $\sum_j (1 + \mu_j)^s |u_j|^2$ is finite. The norm in H^s is defined as

$$\|u\|_{\mathrm{H}^s} = \left\|(\mathrm{I} - \Delta_\Omega)^{s/2} u\right\|_{\mathrm{L}^2}, \qquad (\mathrm{I} - \Delta_\Omega)^{s/2} u = \sum_j (1 + \mu_j)^{s/2} u_j Y_j \,. \tag{4.10}$$

A linear operator B on H^s will be called a (real) Fourier multiplier, if there exists a sequence $j \mapsto b_j$ of (real) numbers such that $BY_j = b_j Y_j$ for all j.

In what follows, we only consider $s > 0$. Then H^s is compactly embedded in L^2.

Corollary 4.2. *Assume that $A : \mathrm{L}^2 \to \mathrm{L}^2$ is self-adjoint and admits a factorization*

$$A = (\mathrm{I} - \Delta_\Omega)^{-r/2} B \,, \tag{4.11}$$

with $r > 0$ and $B : \mathrm{L}^2 \to \mathrm{L}^2$ a bounded real Fourier multiplier. If A satisfies the spectral condition in Lemma 4.1, and if f is as described earlier, then the equation $u = Af(u)$ has a nonzero solution in H^r.

The function u is obtained from a nonzero solution $w \in \mathrm{L}^2$ of the equation $Aw = g(w)$ by setting $u = g(w)$. Then $u = Af(u)$, and $u = (\mathrm{I} - \Delta_\Omega)^{-r/2} Bw$ belongs to H^r.

Lemma 4.3. *Assume that Ω has dimension $n \ge 2$. Then the solution of $u = Af(u)$ described in Corollary 4.2 belongs to H^s for every $s < 1 + r$.*

Proof. Let $0 < \sigma < 1$. Then the norm (4.10) on H^σ is known to be equivalent [23,29] to the norm $\|.\|_\sigma$ defined by the equation

$$\|u\|_\sigma^2 = \|u\|_{\mathrm{L}^2}^2 + [u]_\sigma^2 \,, \qquad [u]_\sigma^2 \stackrel{\text{def}}{=} \iint_{\Omega \times \Omega} \frac{|u(x) - u(y)|^2}{d(x, y)^{n + 2\sigma}} d\mu(x) d\mu(y) \,. \tag{4.12}$$

Assume that u belongs to H^σ. Then

$$|f(u(x)) - f(u(y))| \le b|u(x) - u(y)| \,, \qquad x, y \in \mathbb{R} \,, \tag{4.13}$$

with $b = \|f'\|_{\mathrm{L}^\infty}$. So we have $[f(u)]_\sigma \le b[u]_\sigma$, and thus $f(u) \in \mathrm{H}^\sigma$. Assume furthermore that $u = Af(u)$. Given that $w = f(u)$ belongs to H^σ, and that A maps H^σ into $\mathrm{H}^{\sigma + r}$, we conclude that $u = Aw$ belongs to $H^{\sigma + r}$.

By iterating this procedure, if necessary, we find that $u \in \mathrm{H}^s$ for some $s > 1$. Now pick an arbitrary $\sigma < 1$. Applying the above once more yields $u \in \mathrm{H}^{\sigma+r}$. **QED**

Proof of Theorem 1.2. To start with, let us restrict to the choices made in Theorem 1.1.

First, we need to verify the assumptions of Lemma 4.1. By Proposition 3.1, $\mathcal{L}$ has a compact inverse A on a subspace $\mathcal{B}_k$ of $\mathcal{B}$. Here, $\mathcal{B}$ is one of our spaces $\mathcal{B}_\rho(\mathbb{R})$ or $\mathcal{B}_\rho(\mathcal{T}_\varrho)$, depending on whether $\Omega = \mathbb{S}$ or $\Omega = \mathbb{S}\times\mathbb{T}$, respectively. And the subspace $\mathcal{B}_k$ is determined by fixing certain parities.

Let J be a set indexing the eigenvectors Y_j of $A : \mathcal{B}_k \to \mathcal{B}_k$. So we have $\mathcal{L}Y_j = \lambda_j Y_j$ for all $j \in J$, with $\lambda_j \neq 0$ and $|\lambda_j| \to \infty$ as $j \to \infty$ in J. And $AY_j = \lambda_j^{-1} Y_j$ for all $j \in J$. Notice that these functions Y_j are eigenvectors of the Laplacean Δ_Ω as well. So we may assume that they are part of a complete orthonormal system for $\mathrm{L}^2 = \mathrm{L}^2(\Omega)$ that consists of eigenvectors of Δ_Ω.

Denote by W the closure of $\mathcal{B}_k$ in L^2. Clearly A extends to a compact self-adjoint linear operator on W. Now we extend A to all of L^2 by setting $Av = 0$ for all $v \in W^\perp$. The eigenvalue condition $a_0 < \kappa < a_\infty$ needed for Lemma 4.1 follows from the assumption $b_\infty < \lambda < b_0$ in Theorem 1.2, setting $\kappa = \lambda^{-1}$. Thus, Lemma 4.1 guarantees the existence of a nonzero function $w \in \mathrm{L}^2$ that satisfies $Aw = g(w)$.

Clearly $u = g(w)$ belongs to W, so we have a solution $u \in W$ of the equation $u = Af(u)$. In principle, $f(u)$ need not belong to W. However, W is characterized by a finite number of parity conditions $u \circ S_i = \pm u$, where $S_i : \Omega \to \Omega$ is a reflection or rotation. (And each $u \mapsto u \circ S_i$ is unitary.) Thus, since f is odd by assumption, $f(u)$ has the same parities as u. So $f(u) \in W$ whenever $u \in W$. This implies that $\mathcal{L}u = f(u)$ holds in weak sense.

In order to complete the proof of Theorem 1.2, we need to verify that A admits a factorization $A = (\mathrm{I} - \Delta_\Omega)^{-r/2}B$, with $r > 0$ and B a bounded real Fourier multiplier. Then Lemma 4.3 guarantees that u belongs to H^s for every $s < 1 + r$. To this end, it suffices to check that there exists constants $C, r > 0$ such that

$$|\lambda_j| \geq C(1 + \mu_j)^{r/2}\,, \qquad j \in J\,, \tag{4.14}$$

where μ_j is the eigenvalue of $-\Delta_\Omega$ for the eigenvector Y_j. Notice that $\mu_j = l(l+1) + n^2$ for $j = (l, m, n)$ in the case $\Omega = \mathbb{S} \times \mathbb{T}$, and $\mu_j = l(l+1)$ for $j = (l, m)$ in the case $\Omega = \mathbb{S}$.

The bound (4.14) follow from Proposition 3.1. Our discussion of $\mathcal{L}$ is Section 5 shows that, aside from the value of C, these estimates do not depend on any specific choice of symmetries, as long as these symmetries prevent $\mathcal{L}$ from having an eigenvalue zero in the chosen subspace. The same applies to the choice of rational values for $\alpha > 0$. **QED**

For completeness, we list here the values of r that are obtained from Proposition 3.1.
In what follows, ε is a positive real number that can be chosen arbitrarily small. And n denotes the dimension of the domain of u.

- (r2) For periodic rotating waves with $\nu = 2$, we have $r = 4$. So our solutions u are in $\mathrm{H}^{5-\varepsilon}$. Since $n = 2$, these solutions are of class C^3 by standard Sobolev embedding theorems.
- (s2) The same holds for stationary solutions with $\nu = 2$.
- (r1) For periodic rotating waves with $\nu = 1$, we obtain $r = 2$ if $\beta < 1$, or $r = 1$ if $\beta \geq 1$. Here, we assume that β is rational, if larger than 1. So our solutions are in $\mathrm{H}^{3-\varepsilon}$ and of class C^1 for $\beta < 1$, or in $\mathrm{H}^{2-\varepsilon}$ and of class C^0 for $\beta \geq 1$. Notice that $n = 2$.
- (s1) For stationary solutions with $\nu = 1$, we have $r = 2$ and thus $u \in H^{3-\varepsilon} \cap \mathrm{C}^1$.

(p) For ordinary periodic solutions (not rotating waves) with α rational, we have $r = 1$. So our solutions belong to $\mathrm{H}^{2-\varepsilon}$. Since $n = 3$, these solutions are at least of class C^0.

(q2) For quasiperiodic solutions with $\nu = 2$ and $\beta \in \mathbb{R} \setminus \mathbb{Q}$ of bounded type, we obtain $r = 1/2$. So our solutions are in $\mathrm{H}^{3/2-\varepsilon}$. Since $n = 3$, they need not be continuous.

5. The linear operator

In this section we establish lower bounds on the eigenvalues of $\mathcal{L}$ on the spaces $\mathcal{B}_k$. These bounds are used in our proof of Lemma 3.2. They also imply Proposition 3.1.

5.1. The periodic case

Consider the equation (1.4) on a space $\mathcal{B} = \mathcal{B}_\rho(\mathcal{T}_\varrho)$ of real analytic functions on $\mathbb{S} \times \mathbb{T}$, with $\rho, \varrho > 1$ fixed. The eigenvalues of $\mathcal{L} = \alpha^2\partial_t^2 + (-\Delta_{\mathbb{S}})^\nu$ are those given in (1.7), if we set $\beta = 0$. That is,

$$\lambda_{l,n} = l^\nu(l+1)^\nu - (\alpha n)^2 = \bigl(b_l - \alpha n\bigr)\bigl(b_l + \alpha n\bigr)\,, \qquad b_l \stackrel{\text{def}}{=} [l(l+1)]^{\nu/2}\,, \tag{5.1}$$

with $n \geq 0$, since we only consider real eigenfunctions. The goal here is to have $b_l - \alpha n$ bounded away from 0. Since $b_0 = 0$, we restrict to functions that have odd frequency parity, so the value of n belongs to $N = \{1, 3, 5, \ldots\}$. This defines our subspace $\mathcal{B}_0$ of $\mathcal{B}$. As it turns out, no other parity conditions are needed here. But we assume that α is rational, say $\alpha = p/q$ with $p, q > 0$ coprime. Then we have a bound

$$|\lambda_{l,n}| \geq \delta_l(b_l + \alpha n)\,, \qquad \delta_l \stackrel{\text{def}}{=} \inf_{n\in N} |b_l - \alpha n| = q^{-1} \inf_{n\in N} |q b_l - pn|\,. \tag{5.2}$$

To obtain a lower bound of the type described in part (p) of Proposition 3.1, it suffices to show that there exists $\delta > 0$ such that $\delta_l \geq \delta$ for all l.

The plate operator. Consider $\nu = 2$. Here b_l is an even integer, so we can achieve $|qb_l - pn| \geq 1$ for all l by restricting to odd values of p.

The NLW operator. Consider $\nu = 1$. Notice that $\delta_l \neq 0$ for all $l \geq 0$. So it suffices to estimate δ_l for large values of l. Then b_l is close to an odd multiple of $\frac{1}{2}$, so $[b_l q - pn]/q$ can be bounded away from 0 by choosing q odd. To be more precise, write $b_l = \bigl(l + \frac{1}{2}\bigr)\sqrt{1-z}$ with $z = (2l+1)^{-2}$. Expanding in powers of z and estimating the $\mathcal{O}(z^3)$ terms by a geometric series, we obtain

$$b_l = l + \tfrac{1}{2} - \epsilon_l\,, \qquad 0 < \epsilon_l < \frac{1}{4(2l+1)}\left[1 + \frac{1/4}{(2l+1)^2 - 1}\right]. \tag{5.3}$$

Notice that $l \mapsto \epsilon_l$ is decreasing. Choosing $l_0 \geq 0$ in such a way that $q\epsilon_{l_0} < \frac{1}{2}$, we have a bound

$$\delta_l \geq q^{-1}\bigl|\tfrac{1}{2} - q\epsilon_l\bigr| \geq q^{-1}\bigl(\tfrac{1}{2} - q\epsilon_{l_0}\bigr) > 0\,, \qquad l \geq l_0\,. \tag{5.4}$$

Next, we consider the equations (1.3) and (1.2) for periodic rotating waves and stationary solutions, respectively. The spaces used in these cases are $\mathcal{B} = \mathcal{B}_\rho(\mathbb{R})$.

Periodic rotating waves. Here we have $\mathcal{L} = \beta^2\partial_\varphi^2 + (-\Delta_{\mathbb{S}})^\nu$, with eigenvalues

$$\lambda_{l,m} = l^\nu(l+1)^\nu - (\beta m)^2\,, \qquad 0 \leq m \leq l\,. \tag{5.5}$$

This is similar to (5.1), except that (α, n) is replaced by (β, m), with $m \leq l$. The restriction $m \leq l$ makes some cases rather simple: if $\nu = 2$, then $\lambda_{l,n} \sim l^2$ for large values of l; or if $\nu = 1$ and $\beta \leq 1$, then $\lambda_{l,n} \sim l$ for large values of l. In these cases, β need not be rational.

Consider now $\nu = 1$ and $\beta > 1$. In this case, the the restriction $m \leq l$ is not particularly helpful, and the situation is similar to the NLW case described above. So we choose $\beta = p/q$ with $p, q > 0$ coprime and q odd. And m is assumed to satisfy $m \equiv k \pmod{2k}$ for some $k \geq 1$. Then we have a bound $|\lambda_{l,m}| \geq \delta_l(b_l + \beta m)$, with δ_l satisfying (5.4).

Stationary solutions. Here we have $\mathcal{L} = (-\Delta_{\mathbb{S}})^\nu$, with eigenvalues $\lambda_l = l^\nu (l+1)^\nu$ that grow like $l^{2\nu}$ as $l \to \infty$. The eigenvalue $\lambda_0 = 0$ is avoided by requiring k-antisymmetry, meaning that $m \equiv k \pmod{2k}$ for some $k \geq 1$.

5.2. Eigenvalues in the quasiperiodic case

For concrete estimates, it is useful to expand into eigenfunctions for the generator $\alpha\partial_t + \beta\partial_\varphi$ of the quasiperiodic motion in the variable $(\varphi, t) \in \mathbb{T}^2$. Among these eigenfunctions are

$$F_{m,n}^{\pm} = \mathfrak{c}_m \times \mathfrak{c}_n \mp \mathfrak{s}_m \times \mathfrak{s}_n \,, \qquad F_{m,n}^{\pm}(\varphi, t) = \cos(nt \pm m\varphi) \,. \tag{5.6}$$

In particular, we have

$$(\alpha\partial_t + \beta\partial_\varphi)^2 F_{m,n}^{\pm} = -(\alpha n \pm \beta m)^2 F_{m,n}^{\pm} \,. \tag{5.7}$$

A second family of eigenfunctions is given by $G_{m,n}^{\pm} = \mathfrak{c}_m \times \mathfrak{s}_n \pm \mathfrak{s}_m \times \mathfrak{c}_n$. Theses functions are not needed here, since they are odd under the mixed-reflection $(\varphi, t) \mapsto (-\varphi, -t)$.

In what follows, $\mathcal{B}_k$ denotes the space of functions in $\mathcal{B}_\rho(\mathcal{T}_\varrho)$ that have odd frequency-parity, are even under the mixed-reflection, and are k-antisymmetric for some given $k \geq 1$. A function $u \in \mathcal{B}_k$ admits an expansion

$$u(\vartheta, \varphi, t) = \sum_{l,m,n,\pm} u_{l,m,n}^{\pm} \mathcal{P}_l^m(\vartheta) F_{m,n}^{\pm}(\varphi, t) \,, \tag{5.8}$$

where the sum is restricted to positive integers l, $m \leq l$, and n, with $m \equiv k \pmod{2k}$ and n odd. The eigenvalues of the operator $\mathcal{L} = (\alpha\partial_t + \beta\partial_\varphi)^2 + \Delta_{\mathbb{S}}^2$ on this space are given by

$$\begin{aligned} \lambda_{l,m,n}^{\pm} &= l^2(l+1)^2 - (\alpha n \pm \beta m)^2 \\ &= \bigl[l(l+1) - (\alpha n \pm \beta m)\bigr]\bigl[l(l+1) + (\alpha n \pm \beta m)\bigr] \,. \end{aligned} \tag{5.9}$$

We assume that $\alpha = p/q$ with $p, q > 0$ coprime. Assume also that β is a positive irrational number of bounded type. Then the function $(P, Q) \mapsto Q|P - \beta Q|$ on $\mathbb{Z} \times \{1, 2, 3, \ldots\}$ is bounded from below by a positive constant. So the equation

$$\Gamma_N = \inf_{Q \geq N} Q \min_P |P - \beta Q| \tag{5.10}$$

defines positive real numbers $\Gamma_1 \leq \Gamma_2 \leq \Gamma_3 \leq \ldots$. A lower bound on Γ_N will be given in Subsection 5.4, for the values of β considered in Theorem 1.1. As a result of (5.10) we have

$$|ql(l+1) - pn \mp \beta qm| \geq \Gamma_{qm}(qm)^{-1} \,. \tag{5.11}$$

By (5.9) this yields the following eigenvalue bound

$$\bigl|\lambda^{\pm}_{l,m,n}\bigr| \geq \Gamma_{qm} q^{-2} m^{-1} \bigl|l(l+1) + (\alpha n \pm \beta m)\bigr| \geq \Gamma_{qm} q^{-2} \bigl[(l+1) \pm \beta\bigr] . \tag{5.12}$$

The bound represented by the last inequality can be improved by roughly a factor of 2. The idea is that, if $|a-b| \approx 0$, then $|a+b| \approx 2|a|$. To be more precise, we have

$$\bigl|\lambda^{\pm}_{l,m,n}\bigr| \geq 2\Gamma_{qm} q^{-2} \bigl[(l+1) - \beta'\bigr] , \qquad \beta' = \max\bigl\{\beta, \Gamma_{qm}/\bigl(q^2 kl\bigr)\bigr\} . \tag{5.13}$$

The bound with $\beta' = \beta$ is obtained in the case where $\bigl|ql(l+1) - pn \mp \beta qm\bigr| \geq C$, with $C = 2\Gamma_{qm}/(qm)$. The bound with $\beta' = \Gamma_{qm}/\bigl(q^2 kl\bigr)$ is obtained when $\bigl|ql(l+1) - pn \mp \beta qm\bigr| < C$.

The bound (5.13) is used in our programs to estimate terms in the expansion with unspecified frequency n; of course only when $l(l+1) > \beta'$. For fixed values of l and m, bounds that hold for all $n \geq N$ with N fixed can be obtained directly from (5.9).

The first inequality in (5.12) shows that $\mathcal{L}$ has a compact inverse on $\mathcal{B}_k$. But it does not yield a lower bound on $\bigl|\lambda^{\pm}_{l,m,n}\bigr|$ that grows with n, uniformly in l.

The following estimate implies part (q) of Proposition 3.1. First, notice that $|\lambda_{l,m,n}| \geq c_l n^2$ for any fixed value of l. So we may assume that $l \geq l_0$. Then (5.12) yields a bound $\bigl|\lambda^{\pm}_{l,m,n}\bigr| \geq cl$ for some fixed $c > 0$. Consider first the case where $n \leq Cl^2$, for some $C > 0$ to be determined. Then

$$\bigl|\lambda^{\pm}_{l,m,n}\bigr| \geq cl , \qquad \bigl|\lambda^{\pm}_{l,m,n}\bigr| \geq c' n^{1/2} , \tag{5.14}$$

with $c' = cC^{-1/2}$. Next, consider the case $n \geq Cl^2$. If we choose C sufficiently large, then the factor $[l(l+1) - (\alpha n \pm \beta m)]$ in (5.9) is larger than α^{-1} in modulus, and thus $\bigl|\lambda^{\pm}_{l,m,n}\bigr| \geq n$. So a bound of the form (5.14) holds for all values of n and l.

5.3. The inverse of $\mathcal{L}$ in the quasiperiodic case

Explicit expressions. Consider the action of $\mathcal{L}^{-1}$ on the subspace of $\mathcal{B}_k$ spanned by functions $(\vartheta, \varphi, t) \mapsto \mathcal{P}^m_l(\vartheta) F(\varphi, t)$, for fixed values of l and m. The corresponding restriction of $\mathcal{L}^{-1}$ maps a function F of the form

$$F(\varphi, t) = \cos(m\varphi)\mathcal{C}(t) + \sin(m\varphi)\mathcal{S}(t) , \tag{5.15}$$

to a function $\tilde{F}$ of the same form, with $(\mathcal{C}, \mathcal{S})$ replaced by $\bigl(\tilde{\mathcal{C}}, \tilde{\mathcal{S}}\bigr)$. Here $\mathcal{C}$ and $\tilde{\mathcal{C}}$ are even, while $\mathcal{S}$ and $\tilde{\mathcal{S}}$ are odd. The functions $\mathcal{C}$ and $\mathcal{S}$ are represented as Fourier series

$$\mathcal{C} = \sum_n C_n \mathfrak{c}_n , \qquad \mathcal{S} = \sum_n S_n \mathfrak{s}_n , \tag{5.16}$$

where $\mathfrak{c}_n(t) = \cos(nt)$ and $\mathfrak{s}_n(t) = \sin(nt)$. Let

$$b^{\pm}_n = \tfrac{1}{2}\bigl(C_n \mp S_n\bigr) , \qquad F^{\pm}_n = \mathfrak{c}_m \times \mathfrak{c}_n \mp \mathfrak{s}_m \times \mathfrak{s}_n . \tag{5.17}$$

Then $C_n = b^-_n + b^+_n$ and $S_n = b^-_n - b^+_n$, so

$$F = \sum_n C_n \mathfrak{c}_m \times \mathfrak{c}_n + \sum_n S_n \mathfrak{s}_m \times \mathfrak{s}_n = \sum_n b^-_n F^-_n + \sum_n b^+_n F^+_n . \tag{5.18}$$

Under the restricted operator $\mathcal{L}^{-1}$, this function is mapped to

$$\tilde{F} = \sum_n \tilde{b}_n^- F_n^- + \sum_n \tilde{b}_n^+ F_n^+ \,, \qquad \tilde{b}_n^\pm = (\lambda_n^\pm)^{-1} b_n^\pm \,. \tag{5.19}$$

Write

$$\lambda_n^\pm = l^2(l+1)^2 - (\alpha n \pm \beta m)^2 = \sigma_n \mp \delta_n \,, \tag{5.20}$$

with

$$\sigma_n = l^2(l+1)^2 - \alpha^2 n^2 - \beta^2 m^2 \,, \qquad \delta_n = 2\alpha\beta m n \,. \tag{5.21}$$

So $\sigma_n = \frac{1}{2}\bigl[\lambda_n^- + \lambda_n^+\bigr]$ and $\delta_n = \frac{1}{2}\bigl[\lambda_n^- - \lambda_n^+\bigr]$. Similarly, define

$$\Sigma_n = \frac{1}{2}\bigl[(\lambda_n^-)^{-1} + (\lambda_n^+)^{-1}\bigr] = \frac{1}{2}\frac{\lambda_n^+ + \lambda_n^-}{\lambda_n^- \lambda_n^+} = \frac{\sigma_n}{\sigma_n^2 - \delta_n^2} \,, \tag{5.22}$$

and

$$\Delta_n = \frac{1}{2}\bigl[(\lambda_n^-)^{-1} - (\lambda_n^+)^{-1}\bigr] = \frac{1}{2}\frac{\lambda_n^+ - \lambda_n^-}{\lambda_n^- \lambda_n^+} = \frac{-\delta_n}{\sigma_n^2 - \delta_n^2} = \frac{\delta_n}{\delta_n^2 - \sigma_n^2} \,. \tag{5.23}$$

As a Fourier series, we now have

$$\tilde{F} = \sum_n \tilde{C}_n \mathfrak{c}_m \times \mathfrak{c}_n + \sum_n \tilde{S}_n \mathfrak{s}_m \times \mathfrak{s}_n \,, \tag{5.24}$$

where

$$\tilde{C}_n = \tilde{b}_n^- + \tilde{b}_n^+ = \Sigma_n C_n + \Delta_n S_n \,, \qquad \tilde{S}_n = \tilde{b}_n^- - \tilde{b}_n^+ = \Delta_n C_n + \Sigma_n S_n \,. \tag{5.25}$$

High frequencies. We need bounds on Σ_n and Δ_n for large values of n. To this end, pick $n_0 > 0$ such that

$$\bigl(l + \tfrac{1}{2}\bigr)^2 \le c n_0 \,, \tag{5.26}$$

with $c < \alpha$. Consider $n \ge n_0$. Using that $l(l+1) \le \bigl(l + \frac{1}{2}\bigr)^2$, we have

$$|\sigma_n| = \alpha^2 n^2 + \beta^2 m^2 - l^2(l+1)^2 \ge \alpha^2 n^2 - c^2 n_0^2 \ge \bigl(\alpha^2 - c^2\bigr) n^2 \,. \tag{5.27}$$

Furthermore, $\delta_n = 2\alpha\beta m n \le 2\alpha\beta\bigl(l + \frac{1}{2}\bigr) n \le 2\alpha\beta c^{1/2} n^{3/2}$. So

$$\frac{\delta_n}{|\sigma_n|} \le \varepsilon_n \stackrel{\text{def}}{=} \frac{2\alpha\beta c^{1/2}}{\alpha^2 - c^2}\, n^{-1/2} \,, \qquad n \ge n_0 \,. \tag{5.28}$$

Assuming that n_0 has been chosen sufficiently large to have $\varepsilon_n < 1$, this yields the bounds

$$|\Sigma_n| = \frac{|\sigma_n|}{\sigma_n^2 - \delta_n^2} \le \frac{1}{(1 - \varepsilon_n^2)|\sigma_n|} \,, \qquad |\Delta_n| = \frac{\delta_n}{\sigma_n^2 - \delta_n^2} \le \frac{\varepsilon_n}{(1 - \varepsilon_n^2)|\sigma_n|} \,. \tag{5.29}$$

Notice that $n \mapsto |\sigma_n|$ is increasing for $n \ge n_0$.

Low frequencies. For fixed values of l and m, we use that

$$\bigl|\tilde{C}_n\bigr| + \bigl|\tilde{S}_n\bigr| \le \bigl(|\Sigma_n| + |\Delta_n|\bigr)\bigl(|C_n| + |S_n|\bigr) \le \Lambda\bigl(|C_n| + |S_n|\bigr) \,, \tag{5.30}$$

whenever n belongs to $N = \{1, 3, 5, \ldots\}$, where

$$\Lambda = \max_{n \in N} \bigl(|\Sigma_n| + |\Delta_n|\bigr) . \tag{5.31}$$

For finitely many values of n, the sum $|\Sigma_n| + |\Delta_n|$ can be estimated explicitly by using that

$$\begin{aligned} |\Sigma_n| + |\Delta_n| &= \tfrac{1}{2}\bigl|(\lambda_n^-)^{-1} + (\lambda_n^+)^{-1}\bigr| + \tfrac{1}{2}\bigl|(\lambda_n^-)^{-1} - (\lambda_n^+)^{-1}\bigr| \\ &= \max\Bigl\{\bigl|\lambda_n^-\bigr|^{-1}, \bigl|\lambda_n^+\bigr|^{-1}\Bigr\} . \end{aligned} \tag{5.32}$$

Large wavenumbers. So far we have considered fixed values of l. A bound on

$$\Lambda_\ell \stackrel{\text{def}}{=} \max_{l \geq \ell} \max_{m \leq l} \max_{n \in N} \max_{\pm} \bigl|\lambda_{l,m,n}^{\pm}\bigr|^{-1} . \tag{5.33}$$

for large values of ℓ is obtained by using (5.13).

5.4. A Diophantine bound

The goal here is to give an accurate lower bound on the constants Γ_N in equation (5.10) for the quadratic irrationals β considered in Theorem 1.1. The variables k, m, n, p, q in this subsection are unrelated to the ones used in other parts of this paper.

Given an arbitrary irrational number β, define

$$\|n\beta\| = \min_{m \in \mathbb{Z}} |n\beta - m| , \qquad n = 1, 2, 3, \ldots \tag{5.34}$$

The approximant for β with a given denominator $q > 0$ is defined to be the rational number p/q closest to β. The denominators $1 = q_0 < q_1 < q_2 < q_3 < \ldots$ of the best approximants $\beta_k = p_k/q_k$ for β are defined by the condition that $\|q_{k+1}\beta\| < \|q_k\beta\|$, and that

$$\|n\beta\| \geq \|q_k\beta\| \quad \text{for} \quad q_k \leq n < q_{k+1} . \tag{5.35}$$

Notice that $-\beta$ and $b_0 + \beta$ have the same denominators q_k as β, for any integer b_0. So let us restrict now to $0 < \beta < 1$. Consider the continued fraction expansion $\beta = [b_1, b_2, b_3, \ldots] = 1/(b_1 + 1/(b_2 + 1/(b_3 + \ldots)))$. A useful expression for $q_k\|q_k\beta\|$ is

$$q_k\|q_k\beta\| = \bigl(b_k + \hat\beta_{k+1} + \check\beta_{k-1}\bigr)^{-1} , \tag{5.36}$$

where $\hat\beta_k = [b_k, b_{k+1}, b_{k+2}, \ldots]$ and $\check\beta_k = q_k/q_{k+1}$. See e.g. equation (1.15) in [1].

Consider now the choice $\beta = |i + \theta|$, where $\theta > 0$ is given by the equation $\theta = 1/(j + \theta)$. Here i and $j > 0$ are given integers. Notice that the continued fraction denominators q_k for β are independent of the value of i. Thus, assume from now on that $\beta = \theta$.

Clearly $\beta = [j, j, j, \ldots]$. So $\hat\beta_k = \beta$ and $\check\beta_k = \beta_{k+1}$. The identity (5.36) becomes

$$q_k\|q_k\beta\| = (j + \beta + \beta_k)^{-1} . \tag{5.37}$$

Given that $\beta_k \to \beta$ as $k \to \infty$, this shows e.g. that $q_k\|q_k\beta\| \to (j + 2\beta)^{-1} = \bigl(j^2 + 4\bigr)^{-1/2}$.

From (5.35) and (5.37) we obtain the bound

$$n\|n\beta\| \geq q_k\|q_k\beta\| > \frac{1}{j + 2\max(\beta_k, \beta)} , \qquad q_k \leq n < q_{k+1} . \tag{5.38}$$

A basic fact about the best approximants β_k is that $\beta_2 < \beta_4 < \ldots < \beta < \ldots < \beta_3 < \beta_1$. Thus $\max(\beta_k, \beta) \le \max(\beta_k, \beta_{k+1})$, and the sequence $k \mapsto \max(\beta_k, \beta_{k+1})$ is decreasing. So

$$n\|n\beta\| > \frac{1}{j + 2\max(\beta_k, \beta_{k+1})}\,, \qquad n \ge q_k\,, \tag{5.39}$$

holds for all k. A lower bound for the constant Γ_N defined in (5.10) is now given by the right hand side of (5.39) with $n = N$, for some k with $q_k \le n$.

6. Computer estimates

The estimates that are necessary to prove Lemma 3.2 are carried out with the aid of a computer. This part of the proof is written in the programming language Ada [32] and is given in [31]. The following is meant to be a rough guide for the reader who wishes to check the correctness of our programs.

6.1. Enclosures and data types

Bounds on a vector x in a space $\mathcal{X}$, also referred to as enclosures for x, are given here by sets $X \subset \mathcal{X}$ that include x and are representable as data on a computer. The enclosure associated with data `B` will be denoted by $\mathtt{B}_{\mathcal{X}}$. Our basic data type `Ball` consists of a pair $\mathtt{B} = (\mathtt{B.C}, \mathtt{B.R})$, where `B.C` is a representable number [34] and `B.R` a nonnegative representable number (type `Radius`). If $\mathcal{X}$ is a real Banach algebra with unit $\mathbf{1}$, then $\mathtt{B}_{\mathcal{X}} = \{x \in \mathcal{X} : \|x - (\mathtt{B.C})\mathbf{1}\| \le \mathtt{B.R}\}$.

Other types of enclosures depend on the algebra $\mathcal{X}$. At a level where details are irrelevant, we use an unspecified type `Scalar`. For spaces of vectors in $\mathcal{X}^n$ and matrices in $\mathcal{X}^{m\times n}$, we use enclosures of type `Vector` and `Matrix`, respectively, based on arrays of `Scalar`. In what follows, we restrict our description to more problem-specific spaces and enclosures.

For our spaces $\mathcal{T} = \mathcal{T}_\varrho$ with fixed `Radius` $\varrho \ge 1$, we use the following type of enclosure. Given an integer $\mathtt{d} \ge 0$, data of type `CosSin1` consist of a triple `H=(H.P,H.C,H.E)` with $\mathtt{H.P} \in \{0,1\}$, where `H.C` is an `array (0 .. d) of Scalar` and `H.E` is an `array (0 .. 2*d) of Radius`. Here, a `Scalar` is a `Ball` in $\mathbb{R}$. Define $\phi_{0,n}(t) = \cos(nt)$ and $\phi_{1,n}(t) = \sin(nt)$. If $\mathtt{d} > 0$, then the enclosure $\mathtt{H}_{\mathcal{T}} \subset \mathcal{T}$ associated with `H` is the set of all functions

$$h = \sum_{n=p}^{d} h_n \phi_{p,n} + \sum_{j=p}^{2d} e_j\,, \qquad p = \mathtt{H.P}\,, \tag{6.1}$$

with $h_n \in \mathtt{H.C(n)}_{\mathbb{R}}$ and $e_j \in \mathcal{T}$ a function of the form $\sum_{n\ge j} c_n \phi_{p,n}$ whose norm satisfies $\|e_j\| \le \mathtt{H.E(j)}$. If $\mathtt{d} = 0$, then $h = h_0 \in \mathtt{H.C(0)}_{\mathbb{R}}$. In this case, $\mathcal{T}_\varrho$ is simply the space of all constant functions on $\mathbb{T}$. For details we refer to the Ada package `TCosSins1`.

Our type `Spheric` is used for enclosures in the spaces $\mathcal{B} = \mathcal{B}_\rho(\mathcal{T})$ with fixed `Radius` $\rho \ge 1$. Given $\mathtt{D} > 0$, data of this type consist of a pair `U=(U.C,U.E)`, where `U.C` is an `array (0 .. D, 0 .. D) of CosSin1`, and `U.E` is an `array (0 .. 2*D) of Radius`. The enclosure $\mathtt{U}_{\mathcal{B}} \subset \mathcal{B}$ associated with `U` is the set of all functions $u \in \mathcal{B}$ that admit a representation

$$\begin{aligned} u(\vartheta,\varphi,t) &= \sum_{l=0}^{D}\sum_{m=0}^{l} \mathcal{P}_l^m(\vartheta)\cos(m\varphi)A_{l,m}(t) \\ &\quad + \sum_{l=1}^{D}\sum_{m=1}^{l} \mathcal{P}_l^m(\vartheta)\sin(m\varphi)B_{l,m}(t) + \sum_{j=0}^{2D} E_j(\vartheta,\varphi,t)\,, \end{aligned} \tag{6.2}$$

with $A_{l,m} \in \texttt{U.C}(\texttt{l}, \texttt{l}-\texttt{m})_{\mathcal{T}}$ for $m \le l$ and $B_{l,m} \in \texttt{U.C}(\texttt{l}-\texttt{m}, \texttt{l})_{\mathcal{T}}$ for $m < l$, and with E_j a function in $\mathbb{P}_{\{j,j+1,\ldots\}}\mathcal{B}$ that has norm $\|E_j\| \le \texttt{U.E(j)}$.

Here, and in what follows, if J is a set of nonnegative integers, $\mathbb{P}_J : \mathcal{B} \to \mathcal{B}$ denotes the projection operator that acts on a function $u \in \mathcal{B}$ by restricting its expansion (2.11) to indices $l \in J$.

6.2. Bounds and procedures

In the context of enclosures, a bound on a map $f : \mathcal{X} \to \mathcal{Y}$ is a function F that assigns to a set $X \subset \mathcal{X}$ of a given type (`Xtype`) a set $Y \subset \mathcal{Y}$ of a given type (`Ytype`), in such a way that $y = f(x)$ belongs to Y whenever $x \in X$. In Ada, such a bound F can be implemented by defining an appropriate `procedure F(X: in Xtype; Y: out Ytype)`. By definition, `X` belongs to the domain of `F` if no `Exception` is being raised.

Bounds on the basic operations involving the type `Ball` and `CosSin1` are defined in the packages `Flts.Std.Balls` and `TCosSins1`, respectively.

The type `Spheric` and bounds on functions involving this type are defined in the package `TSpherics`. Among the general-purpose procedures in `TSpherics` is the product `Prod` of two `Spheric`. It involves the use of Clebsch-Gordan (CG) coefficients; see also Remark 4. Enclosures for these coefficients are computed and stored in the packages `CG` and `CG.Spher`. The package `CG` is an extension of a homonymous package that was developed in [25]. The extensions are mostly related to the fact that the Zernike functions used in [25] are based on generalized spherical harmonics for half-integer angular momenta l and m, while here, l and m are always integers. For a description of the techniques involved, we refer to Section 5 in [25].

Most procedures in `TSpherics` are specific to the problem at hand. This includes the bounds `NegInvLap`, `RotWave`, `InvPlate`, `InvWave`, and `InvQP`, on the operators $\mathcal{L}^{-1}$ used for stationary solutions, rotating periodic solutions, ordinary periodic solutions (for the plate and wave equations), and quasiperiodic solutions, respectively. These procedures use the estimates given in Section 5. Enclosures for the constants that appear in these estimates are obtained via the procedures in `TSpherics_Aux`. Some other procedures in `TSpherics` are used to plot graphs and are not part of the proof.

`TSpherics` also defines the type `TSMode` that characterizes subspaces like $\mathbb{P}_{\{j\}}\mathcal{B}$ (coefficient modes) and $\mathbb{P}_{\{j,j+1,\ldots\}}\mathcal{B}$ (error modes). Arrays of `TSMode` are used by the function `Make` to define partitions of unity (in the sense of direct sums) for the spaces $\mathcal{B}$ or the symmetric subspaces $\mathcal{B}_k$. This is the problem-specific part of a general infrastructure designed to implement bounds on quasi-Newton maps. Another part is handled by the packages `Linear` and `Linear.Contr` that work with a generic type `Fun` and associated `Modes`. These two packages include all the tools needed to construct (in terms of bounds) a quasi-Newton map $\mathcal{N}$ for a given map G, and to verify bounds like (3.3). The main task is to estimate the derivative of a linear operator such as $D\mathcal{N}(h)$.

The package at the top of our hierarchy is `TSpherics.Fix`. It first implements a bound `GMap` on the map G defined by (3.1), as well as a bound `DGMap` on the derivative of G. These are just compositions of bounds defined in `TSpherics`. Then `TSpherics.Fix` instantiates the packages `Linear` and `Linear.Contr` with `Fun => Spheric` and `Mode => TSMode`. Using the procedures `Op_Norm` and `DContr` from these two packages, implementing a bound `DContrNorm` on the map $h \mapsto \|D\mathcal{N}(h)\|$ is straightforward. This bound is used by `ContrFix` to verify the inequalities in the claim of Lemma 3.2. The ball B_δ mentioned in this lemma has a representable radius $\delta > 0$ and is described by a `Spheric`-type enclosure.

6.3. Organizing the bounds

Our proof of Lemma 3.2 is organized by the program `Run_All`. For each row in Table 1, `Run_All` fetches the necessary parameter values from the package `Params` and then calls the (standalone) procedures `Approx_Fixpt` and `Check_Fixpt` with an argument of type `Param`. Both of these procedures use an approximate fixed point $\bar{u}$ that is provided in a data file [31]. The procedure `Approx_Fixpt` is purely numerical and determines a matrix `M` that defines the operator M used in (3.2). Using the values in `Param`, the procedure `Check_Fixpt` instantiates the package `TSpherics.Fix` with the proper arguments. Then it calls `ContrFix` to verify the inequalities in the claim of Lemma 3.2. Finally, it calls `TSpherics.Leading_Coeffs` to generate the bounds given in Section 7.

Our programs were run successfully on a standard desktop machine, using a public version of the gcc/gnat compiler [33]. Instructions on how to compile and run these programs can be found in the README file that is included with the source text [31].

7. Additional data

Each of our solutions u can be represented as a series

$$u = \sum_{0\le m\le l} \mathcal{P}_l^m \times \mathfrak{c}_m \times A_{l,m} + \sum_{0<m\le l} \mathcal{P}_l^m \times \mathfrak{s}_m \times B_{l,m}\,, \tag{7.1}$$

where $f \times g \times h$ stands for the function $(\vartheta,\varphi,t) \mapsto f(\vartheta)g(\varphi)h(t)$, and h is represented as a Fourier series $h = a_0 + \sum_{n>0}(a_n\mathfrak{c}_n + b_n\mathfrak{s}_n)$. Here $\mathfrak{c}_j = \cos(j.)$ and $\mathfrak{s}_j = \sin(j.)$.

So u can be specified by giving a list of all modes $c_{l,m,n}\mathcal{P}_m^l \times \phi_m \times \phi_n$ that contribute to the sum (7.1), with $c_{l,m,n} \in \mathbb{R}$ and $\phi \in \{\mathfrak{c},\mathfrak{s}\}$. We consider an ordered version of this list, where the modes are listed in non-increasing order of the absolute values $|c_{l,m,n}|$.

The following tables show the first few modes in this ordered list, for each of our solutions u. The top-left entry in each table is the label of the solution u being described. This entry also links to a plot or animation of the solution u.

1	
	$2.3846\ldots * \mathcal{P}_1^1 \times \mathfrak{c}_1 \times \mathfrak{c}_0$
	$0.3061\ldots * \mathcal{P}_3^3 \times \mathfrak{c}_3 \times \mathfrak{c}_0$
	$-0.2371\ldots * \mathcal{P}_3^1 \times \mathfrak{c}_1 \times \mathfrak{c}_0$

2	
	$-4.6095\ldots * \mathcal{P}_2^1 \times \mathfrak{c}_1 \times \mathfrak{c}_0$
	$-0.9063\ldots * \mathcal{P}_4^3 \times \mathfrak{c}_3 \times \mathfrak{c}_0$
	$0.4272\ldots * \mathcal{P}_6^1 \times \mathfrak{c}_1 \times \mathfrak{c}_0$

3	
	$7.7561\ldots * \mathcal{P}_3^2 \times \mathfrak{c}_2 \times \mathfrak{c}_0$
	$-1.1214\ldots * \mathcal{P}_7^2 \times \mathfrak{c}_2 \times \mathfrak{c}_0$
	$1.0315\ldots * \mathcal{P}_7^6 \times \mathfrak{c}_6 \times \mathfrak{c}_0$
	$0.5973\ldots * \mathcal{P}_9^6 \times \mathfrak{c}_6 \times \mathfrak{c}_0$

4	
	$1.1841\ldots * \mathcal{P}_3^3 \times \mathfrak{c}_3 \times \mathfrak{c}_0$
	$-0.0103\ldots * \mathcal{P}_9^9 \times \mathfrak{c}_9 \times \mathfrak{c}_0$
	$-0.0067\ldots * \mathcal{P}_5^3 \times \mathfrak{c}_3 \times \mathfrak{c}_0$

5	
	$4.2884\ldots * \mathcal{P}_1^1 \times \mathfrak{c}_1 \times \mathfrak{c}_1$
	$-0.8516\ldots * \mathcal{P}_1^1 \times \mathfrak{c}_1 \times \mathfrak{c}_3$
	$0.2213\ldots * \mathcal{P}_1^1 \times \mathfrak{c}_1 \times \mathfrak{c}_5$
	$0.0603\ldots * \mathcal{P}_3^3 \times \mathfrak{c}_3 \times \mathfrak{c}_1$

6	
	$1.4762\ldots * \mathcal{P}_3^3 \times \mathfrak{c}_3 \times \mathfrak{c}_1$
	$-1.1434\ldots * \mathcal{P}_3^1 \times \mathfrak{c}_1 \times \mathfrak{c}_1$
	$-0.0230\ldots * \mathcal{P}_1^1 \times \mathfrak{c}_1 \times \mathfrak{c}_1$
	$0.0206\ldots * \mathcal{P}_5^5 \times \mathfrak{c}_5 \times \mathfrak{c}_1$

7	
	$1.9273\ldots * \mathcal{P}_3^2 \times \mathfrak{c}_2 \times \mathfrak{c}_1$
	$-0.0088\ldots * \mathcal{P}_7^2 \times \mathfrak{c}_2 \times \mathfrak{c}_1$
	$0.0081\ldots * \mathcal{P}_7^6 \times \mathfrak{c}_6 \times \mathfrak{c}_1$

8	$2.1316\ldots$	$* \ \mathcal{P}_1^1$	$\times\ \mathfrak{c}_1$	$\times\ \mathfrak{c}_1$
	$2.1316\ldots$	$* \ \mathcal{P}_1^1$	$\times\ \mathfrak{s}_1$	$\times\ \mathfrak{s}_1$
	$-1.5855\ldots$	$* \ \mathcal{P}_3^3$	$\times\ \mathfrak{c}_3$	$\times\ \mathfrak{c}_5$
	$1.5855\ldots$	$* \ \mathcal{P}_3^3$	$\times\ \mathfrak{s}_3$	$\times\ \mathfrak{s}_5$
	$-0.0171\ldots$	$* \ \mathcal{P}_3^1$	$\times\ \mathfrak{c}_1$	$\times\ \mathfrak{c}_1$
	$-0.0171\ldots$	$* \ \mathcal{P}_3^1$	$\times\ \mathfrak{s}_1$	$\times\ \mathfrak{s}_1$

9	$-8.9871\ldots$	$* \ \mathcal{P}_3^1$	$\times\ \mathfrak{c}_1$	$\times\ \mathfrak{c}_5$
	$8.9871\ldots$	$* \ \mathcal{P}_3^1$	$\times\ \mathfrak{s}_1$	$\times\ \mathfrak{s}_5$
	$0.1390\ldots$	$* \ \mathcal{P}_5^3$	$\times\ \mathfrak{c}_3$	$\times\ \mathfrak{c}_{15}$
	$-0.1390\ldots$	$* \ \mathcal{P}_5^3$	$\times\ \mathfrak{s}_3$	$\times\ \mathfrak{s}_{15}$
	$0.0957\ldots$	$* \ \mathcal{P}_1^1$	$\times\ \mathfrak{c}_1$	$\times\ \mathfrak{c}_5$
	$-0.0957\ldots$	$* \ \mathcal{P}_1^1$	$\times\ \mathfrak{s}_1$	$\times\ \mathfrak{s}_5$

10	$-2.8236\ldots$	$* \ \mathcal{P}_1^1$	$\times\ \mathfrak{c}_1$	$\times\ \mathfrak{c}_1$
	$-2.8236\ldots$	$* \ \mathcal{P}_1^1$	$\times\ \mathfrak{s}_1$	$\times\ \mathfrak{s}_1$
	$2.4385\ldots$	$* \ \mathcal{P}_3^3$	$\times\ \mathfrak{c}_3$	$\times\ \mathfrak{c}_5$
	$-2.4385\ldots$	$* \ \mathcal{P}_3^3$	$\times\ \mathfrak{s}_3$	$\times\ \mathfrak{s}_5$
	$1.2649\ldots$	$* \ \mathcal{P}_3^1$	$\times\ \mathfrak{c}_1$	$\times\ \mathfrak{c}_7$
	$-1.2649\ldots$	$* \ \mathcal{P}_3^1$	$\times\ \mathfrak{s}_1$	$\times\ \mathfrak{s}_7$
	$-0.2916\ldots$	$* \ \mathcal{P}_3^1$	$\times\ \mathfrak{c}_1$	$\times\ \mathfrak{c}_9$
	$-0.2916\ldots$	$* \ \mathcal{P}_3^1$	$\times\ \mathfrak{s}_1$	$\times\ \mathfrak{s}_9$

11	$-3.3000\ldots$	$* \ \mathcal{P}_1^1$	$\times\ \mathfrak{c}_1$	$\times\ \mathfrak{c}_1$
	$-3.3000\ldots$	$* \ \mathcal{P}_1^1$	$\times\ \mathfrak{s}_1$	$\times\ \mathfrak{s}_1$
	$1.5528\ldots$	$* \ \mathcal{P}_2^1$	$\times\ \mathfrak{c}_1$	$\times\ \mathfrak{c}_5$
	$-1.5528\ldots$	$* \ \mathcal{P}_2^1$	$\times\ \mathfrak{s}_1$	$\times\ \mathfrak{s}_5$
	$-0.3465\ldots$	$* \ \mathcal{P}_2^1$	$\times\ \mathfrak{c}_1$	$\times\ \mathfrak{c}_7$
	$-0.3465\ldots$	$* \ \mathcal{P}_2^1$	$\times\ \mathfrak{s}_1$	$\times\ \mathfrak{s}_7$
	$-0.1045\ldots$	$* \ \mathcal{P}_3^1$	$\times\ \mathfrak{c}_1$	$\times\ \mathfrak{c}_{11}$
	$0.1045\ldots$	$* \ \mathcal{P}_3^1$	$\times\ \mathfrak{s}_1$	$\times\ \mathfrak{s}_{11}$

References

[1] J.W.S. Cassels, *An introduction to Diophantine approximation*, Cambridge University Press (1957).

[2] P.H. Rabinowitz, *Free vibrations for a semi-linear wave equation*, Comm. Pure Appl. Math. **31**, 31–68 (1978).

[3] H. Brezis, J.-M. Coron, L. Nirenberg, *Free vibrations for a nonlinear wave equation and a Theorem of P. Rabinowitz* Comm. Pure Appl. Math. **33**, 667–684 (1980).

[4] P.H. Rabinowitz, *On nontrivial solutions of a semilinear wave equation*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **8**, 647–657 (1981).

[5] H. Brezis, *Periodic solutions of nonlinear vibrating strings and duality principles*, Bull. Amer. Math. Soc. **8**, 409–426 (1983).

[6] J. Moser, *Minimal solutions of variational problems on a torus*, Ann. Inst. H. Poincar'e Anal. Non Linéaire **3**, 229–272 (1986).

[7] M.L. Blank, *Metric properties of minimal solutions of discrete periodical variational problems*, Nonlinearity **2**, 1–22 (1989).

[8] W. Kryszewski, A. Szulkin, *An infinite dimensional Morse theory with applications*, Trans. Am. Math. Soc. **349**, 3181–3234 (1997).

[9] W. Kryszewski, A. Szulkin, *Generalized linking theorem with an application to a semilinear Schrödinger equation*, Adv. Diff. Eq. **3**, 441–472 (1998).

[10] C. Lee, *Periodic solutions of beam equations with symmetry*, Nonlinear Anal. **42**, 631–650 (2000)

[11] R. de la Llave, *Variational methods for quasi-periodic solutions of partial differential equations*, in: Hamiltonian Systems and Celestial Mechanics, pp. 214–228 (2000).

[12] J.Q. Liu, *Free vibrations for an asymmetric beam equation*, Nonlinear Anal. **51**, 487–497 (2002)

[13] J.Q. Liu, *Free vibrations for an asymmetric beam equation, II*, Nonlinear Anal. **56**, 415–432 (2004)

[14] G. Gentile, V. Mastropietro, M. Procesi, *Periodic solutions for completely resonant nonlinear wave equations with Dirichlet boundary conditions*, Commun. Math. Phys. **256**, 437–490 (2005).

[15] V. Mastropietro, M. Procesi, *Lindstedt series for periodic solutions of beam equations with quadratic and velocity dependent nonlinearities*, Commun. Pure Appl. Anal. 5, **1**, 128 (2006).

[16] M. Berti, *Nonlinear Oscillations of Hamiltonian PDEs*, Progress in Nonlinear Differential Equations and Their Applications, Volume 74, Birkhäuser 2007.

[17] G. Gentile, M. Procesi, *Periodic solutions for a class of nonlinear partial differential equations in higher dimension*, Commun. Math. Phys. **289**, 863–906 (2009).

[18] L.H. Eliasson, B. Grébert, S. Kuksin, *KAM for the non-linear beam equation*, Geom. Funct. Anal. **26**, 1588–1715 (2016).

[19] J-L. Figueras, R. de la Llave, *Numerical computations and computer cssisted proofs of periodic orbits of the Kuramoto–Sivashinsky equation*, SIAM J. Appl. Dyn. Syst. **16**, 834–852 (2016).

[20] G. Arioli, H. Koch, *Families of Periodic Solutions for Some Hamiltonian PDEs*, SIAM J. Appl. Dyn. Syst. **16**, 1–15 (2017).

[21] R. Castelli, M. Gameiro, J.-P. Lessard, *Rigorous numerics for ill-posed PDEs: periodic orbits in the Boussinesq equation*, Arch. Ration. Mech. Anal. **228**, 129–157 (2017).

[22] G. Arioli, H. Koch, *Spectral stability for the wave equation with periodic forcing*, J. Differ. Equations **265**, 2470–2501 (2018).

[23] L. Guo, B. Zhang, Y. Zhang, *Fractional p-Laplacian equations on Riemannian manifolds*, Electron. J. Differ. Equ. **156**, 1–17 (2018).

[24] R. de la Llave, Y. Sire, *An a posteriori KAM theorem for whiskered tori in Hamiltonian partial differential equations with applications to some ill-posed equations*, Arch. Ration. Mech. Anal. **231**, 971–1044 (2019).

[25] G. Arioli, H. Koch, *Non-radial solutions for some semilinear elliptic equations on the disk*, Nonlinear Analysis **179**, 294–308 (2019).

[26] M. Berti, P. Bolle, *Quasi-periodic solutions of nonlinear wave equations on the d-dimensional torus*, EMS Monographs in Mathematic, Volume 11, EMS Press (2020).

[27] G. Arioli, H. Koch, *A Hopf bifurcation in the planar NavierStokes equations*, J. Math. Fluid Mech. **23**:70 14pp (2021).

[28] J. B. van den Berg, M. Breden, J.-P. Lessard, L. van Veen, *Spontaneous periodic orbits in the Navier-Stokes flow*, J. Nonlinear Sci. **31**, 41 (2021).

[29] C.A. Rey, N. Saintier, *Non-local equations and optimal Sobolev inequalities on compact manifolds*, Preprint 2022.

[30] G. Arioli, H. Koch, *Validated numerical solutions for a semilinear elliptic equation on some topological annuli in the plane*, Preprint 2022.

[31] G. Arioli, H. Koch. The source code for our programs, and data files, are available at `web.ma.utexas.edu/users/koch/papers/spheric/`

[32] Ada Reference Manual, ISO/IEC 8652:2012(E), avaliable at `www.ada-auth.org/arm.html`

[33] A free-software compiler for the Ada programming language, which is part of the GNU Compiler Collection; see `gnu.org/software/gnat/`

[34] The Institute of Electrical and Electronics Engineers, Inc., *IEEE Standard for Binary Floating–Point Arithmetic*, ANSI/IEEE Std 754–2008.