Renormalization of Hamiltonian flows and critical invariant tori

(H. Koch)

- introduction
- formal definition of $\mathcal{R}$
- near-integrable Hamiltonians
- connection with commuting maps
- normal form: resonant Hamiltonians
- a non-trivial RG fixed point
- on critical invariant tori

→ DCDS 8, #3 (2002)
→ mp_arc 02-175

Collaborators on closely related problems

J. J. Abad, P. Wittwer

Other related work

- perturbative renormalization

  ((Eliasson), Feldman, Trubowitz, Gallavotti, Chierchia, Falcolini, Ecalle, Valet, Gentile, Mastropietro, ...)

- renormalization à la QFT

  (Bricmont, Gawedzki, Kupiainen, Schenkel, ...)

- commuting maps

  (Kadanoff, Shenker, MacKay, Morrison, ..., Stirnemann)

- approx. renormalization of Hamiltonians

  (Escande, Doveil, Jauslin, Govin, Chandre, Benfatto, Celletti, ...)

- renormalization of torus flows

  (J. Lopes Dias)

See references in mp_arc 02-175

Consider mainly analytic H's

$$H(q,p)\ ,\quad q\in\mathbb{T}^d,\quad p\in B^d$$

$$\dot{F}=\{F,H\}=\nabla_q F\cdot\nabla_p H-\nabla_q H\cdot\nabla_p F$$

$$\dot{q}=\nabla_p H$$

$$\dot{p}=-\nabla_q H$$

and closed orbits,

or invariant ω-tori "centered at $p=0$" *

$$\Gamma:\mathbb{T}^d\times\{0\}\to\mathbb{T}^d\times B^d$$

$$\Phi_t\circ\Gamma=\Gamma\circ\Phi_t^0$$

Integrable example

$$H(q,p)=\omega\cdot p+h(p)\qquad Dh(0)=0$$

$$\Phi_t^0(q,0)=(q+t\omega,0)$$

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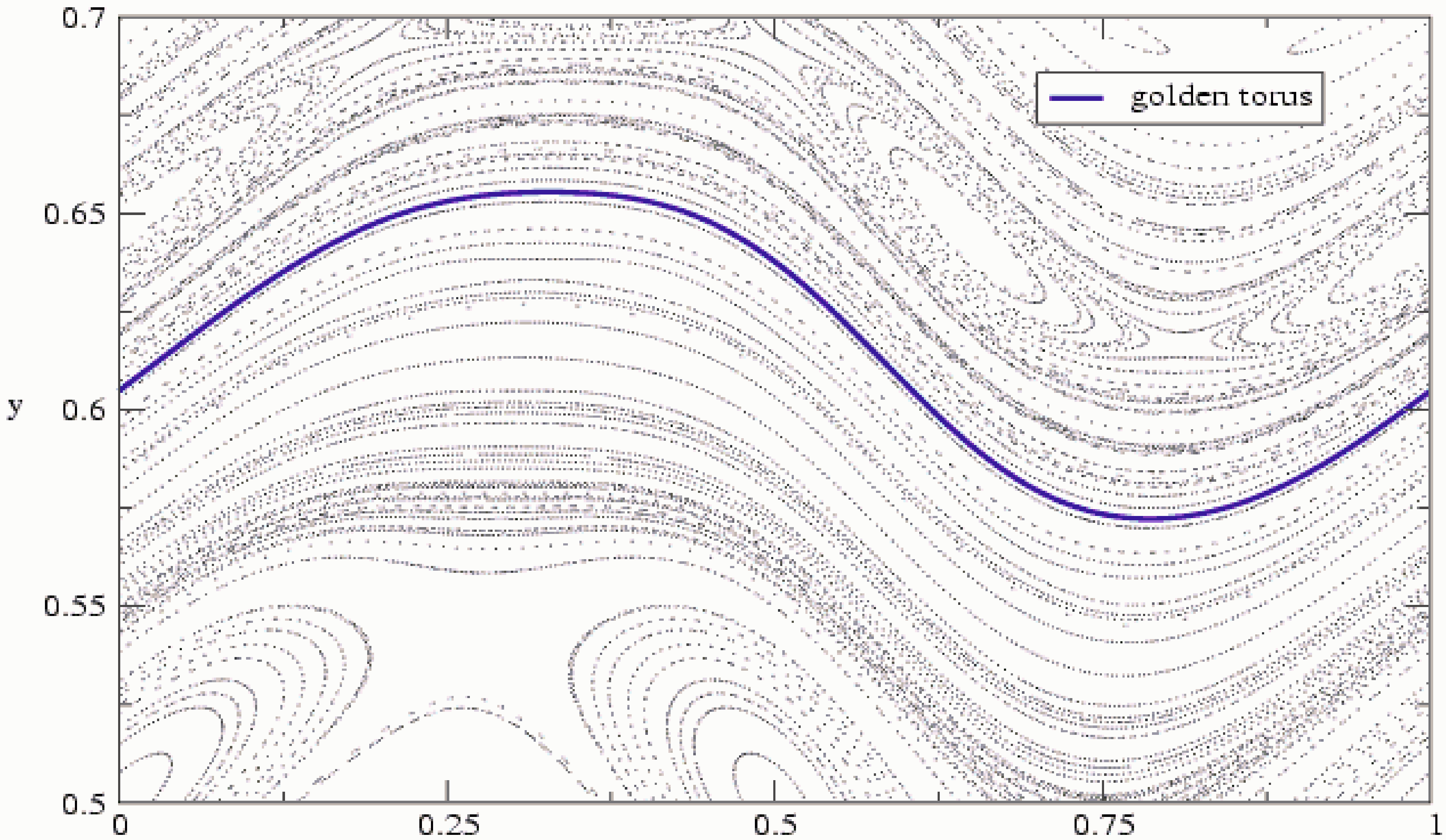
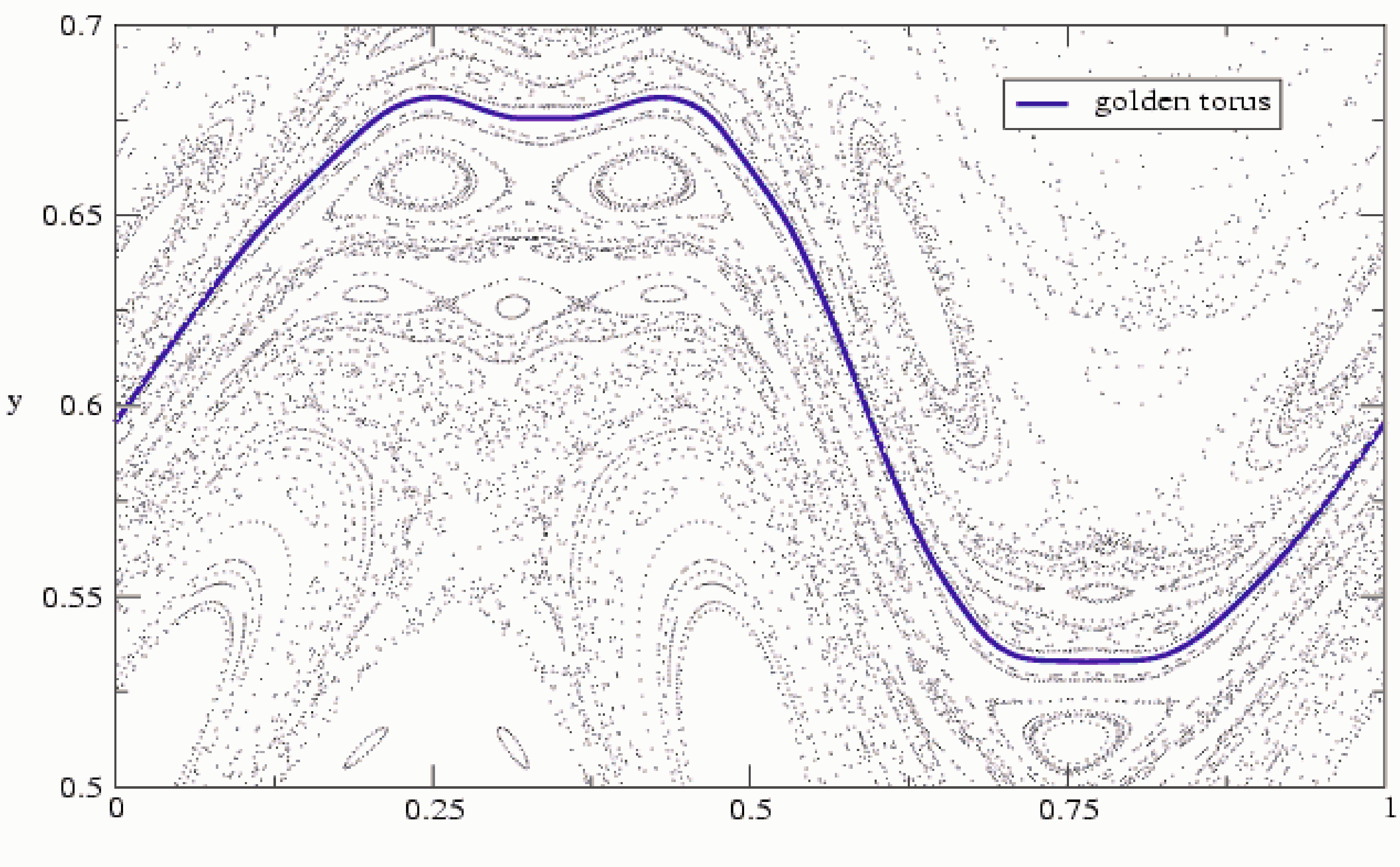
* $\int_\gamma p\cdot dq=0$

standard map orbits (A. Haro)

K= 0.50



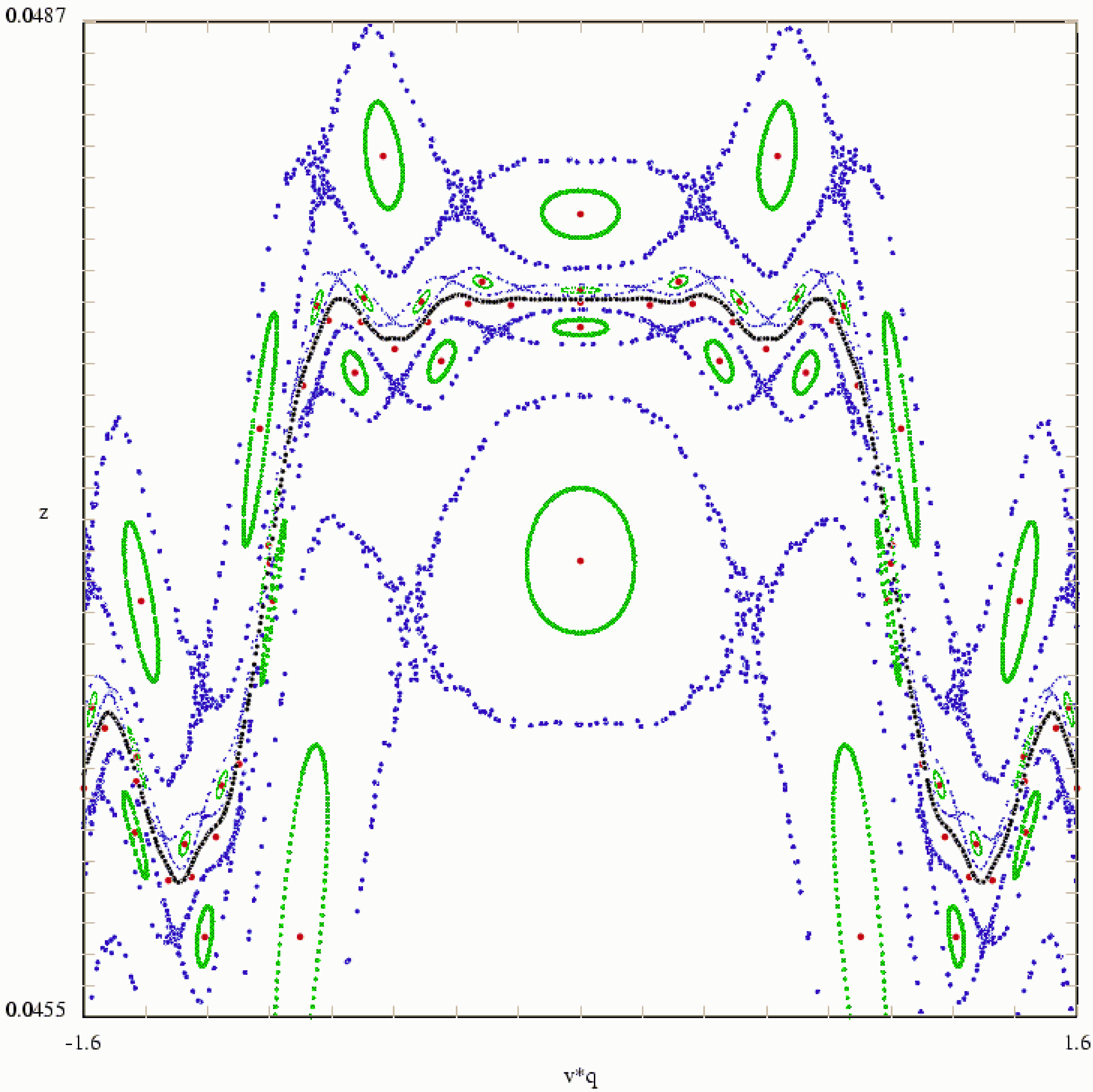
K= 0.93



Periods $\frac{13}{8}$, $\frac{21}{13}$, $\frac{34}{21}$, ... and critical circle for the return map of H_*

(J.J. Abad, H.K., P. Wittwer)

Figure 1



Phenomena considered are invariant under

$\mathcal{G}$: transformations that do not change (direction of) rotation vectors, stability, ...

Considering only

- scaling of the energy (or time)

$$H \mapsto \tau H - E \qquad (\tau, E \text{ const.})$$

- scaling of the action variables

$$H \mapsto \mu^{-1} H(\cdot, \mu\,\cdot)$$

- canonical transformations homotopic to Id

$$H \mapsto H \circ U_\phi$$

$U_\phi : (q,p) \mapsto (q+Q, p+P)$ defined implicitly

$$\begin{aligned} Q(q,p) &= \nabla_p \phi(q, p+P(q,p)) \\ P(q,p) &= -\nabla_q \phi(q, p+P(q,p)) \end{aligned} \qquad (\text{periodic in } q)$$

If ϕ is "small" then

$$H \circ U_\phi = H + \{H, \phi\} + \text{small}^2$$

Other canonical transformations can change (direction of) rotation vectors. Consider

$$\mathcal{T}(q,p) = \left(Tq, (T^*)^{-1}p\right) \qquad T \text{ linear}, \ldots$$

Correspondence:

orbit of H with rotation vector w

—"— $H \circ \mathcal{T}$ —"— $T^{-1}w$

Consider only integer $d \times d$ matrices T with

- $\det(T) = \pm 1$
- one expanding eigenv.: $T\omega = \vartheta\omega$, $\vartheta > 1$
- $d-1$ contracting eigenvalues

Example: golden mean

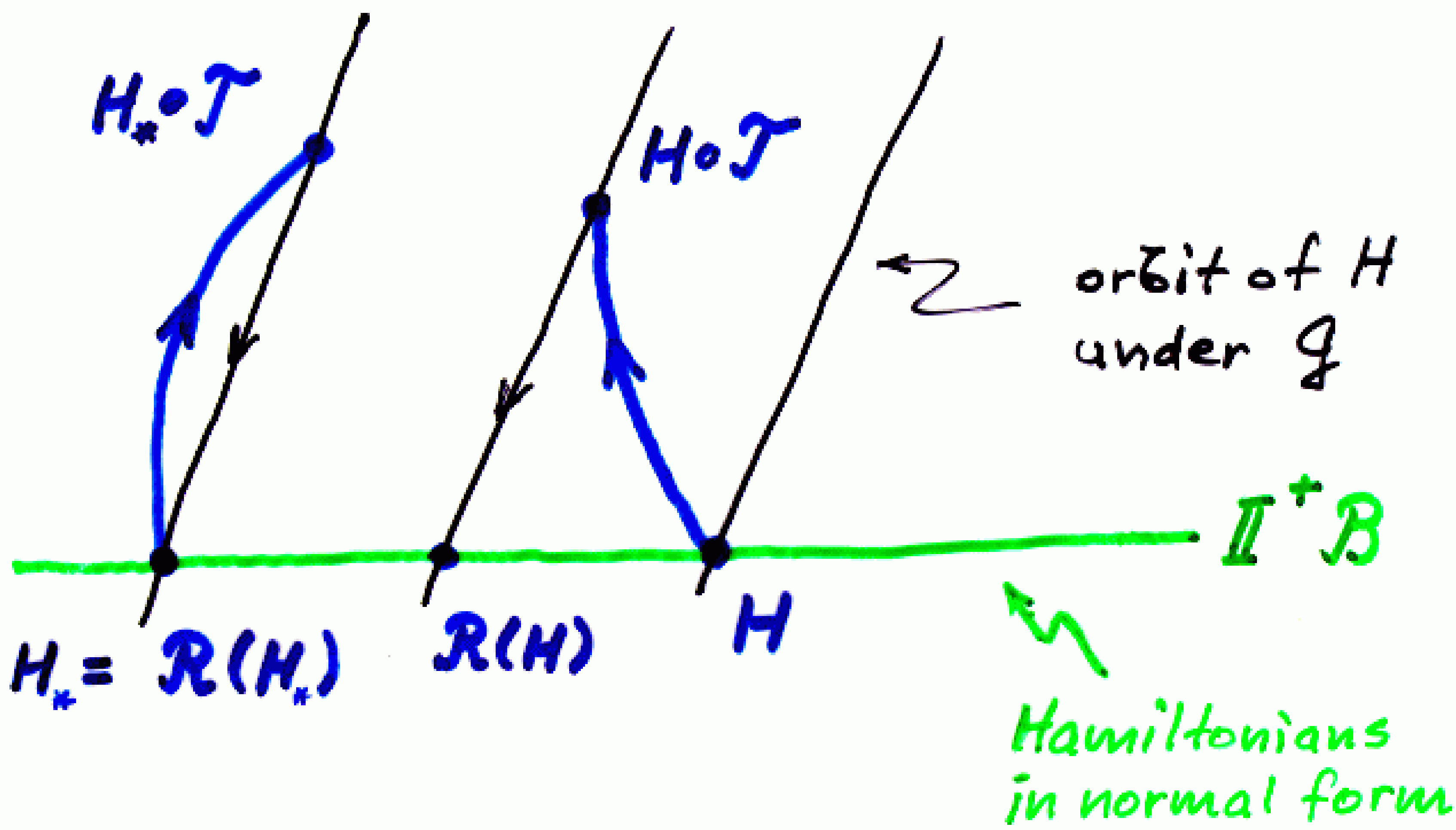
$$T = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad \omega = (1, \vartheta), \quad \vartheta = \frac{\sqrt{5}+1}{2}$$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \to \begin{bmatrix} 1 \\ 2 \end{bmatrix} \to \begin{bmatrix} 2 \\ 3 \end{bmatrix} \to \begin{bmatrix} 3 \\ 5 \end{bmatrix} \to \ldots$$

$$\frac{1}{1}, \quad \frac{2}{1}, \quad \frac{3}{2}, \quad \frac{5}{3}, \ldots \quad \to \vartheta$$

$$\mathcal{R}(H) = H \circ \mathcal{T} \quad (\mathrm{mod}\ \mathcal{G})$$

Choose an appropriate* normal form



$\mathbb{I}^+$: projection onto resonant subspace

* appropriate:

- $\mathcal{R}$ is a hyperbolic dynamical system on $\mathbb{I}^+\mathcal{B}$
- the number of expanding directions is finite, and as small as possible

$$\mathcal{R}(H) = \underbrace{\frac{\tau_H}{\mu} H \circ \mathcal{T}_\mu \circ U_{\phi_0}}_{H'} \circ \mathcal{U}_{H'} - E_H$$

$\mathcal{T}_\mu$: $(q,p) \mapsto (Tq, \mu (T^*)^{-1} p)$ $\quad \mu = \dots$

U_{ϕ_0} : rough normalization
same for all H's in a small n'hood
$= Id$ for analysis of near-integrable H's

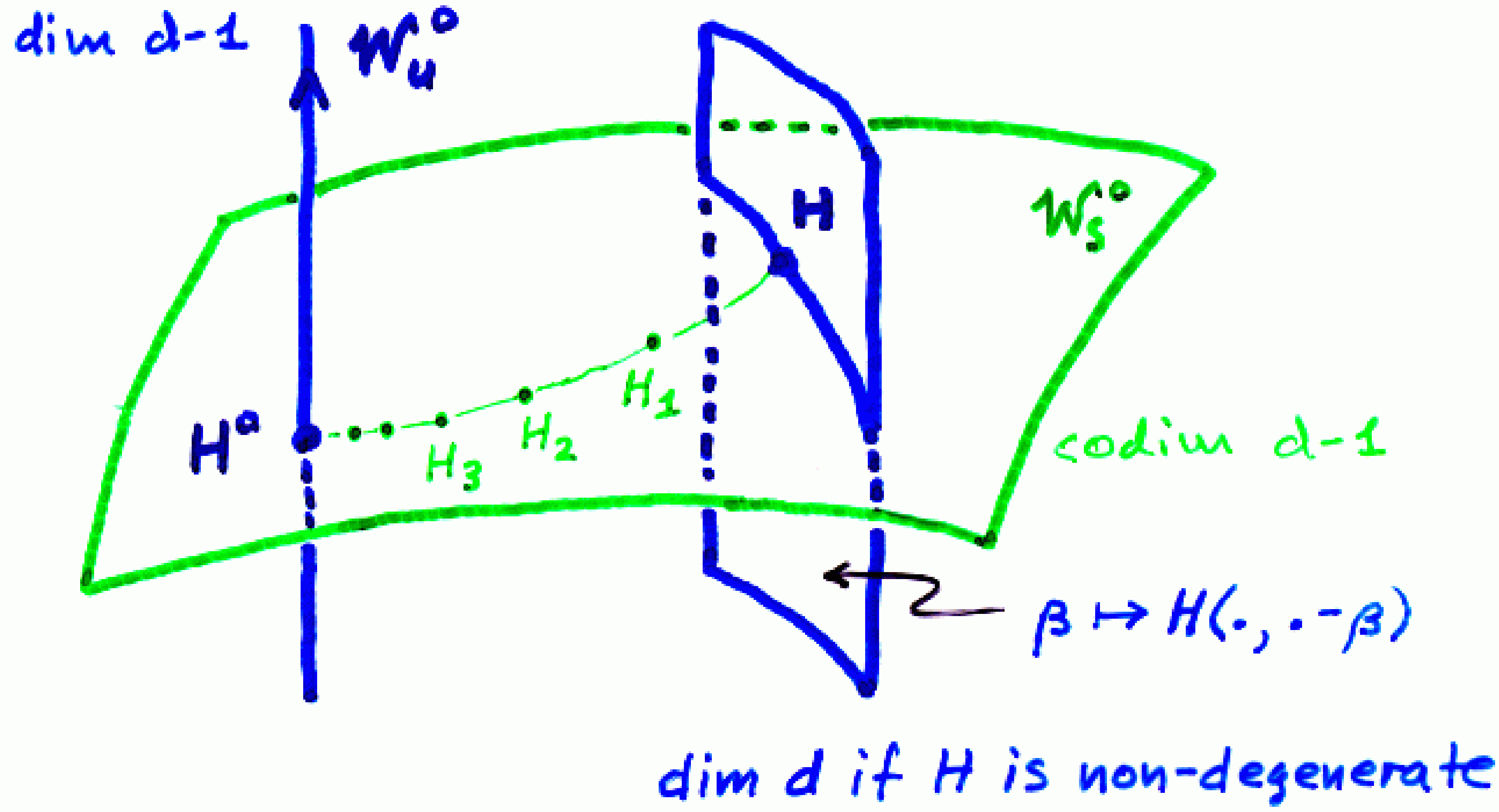
$\mathcal{U}_{H'}$: final normalization
$= U_{\phi_1} \circ U_{\phi_2} \circ \dots$ $\quad \phi_k$ small, non-resonant
eliminates non-resonant modes

$$\mathbb{I}^-(H' \circ \mathcal{U}_{H'}) = 0$$

$\mathbb{I}^- = \mathbb{I} - \mathbb{I}^+$: projection onto non-resonant subspace

(precise definitions later)

Theorem: If ..., then near $H^\circ(q,p) = \omega \cdot p$, $\mathcal{R}$ is well defined and analytic and ...



Application 1: invariant ω-torus

$$\Gamma_H = V_0 \circ V_1 \circ V_2 \circ \ldots$$

$$V_n = \mathcal{T}_\mu^{\,n} \circ \mathcal{U}_{H_n'} \circ \mathcal{T}_\mu^{\,-n}$$

$$H_n = \mathcal{R}^n(H)$$

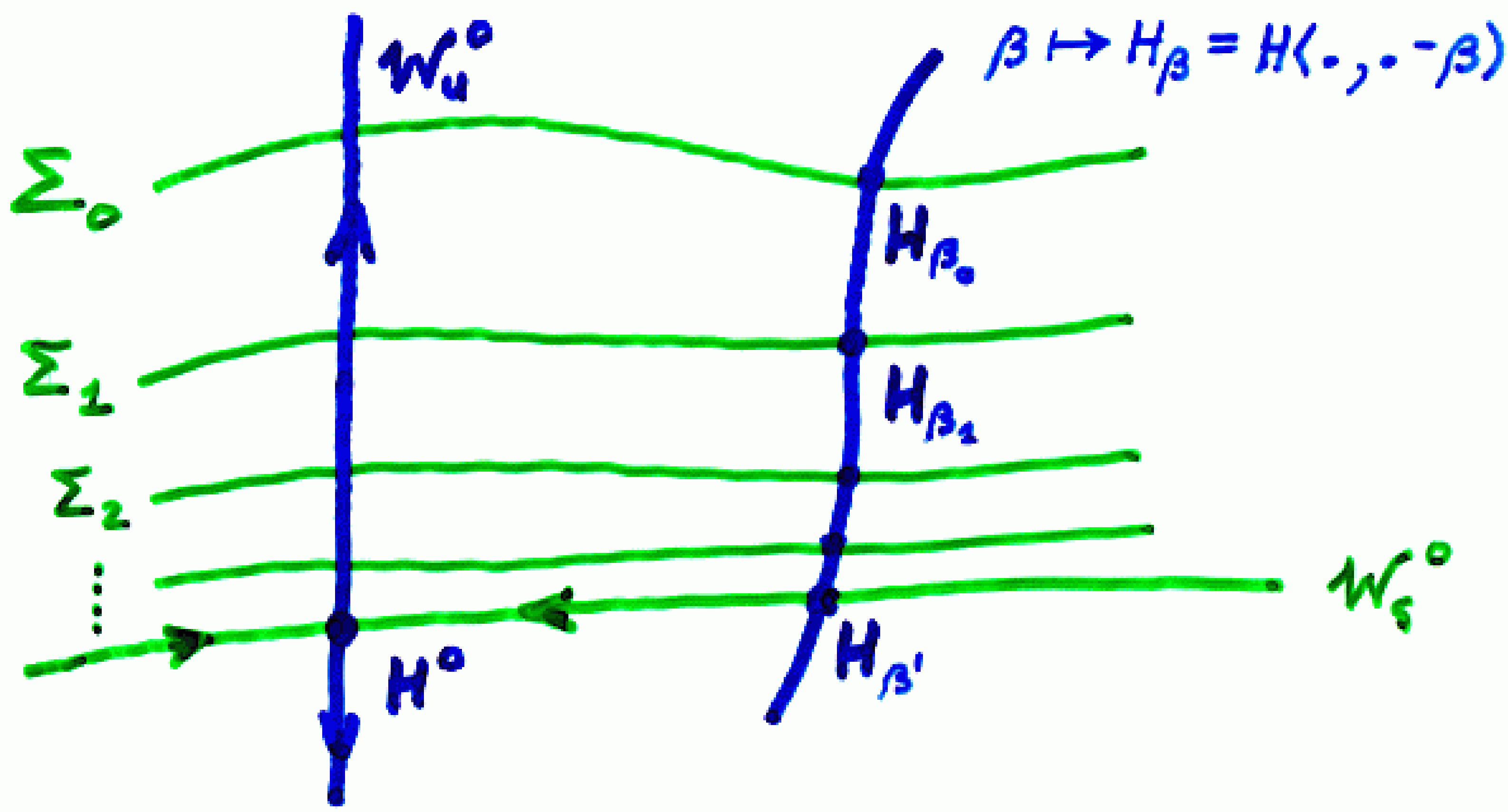
Application 2 (with J.J. Abad)

Consider H's even under $q \mapsto -q$.
Fix w near ω with w_i/w_j rational.
Define a manifold Σ such that H's on

$$\Sigma_n = \mathcal{R}^{-n}(\Sigma \cap \text{Ball})$$

have a closed orbit γ_n with frequency vector proportional to $w_n = T^n w$

Theorem: If ... then

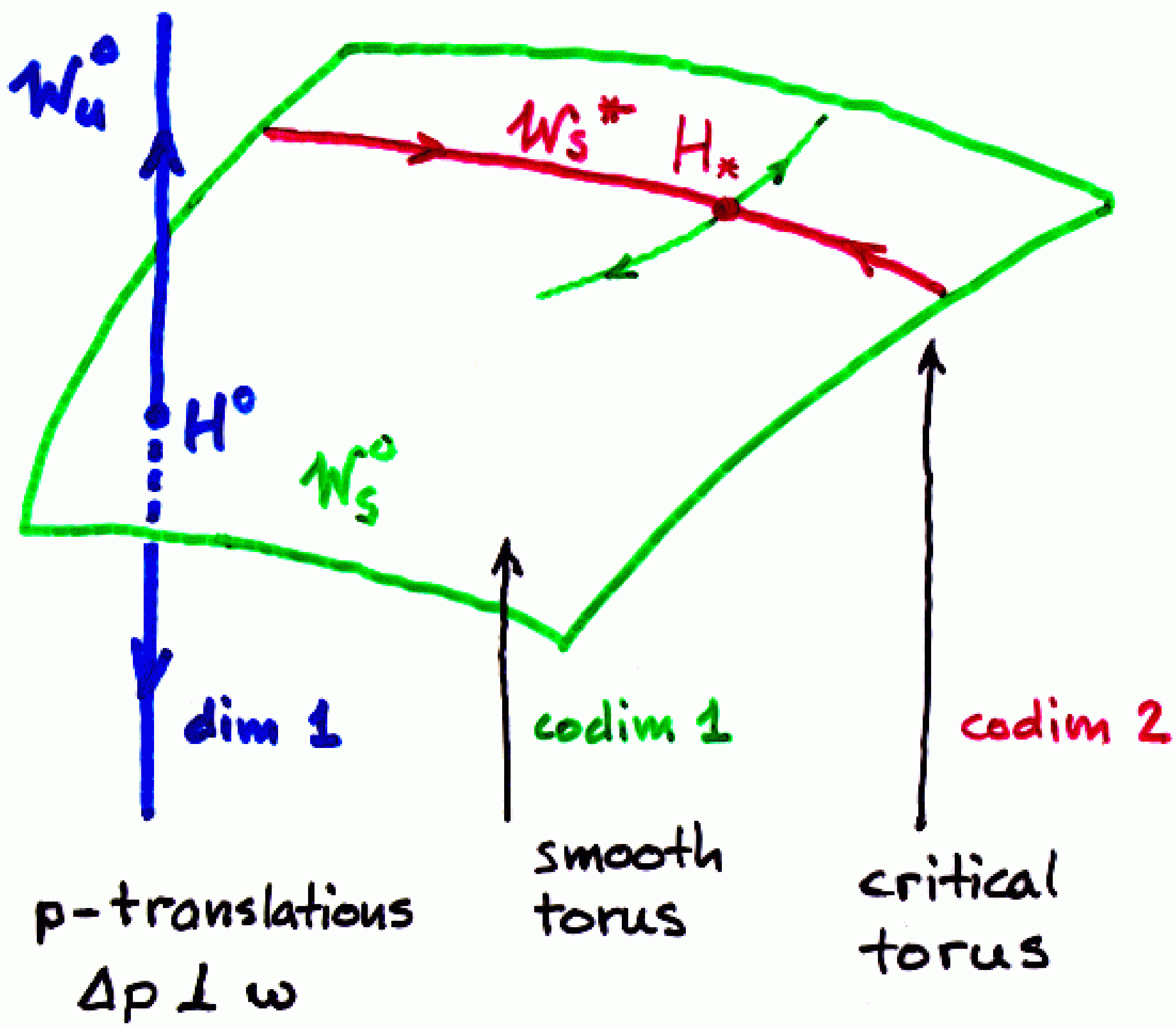


$$-\frac{1}{n}\ln|\gamma_n(0)-\Gamma'(0)| = \ln|\lambda_2| + \mathcal{O}\left(\frac{1}{n}\right)$$

2nd largest eigenvalue of $D\mathcal{R}(H^0)$

For $T = \begin{bmatrix} 0 & 1 \\ 1 & N \end{bmatrix}$ $\left(\vartheta = \dfrac{N + \sqrt{N^2+1}}{2} \right)$

expecting



$\omega = (1, \vartheta)$

Connection with commuting maps:

Let $T^{*}\omega' = \vartheta\omega'$, $\omega'\cdot\omega = 1$. Assume

$$\omega'\cdot\nabla_p H = 1 \qquad (\,\omega'\cdot q \text{ is "time"}\,)$$

Consider only coord. changes leaving $\omega'\cdot q$ invar.

$$\mathcal{R}(H) = \frac{\vartheta}{\mu} H\circ\Lambda \qquad \Lambda = \mathcal{T}_\mu \circ U_{\phi_0} \circ \mathcal{U}_{H'}$$

Define

$$\mathcal{F}_k = \Phi_{2\pi\omega_k'} \circ V(2\pi\delta_k) \qquad k = 1, 2, \dots, d$$

q-translations

<u>Theorem</u>: Assume H diff'ble, Λ diffeo, on $\mathbb{T}^d \times \mathbb{R}^d$.
Then $\mathcal{F}_k$'s commute with each other, and leave $\{(q,p)\in H^{-1}(0) : \omega'\cdot q = 0\}$ invariant.
If $\tilde{H} = \mathcal{R}(H)$ and ... then

$$\tilde{\mathcal{F}}_k = \Lambda^{-1}\circ \mathcal{F}_1^{T_{1,k}} \circ \mathcal{F}_2^{T_{2,k}} \circ \dots \circ \mathcal{F}_d^{T_{d,k}} \circ \Lambda$$

From now on

$$T = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad \vartheta = \frac{\sqrt{5}+1}{2}$$

$$\omega = (\vartheta^{-1}, 1), \quad \omega' = \text{const}\,\omega \quad \text{s.t. } \omega'\cdot\omega = 1$$

$$\Omega = (1, -\vartheta^{-1}), \quad \Omega' = \text{const}\,\Omega \quad \text{s.t. } \Omega'\cdot\Omega = 1$$

Domain D_ϱ and space $\mathcal{B}_\varrho$

$$D_\varrho : \ \operatorname{Im} \omega'\cdot q = 0, \ |\operatorname{Im} \Omega'\cdot q| < \varrho_1, \ |\Omega\cdot p| < \varrho_2$$

$$H(q,p) = \omega\cdot p + \sum_{\nu,k} h_{\nu,k} \cos(\nu\cdot q)\,(\Omega\cdot p)^k$$

$$\|h\|_\varrho = \sum_{\nu,k} |h_{\nu,k}| \cosh(\varrho_1 \Omega\cdot\nu)\, \varrho_2^k$$

Notice

- no decay of coeff's in directions where $\omega\cdot\nu \to \pm\infty$ with $\Omega\cdot\nu$ and k bounded (H's can be rough in direction of flow)
- $h \mapsto h\circ T_\mu$ is bounded from $\mathcal{B}_\varrho$ to $\mathcal{B}_{\vartheta\varrho}$ if $|\mu| \le \vartheta^{-2}$

Solve $\mathbb{I}^-(H\circ\mathcal{U}_H)=0$ iteratively:

$$H = H_1 + f_1\ ,\quad \mathbb{I}^- H_1 = 0\ ,\quad f_1 \text{ small}$$

First step (ϕ_1 small)

$$H\circ U_{\phi_1} = H_1 + \underbrace{f_1 + \{H_1, \phi_1\}} + \text{small}^2$$

Solve $\quad \mathbb{I}^-(f_1 + \{H_1, \phi_1\}) = 0 \quad (\mathbb{I}^+\psi_1 = 0)$

$$\psi_1 = \left[\mathbb{I} + \mathbb{I}^-\mathcal{D}(H_1 - H_0)\right]^{-1}\mathbb{I}^- f_1 \qquad \phi_1 = \frac{1}{\omega\cdot\nabla_q}\,\psi_1$$

$$\mathcal{D}(h_1) = \left[(\Omega'\cdot\nabla_p h_1)\frac{\Omega\cdot\nabla_q}{\omega\cdot\nabla_q} - (\Omega\cdot\nabla_q h_1)\frac{\Omega'\cdot\nabla_p}{\omega\cdot\nabla_q}\right]\mathbb{I}^-$$

Now

$$\tilde{H} = H\circ U_{\phi_1} = \underset{\mathbb{I}^+\tilde{H}}{\underset{\uparrow}{H_2}} + \underset{\mathbb{I}^-\tilde{H}}{\underset{\uparrow}{f_2}}\ ,\quad f_2 \text{ small}^2$$

iterate ...

$$\mathcal{U}_H = U_{\phi_1}\circ U_{\phi_2}\circ U_{\phi_3}\circ\ldots$$

Require

(1) $\mathcal{D}(h_n)$ bounded, op-norm < 1

(2) functions in $\mathbb{I}^+\mathcal{B}_\rho$ analytic

Define

$$I^+ : \ |\omega\cdot\nu| \le \sigma|\Omega\cdot\nu| \ \underline{\text{or}} \ |\omega\cdot\nu| < \kappa k$$

$$(\mathbb{I}^+ h)(q,p) = \sum_{(\nu,k)\in I^+} h_{\nu,k} \cos(\nu\cdot q)(\Omega\cdot p)^k$$

Then

(1) on I^-, $\frac{\Omega\cdot\nu}{\omega\cdot\nu}$ and $\frac{k}{\omega\cdot\nu}$ are bounded

$$\|\mathcal{D}(h)\|_\rho \le \frac{1}{\sigma}\|\Omega\cdot\nabla_p h\|_\rho + \frac{1}{\kappa\rho_2}\|\Omega\cdot\nabla_q h\|_\rho$$

(2) coeff's of $\mathbb{I}^+h$ decrease exponentially:
in I^+, if $\omega\cdot\nu \to \pm\infty$ then $|\Omega\cdot\nu| \to \infty$ or $k\to\infty$
Specifically, if $\rho' = (\rho_1 + \varepsilon\sigma, e^{\varepsilon\kappa}\rho_2)$ then

$$\cosh(\rho_1'\Omega\cdot\nu)(\rho_2')^k \ge \text{const}\, e^{\varepsilon|\omega\cdot\nu|}\cosh(\rho_1\Omega\cdot\nu)\,\rho_2^k$$
$$(\nu,k)\in I^+$$

$$\begin{aligned}(\mathcal{S}H)(q,p) &= c_{_H} H\bigl(q, p/c_{_H}\bigr)\,,\\ \mathcal{L}H &= \vartheta\mu_{_0}^{-1} H \circ \mathcal{T}_{\mu_0} \circ U_{\phi_0}\,,\\ \mathcal{N}(H) &= H \circ \mathcal{U}_{_H}\,,\end{aligned}$$

where $c_{_H} = 2h_{0,2}$ and $\mu_{_0} = \vartheta^{-3}$.

$$\mathcal{R} = \mathcal{N} \circ \mathcal{L} \circ \mathcal{S}$$

Theorem. *There exists an even real resonant Fourier-Taylor polynomial h_1, an odd real Fourier-Taylor polynomial ϕ_0, and a choice of the parameters σ, κ, ρ, such that $\mathcal{R}$ defines a compact analytic map, from some open neighborhood B of $H_1 = H_0 + h_1$ in $\mathcal{B}'_\rho$, to the resonant subspace of $\mathcal{B}'_\rho$. This map $\mathcal{R}$ has a unique fixed point H_* in B, which is real analytic, and non-trivial, in the sense that $c_{_{H_*}} > 1$ ($c_{_{H_*}} = 1.024332969\ldots$).*

For a (computer-assisted) proof, consider an approx. Newton map $\mathcal{M}$,

$$\mathcal{M}(h) = h + \mathcal{R}(H_1 + Mh) - (H_1 + Mh)$$

$$M \approx \mathbb{I} - D\mathcal{R}(H_1)$$

and show that $\mathcal{M}$ contracts a ball $B(r)$ ...

Problem: estimating $D\mathcal{M}$ "as usual" is computationally prohibitive

Cure: perturbation theory about H_1 (includes the use of $\mathcal{U}_{\phi_0}$)

Affine approximation $\mathcal{N}_1$ of $\mathcal{N}: H \mapsto H \circ \mathcal{U}_H$

$$\mathcal{N}_1(H_1 + f_1) = H_1 + \mathbb{I}^{+}(f_1 + \{H_1, \phi_1\}) \qquad \phi_1 = \ldots$$

defines an approximate RG

$$\mathcal{R}_1 = \mathcal{N}_1 \circ \mathcal{L} \circ \mathcal{S}$$

Expecting $\mathcal{N} - \mathcal{N}_1$ and $\mathcal{R} - \mathcal{R}_1$ of order r^2

Define

$$\tilde{\mathcal{P}}(H_1, f) = \mathcal{P}(H_1 + f) - \mathcal{P}(H_1) - D\mathcal{P}(H_1) f$$

$$\tilde{\mathcal{R}}(H_1, f) = D\mathcal{N}_1(H_1)\, \mathcal{L}\, \tilde{\mathcal{P}}(H_1, f)$$

Then

$$\mathcal{M}(h) = \left(\mathcal{R}_1(H_1) - H_1 \right) \qquad (a)$$

$$+ \left[\mathbb{I} - (\mathbb{I} - D\mathcal{R}_1(H_1)) M \right] h \qquad (b)$$

$$+ \tilde{\mathcal{R}}(H_1, Mh) + (\mathcal{R} - \mathcal{R}_1)(H_1 + Mh) \qquad (c)$$

To get $\mathcal{M} : B(r) \circlearrowleft$, show

(a) $\mathcal{R}_1(H_1) - H_1$ of norm $\varepsilon \ll r$

(b) $[\ldots.]$ has op-norm $K < 1$

(c) $\| \tilde{\mathcal{R}}(\ldots) + (\mathcal{R} - \mathcal{R}_1)(\ldots) \|_\rho \leq C r^2$ with ...

To get contraction

extend bound (c) to $B(2r)$

Cauchy $\rightarrow$ derivatives of non-linear terms bounded by Cr on $B(r)$

Chose

$\sigma \approx 0.85001$, $\kappa \approx \sigma/0.4$

$\rho \approx (0.85, 0.15)$, $\rho^* > \rho$

$r \approx 3 * 10^{-12}$

Estimated

$\varepsilon = \| \mathcal{R}_1(H_1) - H_1 \|_{\rho^*}$ $\quad (< 10^{-14})$

$K = \| \mathbb{I} - (\mathbb{I} - D\mathcal{R}_1(H_1))M \|_{\rho}$ $\quad (< 0.84)$

$K_n = \| \tilde{\mathcal{R}}(H_1, Mh) \|_{\rho^*}$ $\quad h \in B(nr) \quad n = 1, 2$

$K'_n = \| (\mathcal{R} - \mathcal{R}_1)(H_1 + Mh) \|_{\rho^*}$ $\quad h \in B(nr) \quad n = 1, 2$

Verified

$\varepsilon + Kr + K_1 + K'_1 < r$, $\quad K + (K_2 + K'_2)/r < 1$

etc.

- order of operators in $\mathcal{R}$ chosen carefully (e.g. contraction near beginning)
- many "higher order" terms
- variable "degrees"
- optimized first 5 Nash-Moser steps
- parallelized composition estimates
- no switching between FPU modes
- ...

Critical tori :

(applies not only to golden mean, but ...)

consider $H = H_*$

$$\frac{\vartheta}{\mu} H \circ \Lambda = H \qquad \Lambda = \mathcal{T}_\mu \circ \mathcal{U}_{\phi_0} \circ \mathcal{U}_{H'}$$

Then

$$\Lambda \circ \Phi_t = \Phi_{\vartheta t} \circ \Lambda \qquad (\#)$$

If H has an invariant ω-torus Γ
then $(\#)$ and technical assumptions imply that

$$\Lambda \circ \Gamma \circ \mathcal{T}^{-1}$$

is also an invariant ω-torus for H.
Thus, if unique,

$$\Gamma = \Lambda \circ \Gamma \circ \mathcal{T}^{-1}$$

Note: If Λ has a fixed point x, then

$$\Lambda(\Phi_t(x)) = \Phi_{\vartheta t}(x) \qquad \rightarrow \textit{invariant manifold}$$

Conversely,

Theorem: Assume $\Gamma : \mathbb{T}^2 \times \{0\} \to D_\varsigma$ is continuous, "admissible", and satisfies

$$\Gamma = \Lambda \circ \Gamma \circ \mathcal{T}^{-1} \qquad (*)$$

(a) If the derivative of Λ at $x = \Gamma(0)$ has exactly one non-contracting direction, and if $t \mapsto \Gamma(t\omega, 0)$ is C^1, then Γ is an invariant ω-torus for H.

(b) If in addition, $-\vartheta^{-1}$ is not an eigenvalue of $D\Lambda(x)$, then Γ is not C^1.

Proof: Use that Γ maps the unstable manifold of $\mathcal{T}$ at 0 to the unstable manifold of Λ at x.

....

Next goal: Solve (*) and check conditions

Equation (*) is equiv. to $\mathcal{F}(\gamma)=\gamma$,

$$\mathcal{F}(\gamma) = \Lambda \circ \Gamma \circ \mathcal{J}^{-1} - I$$
$$= \mathcal{J}_\mu \circ [u \circ (I+\gamma) + \gamma] \circ \mathcal{J}^{-1}$$

$\Gamma - I$ $\qquad$ $U - I$ $\qquad$ $U = U_{\phi_0} \circ U_{H'} = \mathcal{J}_\mu^{-1} \circ \Lambda$

Crucial: $u = (f\Omega, \ldots)$ and $\gamma = (g\Omega, \ldots)$
have zero component
in the expanding direction of $\mathcal{J}_\mu$

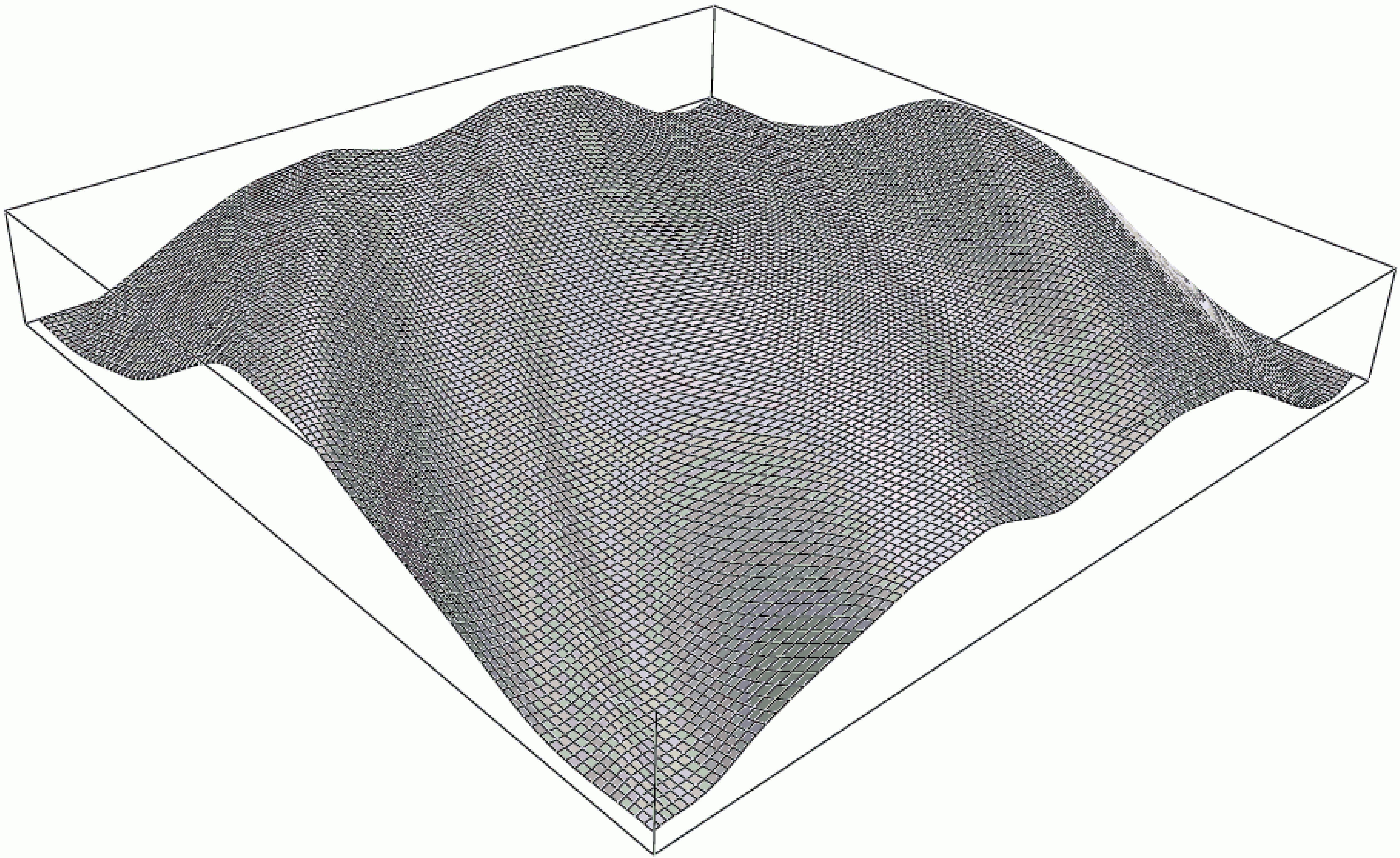
→ Get contraction in a space $\mathcal{A}_r$

$$\|\gamma\|_r = \sum_\nu |\gamma_\nu| \, e^{r|\omega\cdot\nu|} (1 + |\Omega\cdot\nu|)^r$$

with suff. small $r > 0$.

Starting with an approx. solution γ' ...

Approximate numerical solution
of torus equation (*)



Bounds on Λ and γ' that hold numerically and seem provable in the golden mean case

Theorem: If .... then $\mathcal{F}$ has a fixed point γ_* in $\mathcal{A}_r$ near γ', and this fixed point defines an invariant ω-torus $\Gamma_* = I + \gamma_*$ for H_*.
If in addition, $-\vartheta^{-1}$ is not an eigenvalue of $D\Lambda(\Gamma(0))$, then γ_* is not C^1.