Outline Based on
Complex Variables and Applications
Brown and Churchill, 6th ed.

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1 Chapter 1 – Basics

2 Chapter Two – Analytic Functions

- Definition of limit: $\lim_{z\to z_0} f(z) = w_0$
  implies that $|f(z) - w_0| < \epsilon$ whenever $0 < |z - z_0| < \delta$
- Cauchy-Riemann Equations
  If f(z) = u(x,y) + iv(x,y) and $f'(z)$ exists $z_0 = x_0 + iy_0$, then the first-order partial derivatives of u and f must exist at (x_0, y_0), and they must satisfy the Cauchy-Riemann equations:
  $$u_x = v_y \qquad u_y = -v_x$$
- Sufficient Conditions for Differentiability
  Let the function $f(z) = y(x, y) + iv(x, y)$ be defined throughout some e neighborhood of $z_0 = x_0 + iy_0$. Suppose that the first order partial derivatives of the functions u and v with respect to x and y exist everywhere in that neighborhood, and that they are continuous at (x_0, y_0). Then, if those partials derivatives satisfy the Cauchy-Riemann equations at (x_0, y_0), the derivative exists.
- Polar Coordinates Same as theorem in 18, polarized. Polar form of C-R:
  $$u_r = \frac{1}{2}v_\theta \qquad \frac{1}{r}u_\theta = -v_r$$

2.9 Analytic Functions

- A function f of z is analytic in an open set if it has a derivative at each point in that set. In particular, f in analytic at a point z_0 if it is analytic in a neighborhood of z_0.

- An entire function is a function that is analytic at each point in the entire finite plane.
- If a point fails to be analytic a point z_0, but is analytic at some point in every neighborhood of z_0, then z_0 is called a singular point.
- Since the derivatives of the sum and product of two functions exists wherever the functions themselves have derivatives, the sum and product of two analytic functions are themselves analytic. So is the quotient, wherever the denominator does not vanish. So is a composition.
- Reflection Principle: suppose f is analytic in some domain D which contains a segment of the x-axis and is symmetric to that axis. Iff f(x) is real for each point x on the segment, then for each point z in the domain: $\overline{f(z)} = f(\overline{z})$

2.10 Harmonic Functions

- real-valued function H of two real variables x and y is said to be harmonic in a given domain of the xy plane if, throughout that domain it has continuous partial derivatives of first and second order, and satisfies the PDE known as Laplace's equation:

$$H_{xx} + H_{yy} = 0$$

- If a function $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D, then its component functions u and v are harmonic in D.
- If two given functions u and v are harmonic in D, and the FOPD satisfy C-R throughout D, v is said to be a harmonic conjugate of u.
- A function $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D iff v is a harmonic conjugate of u.

3 Chapter 3 – Elementary Functions

- $\sin z = \frac{e^{iz} - e^{-iz}}{2i}$
- $\cos z = \frac{e^{iz} + e^{-iz}}{2}$
- $\sinh z = \frac{e^{z} - e^{-z}}{2}$
- $\cosh z = \frac{e^{z} + e^{-z}}{2}$

3.23 The Logarithmic Function and Its Branches

- $\log z = \ln|z| + iargz$
- $\text{Log}z = \ln|z| + iArgz$
- $\log z = \text{Log}z + 2in\pi$
- branch of a multiple-valued function f is any single-valued function F that is analytic in some domain at each point z of which the value F(z) is one of the values f(z).
- The function $\text{Log}z = \ln r + iArgz$ $(r > 0, -\pi < Argz < \pi)$ is called the principal branch
- A branch cut is a portion of a line or curve that is introduced in order to define a branch F is a multiple-valued function f. Points on the branch cut for F are singular points, and any point that is common to all branch cuts of f is called a branch point.

3.24 Complex Exponents

- When $z \neq 0$ and the exponent c is any complex number, z^c is defined as: $z^c = e^{c \log z}$

  Principal value defined as obvious.

4 Chapter Four – Integrals

4.31 Complex-valued Functions w(t)

- Mean-value theorem for derivatives doesn't apply
- Integrals generally exist if piecewise continuous.

4.32 Contours

- An arc is a simple or Jordan arc if it does not cross itself.
- Similarly, simple closed curve or Jordan curve.
- If the derivatives of the component functions of an arc exist and are continuous, the arc is differentiable.
- smooth are has a continuous derivative on the closed interval and is nonzero on the open interval
- contour is a piecewise smooth arc
- A simple closed contour has only the beginning and end points the same.

4.33 Contour Integrals

- Definition of line or contour integral:

$$\int_C f(z)dz = \int_a^b f[z'(t)]z'(t)dt \leq ML$$

4.34 Antiderivatives

- An antiderivative is, necessarily, an analytic function.
- Theorem: Suppose that a function f is continuous on a domain D. If any one of the following statements true, then so are the others:
  - F has an antiderivative F in D;
  - The integrals of $f(z)$ along contour ling entirely in D and extending from any fixed point z_1 to any fixed point z_2 all have the same value;
  - The integrals of $f(z)$ around closed contours lying entirely in D all have value zero

  Basically, all are true or none are true.
- Cauchy-Gourst Theorem

  If a function f is analytic at all points interior to and on a simple closed contour C, then: $\int_C f(z)dz = 0$

4.35 Proof of the Theorem

- Simply and Multiply-Connected Domains
- A simply connected domain D is a domain such that every simple closed contour within it encloses only points of D.
- A domain that is not simply connected is multiply connected.
- Theorem: If a function f is analytic throughout a simply connected domain D, then: $\int_c f(z)dz = 0$
- Corollary: If C_1 and C_2 denote P.O.S.C.C., where C_2 is interior to C_1, and if function f is analytic in the closed region between and including the two contours, then the integral of f around C_1 equals the integral of f around C_2. Principle of Deformation of Paths

4.36 Cauchy Integral Formula

- Let f be analytic everywhere within and on a simple closed contour C, taken in the positive sense. If z_0 is any point interior to C, then:

$$f(z_0) = \frac{1}{2\pi i} \int \frac{f(z)dz}{(z - z_0)}$$

4.37 Derivatives of Analytic Functions

- Theorem: if a function is analytic at a point, then its derivatives of all orders are also analytic functions at that point.
- Corollary: If a function is analytic at a point, then the component functions u, v have continuous partials of all orders at that point.

4.38 Liouville's Theorem and the Fundamental Theorem of Algebra

- If f is entire and bounded in the complex plane, then f(z) is constant throughout the plane.
- Fundamental Theorem of Algebra
- Cauchy's inequality: $|f^{(n)}(z_0)| \leq \frac{n!M_R}{R^n}$
- $f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + \rho e^{i\theta}) d\theta$

4.39 Maximum Moduli of Functions

- Theorem: if a function f is analytic and not constant in a given domain D, then $|f(z)|$ has no maximum value in D.
- Corollary: essentially, maxima of $|f(z)|$ must occur on the boundary

5 Chapter Five – Series

- Convergence of Sequences and Series
- Convergence, absolute convergence
- Necessary but not sufficient that $z_n \to 0$
- Taylor Series

  Suppose that a function f is analytic throughout an open disk $|z - z_0| < R_0$. Then, at each point z in that disk, f(z) has the series representation: (that is, the power series converges to f(z) whenever $|z - z_0| < R0$:

  Where: $a_n = \frac{f^{(n)}(z_0)}{n!}$

5.43 Laurent Series

Suppose that a function f is analytic throughout an annular domain $R_1 < |z - z_0| < R_2$, and let C denote any positively oriented simple closed contour around z_0 and lying in that domain. Then, at each point z in the domain, $f(z)$ has the series representation: $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$

Where: $a_n = \frac{1}{2\pi i} \int_C \frac{f(z)dz}{(z-z_0)^{n+1}}$
$b_n = \frac{1}{2\pi i} \int_C \frac{f(z)dz}{(z-z_0)^{-n+1}}$
$c_n = \frac{1}{2\pi i} \int_C \frac{f(z)dz}{(z-z_0)^{n+1}}$

5.44 Examples: Useful Series

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$$

$$\frac{1}{1+z} = \sum_{n=0}^{\infty} (-1)^n z^n$$

$$1/z = \sum_{n=0}^{\infty} (-1)^n (z-1)^n$$

5.45 Absolute and Uniform Convergence of Power Series

- If a power series centered at z_0 converges at some point z_1, then it is absolutely convergence at each point z in an open disk centered at z_0 extending out to z_1 in radius.
- Corollary (to an unmentioned theorem): a power series represents a continuous function S(z) at each point inside its circle of convergence.

5.46 Integration and Differentiation of Power Series

- Power series can be integrated term-by-term, and summations pulled in and out of integrals.
- Corollary: the sum of a power series is analytic at each point z interior to the circle of convergence of the series.
- A power series can be differentiated term by term.

5.47 Uniqueness of Series Representations

- If a power series converges to f(z) at all points interior to some circle $|z - z_0| = R$, then it is the Taylor series for f.
- Likewise for Laurent series, but necessarily only in an annular domain about z_0
- Multiplication and Division of Power Series

  Leibniz's rule: $[f(z)g(z)]^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)}(z)$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

$$C_n = \sum_{k=0}^{n} a_k b_{n-k}$$

$$C_n = \frac{1}{b_0}(a_n - \sum_{k=1}^{n} b_k c_{n-k})$$

6 Chapter Six – Residues and Poles

7 Residues

- A singular point is isolated if, in addition, there is a deleted neighborhood throughout which f is analytic.
- The residue of f at z is the coefficient of $\frac{1}{z-z_0}$ in an L-series of f at z.

7.1 Residue Theorems

- Cauchy's Residue Theorem: Let C be a POSCC. If a function f is analytic inside and on C except for a finite number of singular points, then the value of the integral around C is $2i\pi$ the sum of the residues.
- Sometimes, if the function is analytic at each point in the finite plane exterior to C, it is more efficient to evaluate the intergral of f around C by finding a single residue.
- Theorem: If a function f is analytic everywhere in the finite plane except for a finite number of singular points interior to a POSSC C, then:

$$\int_C f(z)dz = 2\pi i \text{Res}_{z=0}[\frac{1}{z^2}f(\frac{1}{z})]$$

7.2 Three types of singular points

- Three types of isolated singular points:
- Pole of order m (where m is the highest power of $(z - z_0)$ in the Laurent series).
- Removeable singular point (when all the b's are zero, so the residue is zero).
- Essential singular point: when an infinite number of b's are nonzero. All hell breaketh loose, taking on every finite value (possibly except zero).

7.3 Residues at Poles

- An isolated singular point z_0 of a function f is a pole of order m iff f(z) can be written: $f(z) = \frac{\phi(z)}{(z-z_0)^m}$ *where* $\phi(z)$ *is nonzero!*
- Moreover, $\text{Res}_{z=z_0} f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!}$
- Is is always true that if z_0 is a pole of a function f, then $\lim_{z\to z_0} f(z) = \infty$
- While the theorem can be useful, often it is better to write directly as Laurent series.

7.4 Zeros and Poles of Order m

- A function that is analytic at a point z_0 has a zero of rder m there iff there exists a function g which is analytic and nonzero at z_0 such that $f(z) = (z - z_0)^m g(z)$
- Zeros of order m are sources of poles of order m. Theorem as a result.
- Corollary: If p and q analytic at point z_0, iff $p(z_0) \neq 0$, $q(z_0) = 0$, and $q'(z_0) \neq 0$:

$$\text{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}$$

- Higher order formulae exit but are not practical.

7.5 Conditions under which $f(z) \equiv 0$

- If $f(z) = 0$ at each point z of a domain or arc containing a point z_0, then $f(z) \equiv 0$ in any neighborhood N_0 of z_0 throughout which f is analytic. That is, $f(z) = 0$ at each point z in N_0.
- Theorem: if a function if analytic throughout a domain D, and the function is zero at each point of a subdomain or arc inside D, then the function is zero throughout D.
- Corollary: a function that is analytic in a domain D is uniquely determined over D by its values over a domain or along an arc contained in D.

7.6 Behavior of f Near Removeable and Essential Singular Points

- A function is anlways analytic and bounded ins ome deleted neighborhood of a removeable singularity.
- Suppose a function is analytic and bounded in some deleted neighborhood of a point z_0; if f is not analytic at z_0, then it's a removeable singularity.
- Essential singularity: not only does hell break loose, but the function assumes values arbitrarily close to any given number.

8 Chapter Seven – Applications of Residues

- Distinction between pair of improper and Cauchy P.V. integrals. If CPV converges not necessarily true that other does.
- Method of evaluating improper integrals

8.61 Improper Integrals Involving Sines and Cosines

- Jordan's inequality: $\int_0^\pi e^{-R\sin\theta}d\theta < \frac{\pi}{R} \quad (R > 0)$

8.62 Definite integrals involving sines and cosines

8.63 Indented paths

8.64 Integration Along a Branch Cut

8.65 Argument PRinciple and Rouche's THeorem

- Meromorphic $\equiv$ analytic in a domain except possibly poles
- Winding number theorem: $\#Z - \#P$
- Roche's Theorem: Let two functions f and g be analytic inside and on a SCC C, and suppose that $|f(z)| > |g(z)|$ at each point on C. Then $f(z)$ and $f(z) + g(z)$ have the same number of zeros, counting multiplicities, inside C.

8.66 Inverse Laplace Transforms

- Forward transform: $F(s) = \int_0^\infty e^{-st}f(t)dt$
- Backward transform: $f(t) = \frac{1}{2\pi i}\lim_{R\to\infty}\int_{L_R} e^{st}F(s)ds \; t > 0$
- Backward transform: $f(t) = \frac{1}{2\pi i}\text{P.V.}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}F(s)ds \; t > 0$
- $f(t) = \sum_{n=1}^{N} \text{Res}_{s=s_n}[e^{st}F(s)] \quad (t > 0)$
- $\text{Res}_{s=s_0}[e^{st}F(s)]+\text{Res}_{s=\overline{s_0}}[e^{st}F(s)] = 2e^{\alpha t}\text{Re}\{e^{i\beta t}[b_1+\frac{b_2}{1!}t+\ldots+\frac{b_m}{(m-1)!}t^{m-1}]\}$