LECTURE NOTES FOR AMATH 351
INTRODUCTION TO DIFFERENTIAL EQUATIONS*

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Abstract. Introductory survey of ordinary differential equations. Linear and nonlinear equations. Taylor series. Laplace transforms. Emphasis on formulation, solution, and interpretation of results. Strong emphasis on examples derived from physical, biological, and engineering sciences.

1. Introduction. The Syllabus

The course follows closely the text by W. E. Boyce and R. C. DiPrima, *Elementary Differential Equations* (6th Ed.). Included in the syllabus are lectures on Chapter 2 (3 lectures), Chapter 3 (5 lectures), Chapter 5 (3 lectures), Chapter 6 (3 lectures), Chapter 7 (6 lectures), and Chapter 9 (6 lectures). In each chapter, emphasis is given for the formulation, solution techniques, and applications.

It is assumed that the student is proficient with methods of differentiation and integration. Along with the ability to manipulate algebraic equations, these skills are sufficient for success in the course. Following is a list of contents.

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2. Chapter 2. First Order Differential Equations

This first chapter is concerned with what are called *First Order Differential Equations.* To be more precise, a first order differential equation is written in the following form:

$$\frac{dy}{dt} = f(t, y) \tag{1}$$

where $f(t, y)$ is a given function of t and y. In general, we cannot solve this equation with arbitrary $f(t, y)$. However, there are many forms of $f(t, y)$ for which we can solve the first order differential equation (1) explicitly. This chapter considers the various forms of $f(t, y)$ for which we can generate a solution to Eq. (1).

2.1. Lec. 1. Method of Integrating Factors and Separable Equations

We begin this chapter by considering the *linear* equation:

$$\frac{dy}{dt} + p(t)y = g(t)\,. \tag{2}$$

This equation is *linear* since $f(t, y) = g(t) - p(t)y$ is linear in the variable y, i.e. things like y^2, $\exp(y)$, or $\cos(y)$ are *nonlinear.*

With a general $p(t)$ and $g(t)$, this equation is still too difficult to solve. We therefore let

$$p(t) = -A \qquad \text{and} \qquad g(t) = B$$

so that we have

$$\frac{dy}{dt} = Ay + B$$

where A and B are constants. After some simple manipulation, we can rewrite this as (divide by $Ay + B$ and multiply through by A):

$$\frac{\frac{dy}{dt}}{y + \frac{B}{A}} = A \qquad (y \neq -B/A)\,.$$

Now we make the following observation:

$$\frac{d}{dt}\left(\ln\left|y + \frac{B}{A}\right|\right) = \frac{1}{y + \frac{B}{A}} \times \frac{dy}{dt}$$

where we have made use of the chain rule. But this is exactly what we have on the left hand side of the previous equation! So we can rewrite our previous simplification as:

$$\frac{d}{dt}\left(\ln\left|y + \frac{B}{A}\right|\right) = A$$

By integrating both sides we have

$$\int \frac{d}{dt}\left(\ln\left|y + \frac{B}{A}\right|\right) dt = \int A dt$$

which gives

$$\ln\left|y+\frac{B}{A}\right| = At + c$$

where c is some constant of integration. Exponentiating both sides results in

$$y+\frac{B}{A} = \exp(At+c) = \exp(At)\exp(c) = C\exp(At)$$

which gives finally

$$y = -\frac{B}{A} + C\exp(At)$$

where the arbitrary constant C can be determined with the aid of the initial condition $y(t_0) = y_0$.

Thus with $p(t)$ and $g(t)$ being constant, we can solve the first order equation. This gives us hope that perhaps we can solve the equation even when $p(t)$ and $g(t)$ are more complicated functions. Recall that the **key** step in the above was to notice that the *left hand side was a derivative of something*! We will utilize this in what follows. We once again consider

$$\frac{dy}{dt} + p(t)y = g(t)\,.$$

We notice that the left hand side is *not* a derivative of something. However, we can try to make it that way by multiplying by some arbitrary function $\mu(t)$.

$$\mu(t)\frac{dy}{dt} + \mu(t)p(t)y = \mu(t)g(t)$$

From the product rule for differentiation, we note that

$$\frac{d}{dt}\left(\mu(t)y\right) = \mu(t)\frac{dy}{dt} + \frac{d\mu(t)}{dt}y$$

which is much like the previous equation on the left hand side. In fact, we can make it exactly like the left hand side of the previous equation if we choose $\mu(t)$ so that

$$\frac{d\mu}{dt} = \mu(t)p(t) \qquad\qquad \frac{1}{\mu}\times\frac{d\mu}{dt} = p(t)\,.$$

We once again notice that the left hand side of the resulting equation for $\mu(t)$ is a derivative of something so that:

$$\frac{1}{\mu}\frac{d\mu}{dt} = p(t) \quad\rightarrow\quad \frac{d}{dt}(\ln\mu(t)) = p(t)$$

which upon integrating both sides with respect to t gives

$$\ln(\mu(t)) = \int p(t)dt \quad\rightarrow\quad \mu = \exp\left(\int p(t)dt\right)$$

where for the present, we have left off the constant of integration. By utilizing this $\mu(t)$ in our original equation, we then have:

$$\frac{d}{dt}\left(\mu(t)y\right) = \mu(t)g(t)$$

which upon integrating gives

$$\mu(t)y = \int \mu(t)g(t)dt + C$$

$$y = \frac{1}{\mu(t)}\left(\int \mu(t)g(t)dt + C\right)$$

which is the general solution of the problem and includes an arbitrary constant of integration C. So if we can perform two integrations (which may not be trivial), we have the general solution to the problem.

Example: Solve $y' + 2ty = t$ with $y(0) = 0$

We note from this problem that $p(t) = 2t$ and $g(t) = t$ so that

$$\mu(t) = \exp\left(\int p(t)dt\right) = \exp\left(\int 2tdt\right) = \exp\left(t^2\right)$$

and

$$y = \frac{1}{\mu(t)}\left(\int \mu(t)g(t)dt + C\right) = \exp\left(-t^2\right)\left(\int t\exp\left(t^2\right)dt + C\right)$$

$$= \exp\left(-t^2\right)\left(\frac{1}{2}\exp\left(t^2\right) + C\right) = \frac{1}{2} + C\exp\left(-t^2\right)$$

Making use of our initial condition we find:

$$y(0) = 0 \quad \rightarrow \quad 0 = \frac{1}{2} + C \quad \rightarrow \quad C = -\frac{1}{2} \quad \rightarrow \quad y = \frac{1 - \exp\left(-t^2\right)}{2}$$

The following theorem applies for first order differential equations of the form we are considering.

THEOREM: *If $p(t)$ and $g(t)$ are continuous on some interval I $(\alpha < t < \beta)$ containing t_0, then there exists a unique solution satisfying $y' + p(t)y = g(t)$ with $y(t_0) = y_0$.*

Example: Solve $ty' + 2y = 4t^2$ and find the range of validity

This is a nice example of a problem where things behave nicely so long as $t \neq 0$. To see this, we simply put the equation in the form of the above theorem.

$$\frac{dy}{dt} + \frac{2}{t}y = 4t\,.$$

This gives us an integrating factor of

$$\mu(t) = \exp\left(\int \frac{2}{t}dt\right) = \exp\left(2\ln t\right) = \exp\left(\ln t^2\right) = t^2$$

which in turn gives us

$$y = \frac{1}{t^2}\left(\int t^2(4t)dt + C\right) = \frac{1}{t^2}\left(t^4 + C\right) = t^2 + \frac{C}{t^2}$$

But we note that $y(t \to 0) = \pm\infty$ depending on the value of C. So the differential equation does not make sense for $t = 0$ as is plainly seen in the equation since $p(t \to 0) = \infty$. This shows us where the above theorem breaks down.

Another class of equations which can be handled quite nicely are what are called *separable* equations. In this case, we write the differential equation in the form:

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0\,.$$

This can be done from $y' = f(t,y)$ by letting $M = -f$ and $N = 1$. If in addition we have

$$M(x,y) = M(x) \qquad \text{and} \qquad N(x,y) = N(y)$$

then

$$M(x) + N(y)\frac{dy}{dx} = 0\,.$$

multiplying through by dx we find

$$M(x)dx + N(y)dy = 0 \quad \to \quad \int M(x)dx + \int N(y)dy = C$$

upon integrating. So provided we can once again perform *two* integrations, we have the general solution to the differential equation.

example: Solve $\frac{dy}{dx} = \frac{y\cos x}{1+2y^2}$ with $y(0) = 1$

Multiplying the above equation through by $(1+2y^2)/y$ gives us the separable form:

$$\cos x dx = \frac{1+2y^2}{y}dy\,.$$

Integrating gives

$$\int \cos x dx + C = \int \frac{1+2y^2}{y}dy \quad \to \quad \sin x + C = \ln|y| + y^2\,.$$

Now if $y(0) = 1$ then $0 + C = 0 + 1$ so that $C = 1$ and

$$\sin x + 1 = \ln|y| + y^2$$

which is the general solution to the problem.

2.2. Lec. 2. Applications: Population Modeling and Mechanics

2.2.1. Population Dynamics. One of the more pertinent applications of first order differential equations is to the modeling of population sizes. In particular, it is reasonable to assume that a population grows according to how big it already is. This makes sense since for large populations there are more of that species to mate and reproduce in comparison to a population with a low number of members. Mathematically, we can write this simple behavior as:

$$\frac{dy}{dt} = \alpha y$$

where y is the population size and α determines the growth rate of a species. Thus this equation allows the rate of change of the population (dy/dt) to be proportional to the existing size of the population. It is easy to verify that the solution to the above equation is simply:

$$y = y_0 \exp(\alpha t) .$$

This implies that populations grow exponentially from an initial population size y_0. The rate of that growth is determined by the parameter α so that for α large the population grows more quickly than for α small. This parameter accounts for the differences in the size and spread of populations of insects and rabbits (large α) versus perhaps elephants and whales (small α).

Unfortunately, most populations do not simply continue to grow exponentially since the populations would go to infinity as time goes to infinity. Thus we need a more realistic model to account for the fact that once a population gets too large, they must compete for food, water, places to live, etc. In some sense, we think of Darwinian theory coming along and destroying that part of the population that just can't compete. A more realistic model then might be:

$$\frac{dy}{dt} = h(y)y = f(y)$$

which is similar to the above except that now the growth rate ($h(y)$) now depends upon the population size itself. This equation is called *autonomous* since the right hand side does not depend explicitly on time. To build a more realistic model, we conjecture the following two behaviors:

- when y is small: the population grows ($h(y) > 0$)
- when y is large: the population declines ($h(y) < 0$)

The simplest way to implement this is by letting

$$h(y) = \alpha - ay$$

so that when y is small, $h(y) \approx \alpha > 0$ and when y is large, $h(y) \approx -ay < 0$. This then gives the population equation

$$\frac{dy}{dt} = \alpha \left(1 - \frac{y}{K}\right) y$$

which is known as the *logistic equation* and which was proposed by the Belgium mathematician Verhulst in 1838 (note that $K = \alpha/a$).

Of particular interest are solutions for which the derivative (or population) is neither growing or decreasing. This happens when

$$\frac{dy}{dt} = 0 \qquad\qquad \text{EQUILIBRIUM}\,.$$

We call this *equilibrium* since the solution is not growing or decreasing. This however, does not mean that the solutions are *stable*. For the logistic equation, equilibrium is attained when:

$$\frac{dy}{dt} = 0 \quad \rightarrow \quad \left(1 - \frac{y}{K}\right) y = 0$$

which gives the equilibrium solutions:

$$y = 0 \qquad \text{and} \qquad y = K\,.$$

More generally, equilibrium solutions can be sought for any equation of the form:

$$\frac{dy}{dt} = f(y)$$

The equilibrium solutions y_e are given by

$$f(y_e) = 0\,.$$

The equilibrium are also called *critical points*.

We now return to the logistic equation and its critical points. In particular, we would like to understand the *stability* of each of them. Recall that for y small, we have growth, whereas for y large we have decay (decline of the population). In particular we have the following:

$$\frac{dy}{dt} > 0 \quad \text{(growth)} \quad \text{when} \quad 0 < y < K$$
$$\frac{dy}{dt} < 0 \quad \text{(decay)} \quad \text{when} \quad y > K$$

where we only consider $y > 0$ since we are considering a population's size. These dynamics can be very nicely depicted in Fig. 1a which shows solutions in the y versus t plane. The equilibrium solution $y = K$ is called *asymptotically stable* since all solutions go to $y = K$ for $t \rightarrow \infty$. In contrast, the $y = 0$ equilibrium solution is *unstable* since all solutions go away from it.

Another way to look at this problem is by simply plotting $f(y)$ versus y. Whenever we find that $f(y) > 0$, we have growth. Likewise, when $f(y) < 0$ then we have decay. Figure 1b gives a nice illustration of this behavior for the logistic equation. Notice that the *equilibrium* solutions correspond to when the solution $f(y)$ crosses zero. We note that solutions below and above the equilibrium $y = K$ both go to $y = K$ as $t \rightarrow \infty$. In this case, K is called the *saturation level* (or environmental carrying capacity).

Although we now know almost everything qualitatively about the behavior of the equation, we note a couple of things:

- We haven't solved anything yet!
- This model behaves markedly different than the *linear* case.

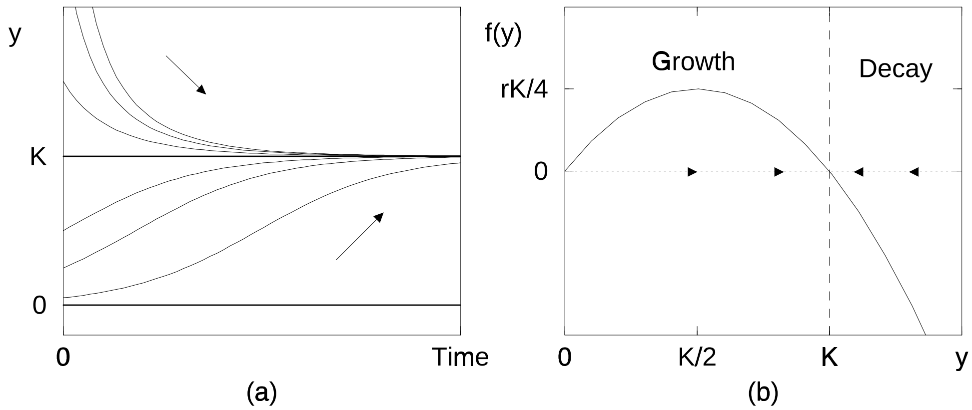

FIG. 1. *Skematic of the solution field y versus t in (a) and plot of $f(y)$ versus y in (b). Note that from these sketches, we can understand the stability of the equilibrium $y = 0$ and $y = K$.*

We can now proceed to solve the logistic equation. By the method of separation, we find

$$\frac{dy}{(1-y/K)y} = \alpha dt \quad \rightarrow \quad \left(\frac{1}{y} + \frac{1/K}{1-y/K}\right) dy = \alpha dt$$

which yields upon integration

$$\ln|y| - \ln\left|1 - \frac{y}{K}\right| = \alpha t + c \quad \rightarrow \quad \frac{y}{1-y/K} = C\exp(\alpha t)\,.$$

Assuming that y_0 is the initial population, we then have

$$y = \frac{y_0 K}{y_0 + (K - y_0)\exp(-\alpha t)}\,.$$

Note that it is clear from this solution that $y \rightarrow K$ as t gets large. So we see once again that $y = K$ is an asymptotically stable equilibrium and $y = 0$ is an unstable equilibrium.

The logistic equation provides a nice population model since it both accounts for the growth of a species and its eventual reaching of a population limit. We can also do more sophisticated things with this kind of modeling. For instance, some species have been found to only reproduce provided there is a sufficiently large population (i.e. the carrier pigeon). In this case, if the population is below a certain level, then the population decays. To incorporate this into the logistic equation, we modify it to be of the following form:

$$\frac{dy}{dt} = -\alpha\left(1 - \frac{y}{K}\right)\left(1 - \frac{y}{A}\right) y = f(y)\,.$$

In Fig. 2a and 2b, we plot the analog of Fig. 1a and 1b. This simple modification allows the population to become extinct (go to zero) if the initial population is below $y = K$. Further, if the population is greater than K, then the population saturates to the equilibrium point $y = A$ (i.e. the environmental carrying capacity).

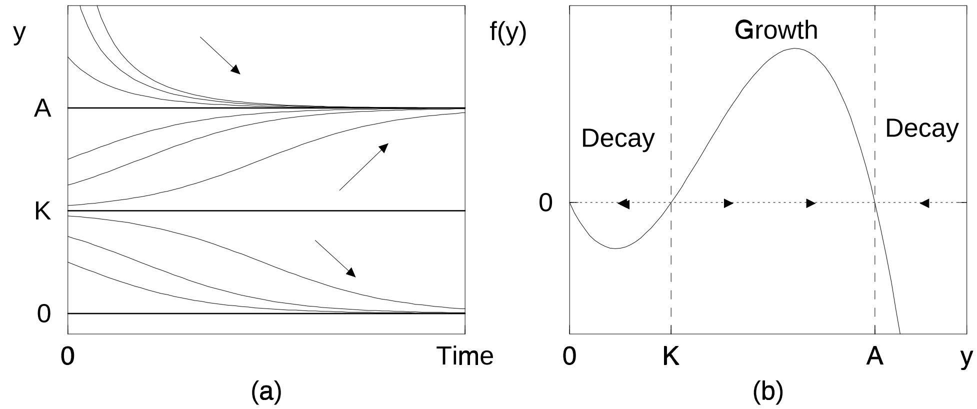

FIG. 2. *Schematic of the solution field y versus t in (a) and plot of $f(y)$ versus y in (b) for population model with threshold phenomena. Note that from these sketches, we can understand the stability of the equilibrium $y = 0$, $y = K$, and $y = A$.*

2.2.2. Classical Mechanics. Changing gears, we move on to an example from physics. Recall that Newton's Law is given by:

$$F = ma$$

where F is the sum of the forces, m is mass, and a is acceleration. We further recall that acceleration is the time rate of change of velocity ($v(t)$) which is in turn the time rate of change of position ($x(t)$) so that:

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2} \qquad\qquad v = \frac{dx}{dt} .$$

Suppose we drop an object which has a wind/air resistance proportional to the speed $|v|$. Then

$$\sum F = (\text{gravity}) - (\text{air resistance}) = mg - \alpha v$$

where α measure the amount of resistance and g is the gravitational acceleration. From Newton's Law, we then have the differential equation:

$$m\frac{dv}{dt} = mg - \alpha v$$

which is rewritten

$$\frac{dv}{dt} + \frac{\alpha}{m}v = g$$

which can be easily solved by either the integrating factor method or separation to yield:

$$v = \frac{mg}{\alpha}\left(1 - \exp(-\alpha t/m)\right) .$$

We note that as $t \to \infty$, then $v = mg/\alpha$ which is the *terminal velocity* of the object. We can then easily find the position of the object by integrating up v:

$$x(t) = \int v(t)dt = x_0 + \frac{mg}{\alpha}t - \frac{m^2 g}{\alpha^2}\left(1 - \exp(-\alpha t/m)\right)$$

where x_0 is the initial position.

2.3. Lec. 3. Exact Equations and Homogeneous Equations

In the first lecture, we learned how to solve equations if they are either separable or have an integrating factor. However, there are many differential equations that do not fit into either of these categories. So let's once again consider the equation:

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

where $M(x,y) \neq M(x)$ and $N(x,y) \neq N(y)$ so that the method of separation fails.

To solve this equation, we consider a function

$$\psi(x, y(x)) = c$$

where c is a constant and $\psi(x,y)$ is just some function of x and $y(x)$ which we will determine. Note that by the chain rule:

$$\frac{d}{dx}\left[\psi(x,y(x))\right] = \frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial y}\frac{dy}{dx} = 0\,.$$

If we relate this to the differential equation we have above, then we notice that

$$\frac{\partial\psi}{\partial x} = M(x,y) \qquad \text{and} \qquad \frac{\partial\psi}{\partial y} = N(x,y)\,.$$

So if the above relations hold, then $\psi(x,y) = c$ is the solution to the differential equation!

Two questions come to mind immediately:

- When can we apply such a technique?
- How do we actually solve for $\psi(x,y)$

The first question can be easily answered by recalling from multivariable calculus that

$$\frac{\partial^2\psi}{\partial x\partial y} = \frac{\partial^2\psi}{\partial y\partial x}\,.$$

Thus we find that

$$\frac{\partial}{\partial y}\left(\frac{\partial\psi}{\partial x}\right) = \frac{\partial M}{\partial y} \qquad \text{and} \qquad \frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial y}\right) = \frac{\partial N}{\partial x}\,.$$

But since these are equal, we then must have

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}\,.$$

This condition must be satisfied in order for $\psi(x,y)$ to be a solution to our problem.

Example: Solve $(y\cos x + 2x\exp(y)) + (\sin x + x^2\exp(y) - 1)y' = 0$

We first note that:

$$M = y\cos x + 2x\exp(y) \quad \rightarrow \quad M_y = \cos x + 2x\exp(y)$$

$$N = \sin x + x^2\exp(y) - 1 \quad \rightarrow \quad N_x = \cos x + 2x\exp(y)$$

so that $M_y = N_x$ and the equation is exact. We can then evaluate our solution $\psi(x,y)$ by recalling that:

$$\frac{\partial\psi}{\partial x} = M = y\cos x + 2x\exp(y)$$

$$\frac{\partial\psi}{\partial y} = N = \sin x + x^2\exp(y) - 1$$

We can integrate up the first equation with respect to x. This gives

$$\psi(x,y) = y\sin x + x^2\exp(y) + h(y)$$

where $h(y)$ is some arbitrary constant which can depend upon y. But from the second equations above, we know what ψ_y must be. So we differentiate our solution for $\psi(x,y)$ above which gives:

$$\frac{\partial\psi}{\partial y} = \sin x + x^2\exp(y) + \frac{dh}{dy}$$

It only remains to equate the two equations for ψ_y. Upon comparing the two, we find

$$\frac{dh}{dy} = -1$$

which can be integrated to yield

$$h(y) = -y\,.$$

Inserting this value of $h(y)$ back into our solution for ψ we find:

$$\psi(x,y) = y\sin x + x^2\exp(y) - y = C$$

where the constant C is determined by initial conditions.

The concept of integrating factors can also play a role in solving by the method of exactness. To see this, consider once again the governing equation:

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

which is not exact, i.e. $M_y \neq N_x$. We can once again try to multiply through by an appropriate function which makes it exact. So then multiply through by a function $\mu(x,y)$ so that

$$\mu(x,y)M(x,y) + \mu(x,y)N(x,y)\frac{dy}{dx} = 0\,.$$

Exactness is now established if

$$(\mu M)_y = (\mu N)_x$$

which gives

$$M\mu_y - N\mu_x + (M_y - N_x)\mu = 0\,.$$

In general, this is very difficult to solve for μ. However, consider the case for which $\mu(x,y) = \mu(x)$. This then simplifies to

$$\frac{d\mu}{dx} = \frac{M_y - N_x}{N}\mu$$

where $(M_y - N_x)/N$ should be a function of x alone. If this is so, then the integrating factor can be easily calculated and the problem becomes exact.

Another class of first order problems which can be solved in closed form involves a differential equation of the form:

$$\frac{dy}{dx} = f(x,y) = f\left(\frac{y}{x}\right)\,.$$

Equations of this form are called *homogeneous*. The easiest way to solve such problems is by defining a new variable

$$z = \frac{y}{x}$$

so that the equation becomes:

$$x\frac{dz}{dx} + z = f(z)$$

where we have made use of the chain rule ($y = xz$) in calculating the left-hand side $dy/dx = dy/dz \cdot dz/dx + dy/dx = x \cdot dz/dx + z$. In this form, the equation can always be separated to yield:

$$\frac{dx}{x} = \frac{dz}{f(z) - z}$$

which can be easily solved by integrating both sides.

Example: Solve $dy/dx = (y^2 + 2xy)/x^2$

We notice that this equation can be written

$$\frac{dy}{dx} = \frac{y^2 + 2xy}{x^2} = \frac{y^2}{x^2} + 2\frac{y}{x}\,.$$

Thus it is in homogeneous form and we can define $z = y/x$ so that we have:

$$\frac{dy}{dx} = z^2 + 2z \quad \rightarrow \quad x\frac{dz}{dx} + z = z^2 + 2z\,.$$

Rearranging gives

$$\frac{dx}{x} = \frac{dz}{z^2+z} = \left(\frac{1}{z} - \frac{1}{z+1}\right) dz\,.$$

Upon integrating we finally find that

$$\ln|x| + c = \ln|z| - \ln|z+1| \quad \rightarrow \quad cx = \frac{z}{z+1}\,.$$

Using the fact that $z = y/x$ we find the solution

$$y = \frac{cx^2}{1-cx}\,.$$

where the constant c is determined by the initial conditions.

2.4. Chapter 2. Summary

There are four methods which are key to this chapter. Each one can handle a particular type of differential equation, and each one requires that we perform two integrations. These integrations are often the most difficult step in the solution process. In some cases, a given differential equation can be solved by one or more of the following methods:

Method of Integrating Factors

In this method, we aim to solve the differential equation

$$\frac{dy}{dt} + p(t)y = g(t)\,.$$

In order to do so, we define the integrating factor

$$\mu(t) = \exp\left(\int p(t)dt\right)$$

which allows us to find the solution

$$y = \frac{\int \mu(t)g(t)dt + c}{\mu(t)}\,.$$

This method is very powerful and should always be kept in mind if you find a differential equation of the above form.

Separable Equations

The separable method is also an extremely powerful tool for considering differential equations of the form

$$M(x) + N(y)\frac{dy}{dx} = 0\,.$$

Upon rewriting we find

$$M(x)dx + N(y)dy = 0$$

which can be integrated to yield the solution

$$\int M(x)dx + \int N(y)dy = c$$

Provided we can once again solve two integrals, we have a general solution to the problem.

Exact Equations

Yet another form of differential equations is

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0\,.$$

If $M_y = N_x$, then the equation is said to be exact and the solution is

$$\psi(x,y) = c$$

where

$$\frac{\partial \psi}{\partial x} = M(x,y) \qquad \text{and} \qquad \frac{\partial \psi}{\partial y} = N(x,y).$$

Using these conditions, it is relatively easy to solve for $\psi(x,y)$ itself. Unlike the first two methods, this technique is somewhat limited since we require $M_y = N_x$. Regardless, this technique can solve many difficult problems.

Homogeneous Equations

Our final method concerns equations of the form:

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)\,.$$

By making the transformation $z = y/x$ and utilizing the chain rule we find that the equation becomes separable:

$$\frac{dx}{x} = \frac{dz}{f(z) - z}\,.$$

Integrating both sides and substituting back in $z = y/x$ gives us the solution to the homogeneous equation.

3. Chapter 3. Second Order Linear Differential Equations

In this chapter we move on to consider second-order linear differential equations which take the generic form

$$y'' + p(t)y' + q(t)y = g(t)$$

where $p(t), q(t)$, and $g(t)$ are continuous functions on some interval of time I. In contrast to first-order systems, the unique solution of a second order system requires that we specify two initial conditions, i.e. we now require both $y(t_0) = y_0$ and $y'(t_0) = y_0'$. Fortunately for us, second-order systems tend to be much easier to solve than first order systems. We just need to keep in mind three simple rules:

- $\left(e^{\lambda t}\right)' = \lambda e^{\lambda t}$
- $\left(e^{\lambda t}\right)'' = \lambda^2 e^{\lambda t}$
- $a\lambda^2 + b\lambda + c = 0 \quad \rightarrow \quad \lambda_\pm = \left(-b \pm \sqrt{b^2 - 4ac}\right)/2a$

All of this chapter boils down to the above rules and algebra, so we should be able to do quite well with these types of equations.

In general, second order systems are more important than first order systems since they describe a much larger variety of phenomena. For instance, we will show in the course of the lectures that simple devices like pendulums, spring-mass systems, and other oscillatory phenomena can be properly described by second-order systems.

3.1. Lec. 1. Homogeneous, Constant Coefficient Equations

We begin by considering what is called a *constant coefficient* and *homogeneous* equation. A constant coefficient equation takes the functions $p(t)$ and $q(t)$ to be constants while a homogeneous equation takes $g(t) = 0$. The resulting differential equation is then written as

$$ay'' + by' + cy = 0 \, .$$

As an illustrative example of second-order behavior, consider the case for which $b = 0$ and $a = 1$. Thus we have

$$y'' + cy = 0 \, .$$

If $c = -1$, then we are looking for a solution for which $y'' = y$. What kind of function, when you take the derivative twice, gives back the original function itself? You need look no further than a regular exponential. In fact, we have the following:

$$y'' - y = 0 \qquad \text{then} \qquad y = c_1 \exp(t) \quad \text{or} \quad y = c_2 \exp(-t) \, .$$

Interestingly enough, we now have *two* solutions to this problem. Similarly, if $c = 1$ then it is easy to verify the following:

$$y'' + y = 0 \qquad \text{then} \qquad y = c_1 \cos(t) \quad \text{or} \quad y = c_2 \sin(t) \, .$$

Again we generate two solutions to the equation. This is markedly different than the first-order systems which only generate a single solution with a single unknown constant. The reason for this is that we have now *two* initial conditions. Thus our general solution has to produce two arbitrary constants so that we can satisfy the initial conditions.

To be more precise, there exists a very simple method by which we can generate a general solution to the homogeneous, constant coefficient problem. Recall that we are considering

$$ay'' + by' + cy = 0 \, .$$

By guessing or attempting a solution of the form

$$y = \exp(\lambda t)$$

we find the following quadratic equation, called the *characteristic equation*, for λ:

$$a\lambda^2 + b\lambda + c = 0 \qquad \text{then} \qquad \lambda_\pm = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\,.$$

Thus our two solutions are $y_1 = \exp(\lambda_+ t)$ and $y_2 = \exp(\lambda_- t)$, and our general solution can be expressed as

$$y = c_1 y_1 + c_2 y_2 = c_1 e^{\lambda_+ t} + c_2 e^{\lambda_- t}.$$

The constants c_1 and c_2 are then evaluated from our initial conditions $y(t_0)$ and $y'(t_0)$. A simple example will serve to illustrate this point a little better.

Example: Solve $y'' + 5y' + 6y = 0$ with $y(0) = 2$ and $y'(0) = 3$.

We try our solution of the from $y = \exp(\lambda t)$ and derive the characteristic equation

$$\lambda^2 + 5\lambda + 6 = 0$$

which by factorization can be rewritten $(\lambda + 2)(\lambda + 3) = 0$ which is only satisfied if $\lambda = -2$ or -3. We note that we derive the same result by using the quadratic formula. Our general solution then is

$$y = c_1 e^{-2t} + c_2 e^{-3t}\,.$$

Applying the boundary conditions, we find that

$$y(0) = 2 = c_1 + c_2 \qquad \text{and} \qquad y'(0) = 3 = -2c_1 - 3c_2\,.$$

From the first relation, we have $c_1 = 2 - c_2$ which can then be inserted into the second relation. What we find is that $c_1 = 9$ and $c_2 = -7$ so that our *unique* solution is

$$y = 9e^{-2t} - 7e^{-3t}\,.$$

So one can see that both constants are necessary in order to generate the solution.

We can make this whole discussion a bit more formal by introducing the notation

$$L[\phi] = \phi'' + p\phi' + q\phi$$

where p and q are continuous on some interval I (i.e. $\alpha < t < \beta$). What can be proved is that if $L[y] = 0$ (homogeneous equation) with $y(t_0) = y_0$ and $y'(t_0) = y_0'$ then a solution *exists* and is *unique*. Further, the solution is twice differentiable which is rather obvious considering that our differential equation takes two derivatives.

In addition to this existence and uniqueness theorem, we have a fundamental concept to discuss: that of *superposition*. Recall that we found *two* solutions to the above problems. And in the example, I added (or superimposed) these two solutions to make my general solution. Why does this work? Consider the following:

- if y_1 is a solution then $L[y_1] = 0$
- if y_2 is a solution then $L[y_2] = 0$

Then if we ask if

$$y = c_1 y_1 + c_2 y_2$$

is a solution, we find that:

$$\begin{aligned}
L[y] &= L[c_1 y_1 + c_2 y_2] \\
&= (c_1 y_1 + c_2 y_2)'' + p(c_1 y_1 + c_2 y_2)' + q(c_1 y_1 + c_2 y_2) \\
&= c_1(y_1'' + p y_1' + q y_1) + c_2(y_2'' + p y_2' + q y_2) \\
&= c_1 L[y_1] + c_2 L[y_2] \\
&= 0 + 0 = 0
\end{aligned}$$

which says that $L[y] = 0$ and y is thus a solution. Additionally, the unique solution must satisfy the initial conditions. If we have $y(t_0) = y_0$ and $y'(t_0) = y_0'$ then we find:

$$y_0 = c_1 y_1(t_0) + c_2 y_2(t_0) \qquad \text{and} \qquad y_0' = c_1 y_1'(t_0) + c_2 y_2'(t_0)$$

This is a two by two system of equations for the unknowns c_1 and c_2. Solving for these constants gives

$$\begin{aligned}
c_1 &= \frac{y_0 y_2'(t_0) - y_0' y_2(t_0)}{y_1(t_0) y_2'(t_0) - y_1'(t_0) y_2(t_0)} \\
c_2 &= \frac{-y_0 y_1'(t_0) + y_0' y_1(t_0)}{y_1(t_0) y_2'(t_0) - y_1'(t_0) y_2(t_0)} .
\end{aligned}$$

So as long as

$$W(y_1, y_2) = y_1(t_0) y_2'(t_0) - y_1'(t_0) y_2(t_0) \neq 0$$

then we can find values for c_1 and c_2. If $W = 0$, then we notice that the denominator goes to zero and c_1 and c_2 go to infinity, which does not make much sense. Further, we require that $W \neq 0$ for all t since t_0 above was chosen arbitrarily. The quantity W is called the *Wronskian* and is very important in differential equations.

THEOREM: If y_1 and y_2 are solutions and

$$L[y] = 0 \qquad \text{and} \qquad W(y_1, y_2) \neq 0 \quad \text{for some } t_0$$

then

$$y = c_1 y_1(t) + c_2 y_2(t)$$

is a *general solution* where the arbitrary constants c_1 and c_2 include every possible solution of $L[y] = 0$.

Example: Calculate the Wronskian of the previous example.

Recall from the previous example that $y_1 = \exp(-2t)$ and $y_2 = \exp(-3t)$ so that

$$W(y_1, y_2) = y_1 y_2' - y_2 y_1' = e^{-2t}\left(e^{-3t}\right)' - e^{-3t}\left(e^{-2t}\right)' = -e^{-5t} \neq 0$$

Thus we know that the y_1 and y_2 truly form the basis of the solution to the previous example.

3.2. Lec. 2. Linear Dependence and the Wronskian

We now address the important concept of *linear dependence* and *linear independence* and show that these ideas are closely related to the Wronskian. We begin with some basic definitions concerning the linear dependence or independence of two functions f and g:

$$\text{linear dependence:} \qquad c_1 f + c_2 g = 0 \qquad \text{for } c_1, c_2 \neq 0\,.$$

Since c_1 and c_2 are not zero, then we can solve for f in terms of g. Thus $f = -(c_2/c_1)g$, i.e. f is the same as g except for a constant factor. So f and g are thought of as dependent since they are essentially the same function. In contrast, we define linear independence as follows:

$$\text{linear independence:} \qquad c_1 f + c_2 g = 0 \qquad \text{only if } c_1 = c_2 = 0\,.$$

So we can no longer write f as a function of g, i.e. they are completely independent functions!

Example: Are $f = \sin(t)$ and $g = \cos(t - \pi/2)$ linearly dependent or independent? What about e^t and e^{2t}?

To answer this, we start with our definitions above:

$$c_1 \sin(t) + c_2 \cos(t - \pi/2) = 0\,.$$

By noting that $\cos(t - \pi/2) = \cos(t)\cos(\pi/2) + \sin(t)\sin(\pi/2) = \sin(t)$, we then find that

$$c_1 \sin(t) + c_2 \sin(t) = 0 \quad \rightarrow \quad c_1 = -c_2\,.$$

Thus $c_1, c_2 \neq 0$ and the two functions are linear dependent. In contrast, the second case gives

$$c_1 e^t + c_2 e^{2t} = 0$$

for which we can find no constant values of c_1 and c_2 which will make the e^t term cancel out the e^{2t} term. So we are forced to conclude that $c_1 = c_2 = 0$ and the functions are linearly independent.

Although this technique of inspecting the functions works, we would like to have a more precise method for determining linear dependence and independence. We develop this by considering once again our two differentiable functions f and g on some interval of time:

$$c_1 f + c_2 g = 0$$

Now suppose we evaluate this quantity at some specific time t_0 on our given interval and also calculate its derivative:

$$c_1 f(t_0) + c_2 g(t_0) = 0 \quad \rightarrow \quad c_1 f'(t_0) + c_2 g'(t_0) = 0\,.$$

This then gives us two equations and two unknowns for c_1 and c_2. From the second equation we find

$$c_2 = -c_1 \frac{f'(t_0)}{g'(t_0)}$$

which upon substituting back into the first equation gives

$$c_1 \left(f(t_0) - \frac{f'(t_0)}{g'(t_0)} g(t_0) \right) = 0 \,.$$

By multiplying through by $g'(t_0)$ we find

$$c_1 \left(f(t_0)g'(t_0) - f'(t_0)g(t_0) \right) = c_1 W(f(t_0), g(t_0)) = 0$$

So we have two possibilities:

- if $W \neq 0$ then $c_1 = 0$ which implies $c_2 = 0$: linear independence
- if $W = 0$ then $c_1 \neq 0$ and $c_2 \neq 0$: linear dependence

So if the Wronskian is zero for some arbitrary t_0, we have linear dependence, and alternatively if the Wronskian is nonzero, then the function f and g are linearly independent.

ABEL'S THEOREM: Let y_1 and y_2 be solutions of

$$L[y] = y'' + p(t)y' + q(t)y = 0$$

where p, q are continuous on some interval I, then

$$W(y_1, y_2) = C \exp\left(- \int p(t) dt \right)$$

so that W is either zero for all t in I ($C = 0$) or it is never zero ($C \neq 0$).

The proof of this theorem is relatively straightforward. We begin by considering the solutions y_1 and y_2 which satisfy:

$$y_1'' + py_1' + qy_1 = 0 \qquad \text{and} \qquad y_2'' + py_2' + qy_2 = 0 \,.$$

Multiplying the first equation by $-y_2$ and second by y_1 gives

$$-y_2y_1'' - py_2y_1' - qy_2y_1 = 0 \qquad \text{and} \qquad y_1y_2'' + py_1y_2' + qy_1y_2 = 0 \,.$$

By adding the two equations together we find

$$(y_1y_2'' - y_2y_1'') + p(t)(y_1y_2' - y_2y_1') = 0$$

which upon noticing that $W = y_1y_2' - y_2y_1'$ and $W' = y_1y_2'' - y_2y_1''$ gives

$$W' + p(t)W = 0 \,.$$

This is just a first order differential equation which we learned how to solve in the last chapter. The solution is found to be

$$W = C \exp\left(- \int p(t) dt \right) \,.$$

So either $C = 0$ which makes the Wronskian zero, or $C \neq 0$ and the Wronskian is non-zero for all time in the given interval and y_1 and y_2 are linearly independent.

With this concept of linear independence in hand, we now return to our constant coefficient, second-order problem. In particular we return to considering:

$$ay'' + by' + cy = 0\,.$$

Recall that we tried a solution of the form $y = \exp(\lambda t)$ which gave

$$a\lambda^2 + b\lambda + c = 0\,.$$

The roots of this characteristic equation were simply:

$$\lambda_{\pm} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\,.$$

In the previous examples, we always had $b^2 - 4ac > 0$. However, we can also have $b^2 - 4ac < 0$ or $b^2 - 4ac = 0$. Each of these three cases is fundamentally different than each other. Therefore we will consider each in turn. Since we have already considered $b^2 - 4ac > 0$ in the previous lecture, we move on to consider $b^2 - 4ac < 0$. In this case, we have to take the square-root of a negative number. This of course gives us an imaginary number since $\sqrt{-1} = i$. In this case, we find the two roots of our characteristic equation to be:

$$\lambda_{\pm} = \beta \pm i\mu$$

where $\beta = -b/2a$ and $\mu = \sqrt{4ac - b^2}/2a$. This implies that our solution looks like

$$y = c_1 e^{(\beta + i\mu)t} + c_2 e^{(\beta - i\mu)t}$$

where $y_1 = \exp((\beta + i\mu)t)$ and $y_2 = \exp((\beta - i\mu)t)$. The question now arises about what is $\exp(i\alpha t)$? Without going into details, I will simply tell you that:

$$e^{\pm i\alpha t} = \cos(\alpha t) \pm i\sin(\alpha t)\,.$$

What may also be familiar to you is that $\sin(t) = (e^{it} - e^{-it})/2i$ and $\cos(t) = (e^{it} + e^{-it})/2$. Using this we find

$$\begin{aligned} y_1 &= e^{(\beta + i\mu)t} = e^{\beta t}(\cos\mu t + i\sin\mu t) \\ y_2 &= e^{(\beta - i\mu)t} = e^{\beta t}(\cos\mu t - i\sin\mu t)\,. \end{aligned}$$

These equations are still a little unwieldy, so we simplify them by defining two new solutions Y_1 and Y_2 in the following way

$$Y_1 = \frac{y_1 + y_2}{2} = e^{\beta t}\cos\mu t \qquad \text{and} \qquad Y_2 = \frac{y_1 - y_2}{2i} = e^{\beta t}\sin\mu t$$

so that our solution is given by

$$y = c_1 Y_1 + c_2 Y_2 = c_1 e^{\beta t}\cos\mu t + c_2 e^{\beta t}\sin\mu t$$

where the constants are again determined by initial conditions. Why can we do this? Simply put, both y_1 and y_2 and the new Y_1 and Y_2 are linearly independent solutions.

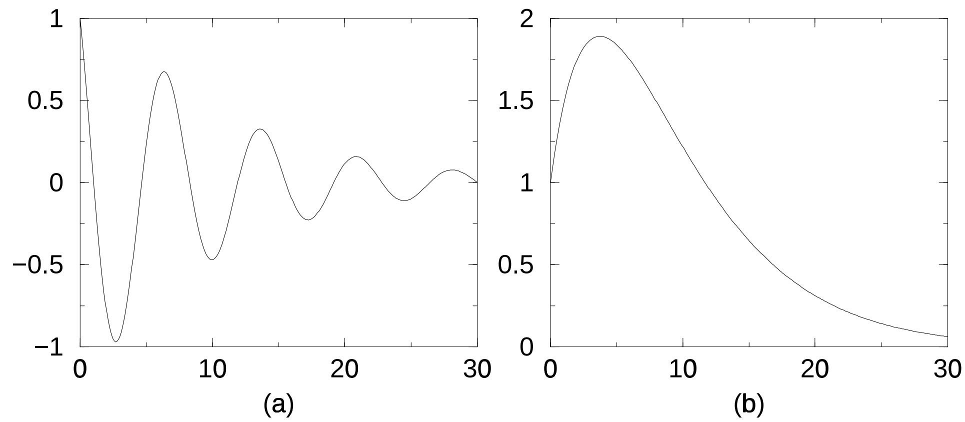

FIG. 3. *Characteristic behavior of damped oscillations which arise when $\lambda = \beta \pm i\mu$ (a), and characteristic behavior when double roots arise (b).*

It is straightforward to verify, for instance, that $W(Y_1, Y_2) = \mu \exp(2\beta t)$ so that some combination of Y_1 and Y_2 describe all solutions of the differential equation.

Example: Solve $y'' + y' + y = 0$.

We begin by plugging in $y = \exp(\lambda t)$. This gives

$$\lambda^2 + \lambda + 1 = 0$$

whose roots are

$$\lambda_\pm = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm i\,\frac{\sqrt{3}}{2}\,.$$

Thus our general solution is given by

$$y = c_1 e^{-t/2} \cos\left(\frac{\sqrt{3}}{2}t\right) + c_2 e^{-t/2} \sin\left(\frac{\sqrt{3}}{2}t\right)$$

Which gives us damped oscillations as depicted in Fig. 3a.

The final case of interest corresponds to the case when we have a double root at $b^2 - 4ac = 0$. Thus we find that $\lambda = -b/2a$ and we have only one solution:

$$y = c_1 y_1 = c_1 e^{-(b/2a)t}\,.$$

But since this is a second order equation, we know that we need two solutions to make up a general solution. To find the second solution, we guess a solution of the form

$$y = v(t) y_1(t)$$

where we have replaced the constant c_1 by the function $v(t)$. Our aim is then to find some equation for $v(t)$ so that we can get a second solution. This method is known as *reduction of order* since it allows you to get a second solution once you have one solution. To proceed, we first note that

$$y' = v'y_1 + vy_1' \qquad \text{and} \qquad y'' = v''y_1 + 2v'y_1' + vy_1''\,.$$

Plugging this into our differential equation we find

$$av''y_1 + 2av'y_1' + avy_1'' + bv'y_1 + bvy_1' + cvy_1 = 0\,.$$

Rearranging we find

$$v\left(ay_1'' + by_1' + cy_1\right) + v'\left(2ay_1' + by_1\right) + v''\left(ay_1\right) = 0$$

But since $ay_1'' + by_1' + cy_1 = 0$ by the fact that y_1 is a solution, we then find after simplifying:

$$v'' + v'\left(2\frac{y_1'}{y_1} + \frac{b}{a}\right) = 0$$

which is a *first-order equation* for v'. Letting $u = v'$ we then have

$$u' + \left(2\frac{y_1'}{y_1} + \frac{b}{a}\right)u = 0$$

which provided we have y_1, we can simply solve this using either the method of integrating factors or separation. Note that this is a general method! We have nowhere assumed anything special about the solution y_1 or the differential equation. At this point however, we do make use of our solution y_1 to simplify things further. We note that

$$\left(2\frac{y_1'}{y_1} + \frac{b}{a}\right) = 2\frac{-(b/2a)\exp(-(b/2a)t)}{\exp(-(b/2a)t)} + \frac{b}{a} = -\frac{b}{a} + \frac{b}{a} = 0$$

which results in

$$u' = 0 \quad \rightarrow \quad v'' = 0\,.$$

Integration then gives

$$v(t) = c_2 t + c_3$$

Recall that our solution is $y = v(t)y_1$ which generates the general solution for the double-root case:

$$y = c_1 e^{-(b/2a)t} + c_2 t e^{-(b/2a)t}\,.$$

We note that we folded the constant c_3 in with c_1. A depiction of this behavior is shown in Fig. 3b. Finally, we comment that the Wronskian of these solutions can be calculated to be $W(y_1, y_2) = \exp(-(b/a)t) \neq 0$ so that the solutions are linearly independent.

3.3. Lec. 3. Nonhomogeneous Equations: Undetermined Coefficients

We now turn our attention to the nonhomogeneous equation:

$$L[y] = y'' + p(t)y' + q(t)y = g(t)$$

where $p(t), q(t)$, and $g(t)$ are continuous over some interval I. In this case, we have the following important theorem:

Theorem: if Y_1 and Y_2 solve the nonhomogeneous equation, then $Y_1 - Y_2$ solves the homogeneous equation. And if y_1 and y_2 are the fundamental set of solutions for the homogeneous equation, then

$$Y_1 - Y_2 = c_1 y_1 + c_2 y_2$$

where c_1 and c_2 are constants.

To see that this is true, we simply note that by definition

$$L[Y_1] = g(t) \qquad \text{and} \qquad L[Y_2] = g(t)\,.$$

Subtracting these two gives:

$$L[Y_1] - L[Y_2] = g(t) - g(t) \quad \rightarrow \quad L[Y_1 - Y_2] = 0$$

which implies that $Y_1 - Y_2$ solves the homogeneous equation and thus

$$Y_1 - Y_2 = c_1 y_1 + c_2 y_2$$

where c_1 and c_2 are constants. We can use this theorem to prove the following theorem.

Theorem: The general solution of the nonhomogeneous equation can be written as

$$y = \phi(t) = c_1 y_1 + c_2 y_2 + Y(t)$$

where y_1 and y_2 are the fundamental set of solutions of the homogeneous equation, c_1 and c_2 are constants, and $Y(t)$ is a specific solution of the nonhomogeneous equation.

This theorem follows directly from the last theorem by letting $Y_1 = \phi(t)$ and $Y_2 = Y(t)$ so that

$$Y_1 - Y_2 = \phi(t) - Y(t) = c_1 y_1 + c_2 y_2$$

so that solving for $\phi(t)$ gives

$$\phi(t) = c_1 y_1 + c_2 y_2 + Y(t)$$

as stated in the theorem.

These theorems give us a recipe to follow in generating our solution to the homogeneous problem

1. Find the general solution of the homogeneous equation
2. Find a single solution to the nonhomogeneous problem

3. Add the two together
4. Determine c_1 and c_2 from the initial conditions

We certainly know how to calculate the homogeneous solution (also called the *complimentary solution* y_c). However, we have not yet learned how to calculate a specific solution to the nonhomogeneous equation (also known as the *particular solution* y_p). In order to get the particular solution, we will rely on our most successful trick: smart guessing. It is best to illustrate the concept by an example.

Example: Find a particular solution to $y'' - 3y' - 4y = 3\exp(2t)$.

To determine a particular solution, we need to guess a solution which generates $\exp(2t)$ on the right-hand side. Therefore we guess:

$$y_p = Ae^{2t}$$

where A is an arbitrary constant and $\exp(2t)$ is used since taking derivatives of it only drops down factors of 2. We first calculate

$$y_p' = 2Ae^{2t} \qquad \text{and} \qquad y_p'' = 4Ae^{2t}$$

which when plugged into our differential equation gives

$$4Ae^{2t} - 3 \cdot 2Ae^{2t} - 4 \cdot Ae^{2t} = 3e^{2t}\,.$$

Removing the common factor of $\exp(2t)$ yields

$$4A - 6A - 4A = 3 \quad \rightarrow \quad A = -\frac{1}{2}$$

so that

$$y_p = -\frac{1}{2}e^{2t}$$

which gives us the particular solution we desire.

Example: Find a particular solution to $y'' - 3y' - 4y = 2\sin t$

In this case, the smart guess would be to try

$$y_p = A\sin t + B\cos t\,.$$

We include both sine and cosine since these both can generate sine terms upon differentiating once or twice in the differential equation. Thus we find:

$$y_p' = A\cos t - B\sin t \qquad \text{and} \qquad y_p'' = -A\sin t - B\cos t$$

which upon insertion into the governing equation gives

$$-A\sin t - B\cos t - 3A\cos t + 3B\sin t - 4A\sin t - 4B\cos t = 2\sin t\,.$$

Collecting terms we find

$$(-B - 3A - 4B)\cos t + (-A + 3B - 4A)\sin t = 2\sin t\,.$$

Equating both sides then gives us

$$B + 3A + 4B = 5B + 3A = 0 \qquad -A + 3B - 4A = 3B - 5A = 2$$

which can be solved to give

$$A = -5/17 \qquad \text{and} \qquad B = 3/17$$

and a particular solution

$$y_p = -\frac{5}{17}\sin t + \frac{3}{17}\cos t\,.$$

Thus a slightly more sophisticated guess gave us the particular solution we were looking for.

There are some relatively simple rules to follow in constructing the particular solution via the method of undetermined coefficients.

- if $g(t) = e^{\alpha t}$ guess $y_p = Ae^{\alpha t}$
- if $g(t) = \cos\alpha t$ or $\sin\alpha t$ guess $y_p = A\cos\alpha t + B\sin\alpha t$
- if $g(t) = a_n t^n + \cdots + a_2 t^2 + a_1 t + a_0$ guess $y_p = A_n t^n + \cdots + A_2 t^2 + A_1 t + A_0$
- if $g(t) = e^{\alpha t}\cos\beta t$ or $e^{\alpha t}\sin\beta t$ guess $y_p = e^{\alpha t}(A\cos\beta t + B\sin\beta t)$
- if $g(t) = g_1(t) + g_2(t)$ guess y_p^1 for g_1, and y_p^2 for g_2 so that $y_p = y_p^1 + y_p^2$

Although this provides a nice general method, problems do arise. Specifically, if your guess is actually a solution of the homogeneous equation, then that guess will never generate a term to satisfy the right hand nonhomogeneous forcing $g(t)$. Again an example will serve to illustrate the point.

Example: Solve $y'' + 4y = 3\cos 2t$.

Since we want the general solution, let's first solve the homogeneous problem

$$y'' + 4y = 0\,.$$

We find the characteristic equation

$$y = e^{\lambda t} \quad \to \quad \lambda^2 + 4 = 0 \quad \to \quad \lambda = \pm 2i$$

which yields the complimentary solution

$$y = c_1\cos 2t + c_2\sin 2t\,.$$

We now would like to get the particular solution by guessing a solution which includes terms like $\cos 2t$ and $\sin 2t$. But these terms actually solve the homogeneous equation! Therefore, if we guess a solution of this form, we will never be able to satisfy the nonhomogeneous part. We then guess the following form of solution:

$$y_p = At\cos 2t + Bt\sin 2t$$

which is the next simplest form of solution which upon differentiation yields $\cos 2t$ and $\sin 2t$ terms. We then calculate

$$y_p' = A\cos 2t + B\sin 2t - 2At\sin 2t + 2Bt\cos 2t$$
$$y_p'' = -4A\sin 2t + 4B\cos 2t - 4At\cos 2t - 4Bt\sin 2t\,.$$

Plugging these into the governing equation yields

$$-4A\sin 2t + 4B\cos 2t - 4At\cos 2t - 4Bt\sin 2t + 4At\cos 2t + 4Bt\sin 2t = 3\cos 2t$$

which simplifies to

$$-4A\sin 2t + 4B\cos 2t = 3\cos 2t \quad \rightarrow \quad A = 0, B = \frac{3}{4}.$$

Thus our general solution is

$$y(t) = c_1\sin 2t + c_2\cos 2t + \frac{3}{4}t\sin 2t$$

where c_1 and c_2 are arbitrary constants.

3.4. Lec. 4. Nonhomogeneous Equations: Variation of Parameters

In the last section, we learned how to solve the nonhomogeneous equation provided the forcing $g(t)$ took on the following form:

$$L[y] = y'' + p(t)y' + q(t)y = g\left(e^t, e^{-t}, \cos\omega t, \sin\omega t, t^n\right)$$

So provided $g(t)$ has cosines, sines, exponentials, or simple polynomials, we can guess the solution via the method of undetermined coefficients. This method is rather limiting however, since it pretty clear that this method will not work if we have something slightly more complicated than the above functions. Thus we would like to find a more general method for getting solutions for a general $g(t)$.

Recall that for the homogeneous solutions we have

$$y(t) = c_1y_1(t) + c_2y_2(t)$$

which satisfies the equations provided $g(t) = 0$. Since in this course we are in the business of judicious guessing, how about we try a solution of the form

$$y(t) = u_1(t)y_1(t) + u_2(t)y_2(t).$$

This guess is motivated by the fact that it is very similar to our homogeneous solutions. And perhaps by letting u_1 and u_2 (which correspond to c_1 and c_2 respectively) vary with time, we can solve the nonhomogeneous equation.

We begin by noting the following:

$$y' = u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2'$$

Now if we take the second derivative of this, then we will get terms like u_1'' and u_2'', but this is highly undesirable since we would be replacing a second order equation with two other second order equations for u_1 and u_2. In order to avoid this, we chose the following:

$$u_1'y_1 + u_2'y_2 = 0.$$

There is no reason why we can't do this. After all, we have the freedom to pick u_1 and u_2 as we desire. This then gives

$$y' = u_1y_1' + u_2y_2'$$

from which it follows that

$$y'' = u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2''.$$

We can then plug these into the governing equations to find

$$u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2'' + p(u_1y_1' + u_2y_2') + q(u_1y_1 + u_2y_2) = g(t)$$

which upon rearranging gives

$$u_1(y_1'' + py_1' + qy_1) + u_2(y_2'' + py_2' + qy_2) + u_1'y_1' + u_2'y_2' = g(t).$$

But the first two terms are zero since both y_1 and y_2 are solutions to the homogeneous equation. This then gives us two equations which must be satisfied

$$\begin{aligned} u_1'y_1' + u_2'y_2' &= g(t) \\ u_1'y_1 + u_2'y_2 &= 0. \end{aligned}$$

The first equation is what is left over from above and the second equation was the condition we imposed on u_1 and u_2 so as not to generate second derivatives.

But we know how to solve two by two systems of this form. In particular, we find from the first equation that

$$u_1' = \frac{g(t) - u_2'y_2'}{y_1'}.$$

And since $-u_2' = u_1'y_1/y_2$ from the second equation we find

$$u_1' = \frac{g(t) + u_1'y_1y_2'/y_2}{y_1'} = \frac{y_2g(t)}{y_2y_1'} + u_1'\frac{y_1y_2'}{y_2y_1'}$$

which can be manipulated to

$$u_1'\left(1 - \frac{y_1y_2'}{y_2y_1'}\right) = \frac{g(t)}{y_1'}.$$

Further simplifying

$$u_1'(y_1y_2' - y_1'y_2) = -y_2g(t) \quad \rightarrow \quad u_1' = -\frac{y_2g(t)}{W(y_1, y_2)}.$$

Plugging back into our expression for u_2' above we find

$$u_2' = \frac{y_1g(t)}{W(y_1, y_2)}.$$

By integrating, we can find the values of u_1 and u_2:

$$u_1 = -\int \frac{y_2g(t)}{W(y_1, y_2)}dt \qquad \text{and} \qquad u_2 = \int \frac{y_1g(t)}{W(y_1, y_2)}dt.$$

We then are ready to write out particular solutions down. It is given by

$$y_p = u_1y_1 + u_2y_2 = -y_1\int \frac{y_2g(t)}{W(y_1, y_2)}dt + y_2\int \frac{y_1g(t)}{W(y_1, y_2)}dt$$

and our complete solution is then

$$y = c_1 y_1 + c_2 y_2 + y_p$$

where the c_1 and c_2 are once again determined by initial conditions. Note that this gives us an extremely general way to get the homogeneous solutions since we only assume that we can integrate up $g(t)$ at the end.

Example: Solve $y'' + 4y = 3\csc t$.

Note that $g(t) = 3\csc t$ is not suited for the method of undetermined coefficients of the last section. So we will use the variation of parameters developed here. We begin by solving the homogeneous equation:

$$y = e^{\lambda t} \quad \to \quad \lambda^2 + 4 = 0 \quad \to \quad \lambda = \pm 2i\,.$$

Thus our complementary solution is

$$y_c = c_1 \cos 2t + c_2 \sin 2t\,.$$

To calculate the particular solution, we first need to find the Wronskian:

$$W(y_1, y_2) = \cos 2t(\sin 2t)' - \sin 2t(\cos 2t)' = 2\,.$$

We can now calculate the particular solution

$$\begin{aligned}
\frac{y_p}{3} &= -\frac{1}{2}\cos 2t \int \sin 2t \csc t dt + \frac{1}{2}\sin 2t \int \cos 2t \csc t dt \\
&= -\cos 2t \int \sin t \cos t \frac{1}{\sin t} dt + \frac{1}{2}\sin 2t \int (\cos^2 t - \sin^2 t)\frac{1}{\sin t} dt \\
&= -\cos 2t \int \cos t dt + \frac{1}{2}\sin 2t \int \left(\frac{\cos^2 t}{\sin t} - \sin t\right) dt \\
&= -\cos 2t \sin t + \cos t \sin 2t + \frac{1}{2}\sin 2t \ln(\tan(t/2))\,.
\end{aligned}$$

So the general solution is

$$y = c_1 \cos 2t + c_2 \sin 2t + \frac{3}{2}\sin 2t \ln(\tan(t/2)) - 3\cos 2t \sin t + 3\cos t \sin 2t$$

where c_1 and c_2 are determined by the initial conditions.

3.5. Lec. 5. Applications: Forced Oscillators and Resonance

Let's consider some applications of the ideas considered in this chapter. A natural place to begin is with Newton's Law:

$$\sum F = ma$$

where $\sum F$ is the sum of the forces, m is the mass, and a is acceleration. To be more precise, we consider, the case of a spring which has a restoring force and a periodic driving term. Thus we find

$$my'' = -ky + f_0 \cos \omega t$$

where the restoring force is given by Hook's Law. We can rewrite this as

$$y'' + \omega_0^2 y = F_0 \cos \omega t$$

where $\omega_0 = \sqrt{k/m}$ and $F_0 = f_0/m$.

This problem is nicely handled by the techniques developed in this section. In particular, we begin by looking for the homogeneous solution:

$$y = e^{\lambda t} \quad \rightarrow \quad \lambda^2 + \omega_0^2 = 0 \quad \rightarrow \quad \lambda = \pm i\omega_0 \, .$$

Thus our solution looks like

$$y_c = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t \, .$$

The particular solution is then found via the method of undetermined coefficients. So we try a solution of the form

$$y_p = A \cos \omega t \, ,$$

which when inserted into our differential equation gives

$$-\omega^2 A \cos \omega t + \omega_0^2 A \cos \omega t = F_0 \cos \omega t \, .$$

We can solve for A and we find

$$A = \frac{F_0}{\omega_0^2 - \omega^2}$$

so that our general solution is

$$y = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + \frac{F_0}{\omega_0^2 - \omega^2} \cos \omega t \, .$$

Note that this solution is valid provided $\omega \neq \omega_0$ since it blows-up if this occurs.

So let's consider for the moment when $\omega \neq \omega_0$. Let's in fact consider the case which has the initial conditions

$$y(0) = 0 \qquad \text{and} \qquad y'(0) = 0 \, .$$

Plugging this in we find that $c_2 = 0$ and $c_1 = -F_0/(\omega_0^2 - \omega^2)$. This then gives the solution as

$$y = \frac{F_0}{\omega_0^2 - \omega^2} (\cos \omega t - \cos \omega_0 t) \, .$$

Utilizing a trigonometry identity for cosine we then find the final solution

$$y = \frac{2F_0}{\omega_0^2 - \omega^2} \sin \left(\frac{\omega_0 - \omega}{2} t \right) \sin \left(\frac{\omega_0 + \omega}{2} t \right) \, .$$

Thus our solution is composed of two distinct frequencies, i.e. $(\omega_0 - \omega)/2$ and $(\omega_0 + \omega)/2$. These two amplitudes form what is called a *beat frequency* between them. Figure 4a depicts qualitatively the beating phenomena which is compromised of two distinct frequencies. Note the rapid and slow frequencies which make up the solution.

In the case where $\omega = \omega_0$, we have already noted that our solution breaks down. This makes sense since in that case the particular solution that we guessed is actually a solution to the homogeneous equation. So let's now consider

$$y'' + \omega_0^2 y = F_0 \cos \omega_0 t$$

This time we guess a solution of the form

$$y_p = At \cos \omega_0 t + Bt \sin \omega_0 t\,.$$

In order to utilize this guess, we first calculate

$$\begin{aligned} y_p' &= A \cos \omega_0 t - \omega_0 At \sin \omega_0 t + B \sin \omega_0 t + \omega_0 Bt \cos \omega_0 t \\ y_p'' &= -2\omega_0 A \sin \omega_0 t - \omega_0^2 At \cos \omega_0 t + 2\omega_0 B \cos \omega_0 t - \omega_0^2 Bt \sin \omega_0 t \end{aligned}$$

and then plug into our governing equation:

$$\begin{aligned} -2\omega_0 A \sin \omega_0 t - \omega_0^2 At \cos \omega_0 t + 2\omega_0 B \cos \omega_0 t - \omega_0^2 Bt \sin \omega_0 t \\ +\omega_0^2 (At \cos \omega_0 t + Bt \sin \omega_0 t) = F_0 \cos \omega_0 t\,. \end{aligned}$$

Collecting terms then gives

$$-2\omega_0 A \sin \omega_0 t + 2\omega_0 B \cos \omega_0 t = F_0 \cos \omega_0 t$$

from which is readily found that

$$A = 0 \qquad \text{and} \qquad B = \frac{F_0}{2\omega_0}$$

Our solution then takes the form

$$y = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + \frac{F_0}{2\omega_0} t \sin \omega_0 t$$

which upon assuming once again $y(0) = y'(0) = 0$ gives

$$y = \frac{F_0}{2\omega_0} t \sin \omega_0 t\,.$$

Thus solutions have unbounded growth, i.e. the system is forced at the *natural frequency* which causes the oscillations to grow like t. A depiction of this behavior is given in Fig. 4b.

The fact that the solution grows to infinity at the resonant frequency seems highly idealized since in fact, we never really see things grow to infinity! In practice, any physical system has some small amount of damping; due perhaps to friction, air resistance, or something of the sort. So we really need to consider the system:

$$my'' + \gamma y' + ky = f_0 \cos \omega t$$

where γ measures our damping force. The homogeneous solution in this case is given by

$$y = e^{\lambda t} \quad \rightarrow \quad m\lambda^2 + \gamma\lambda + k = 0 \quad \rightarrow \quad \lambda_\pm = -\frac{\gamma}{2m} \pm \sqrt{\frac{\gamma^2}{4m^2} - \frac{k}{m}}\,.$$

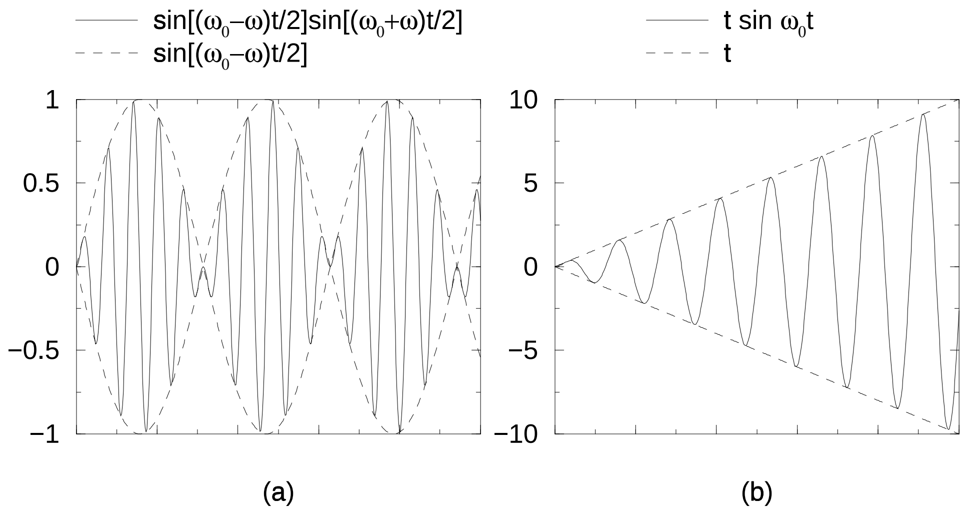

FIG. 4. *(a) Characteristic behavior of beating phenomena which occurs when $\omega \neq \omega_0$ and (b) growth behavior which occurs when ω is at the resonant frequency ω_0*

If $\gamma^2/4m^2 - k/m < 0$ then

$$\lambda_\pm = -\beta \pm i\mu$$

where $\mu = \sqrt{k/m - \gamma^2/4m^2}$ and $\beta = \gamma/2m$. Our homogeneous solution in this case is given by

$$y_c = c_1 e^{-\beta t}\cos \mu t + c_2 e^{-\beta t}\sin \mu t$$

which corresponds to damped oscillations. This is exactly as we expect when we have damping.

The particular solution can once again be found by the variation of parameters or undetermined coefficients methods. So we try

$$y = A\cos\omega t + B\sin\omega t\,.$$

After a bit of algebra, we find

$$B = \frac{\gamma_0\omega F_0}{(\omega_0^2 - \omega^2)^2 + \gamma_0^2\omega^2} \qquad \text{and} \qquad A = \frac{F_0(\omega_0^2 - \omega^2)}{(\omega_0^2 - \omega^2)^2 + \gamma_0^2\omega^2}$$

where $\gamma_0 = \gamma/m$, $\omega_0^2 = k/m$, and $F_0 = f_0/m$. Since the homogeneous solutions die away as $t \to \infty$, we find in that limit that

$$y(t\to\infty) = \frac{F_0}{(\omega_0^2 - \omega^2)^2 + \gamma_0^2\omega^2}\left((\omega_0^2 - \omega^2)\cos(\omega t) + \gamma_0\omega\sin\omega t\right)\,.$$

Note that in this case, the presence of the damping term stops the solution from blowing up when $\omega = \omega_0$. So even the smallest amount of damping has a profound effect. This is illustrated in Fig. 5 where we show the maximum solution for various values of ω.

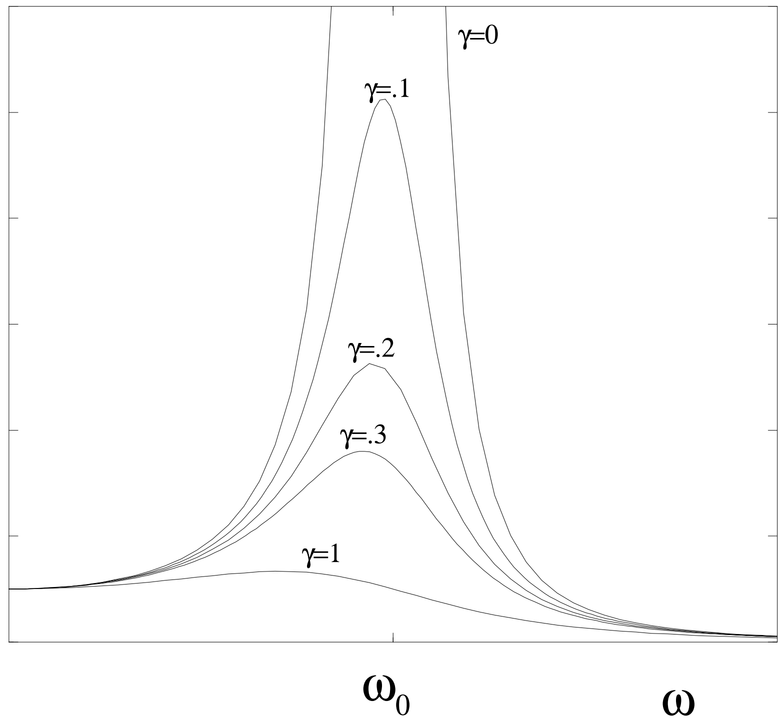

FIG. 5. *Qualitative behavior of the forced vibration with damping: above is the amplitude of the steady-state response versus the driving frequency* ω. *Note that the solution does not blow up at* $\omega = \omega_0$ *with a slight amount of damping.*

As a final note to this application section. This fundamental idea of natural frequencies is the basis of microwave ovens (for which the microwaves are at the natural frequency of the vibrational mode of water molecules), lasers (where light is responsible for the resonant phenomena of stimulated emission), dribbling a basketball (where a resonant frequency of dribbling is required to keep the ball bouncing), and of course, the collapse of the Tacoma Narrows Bridge where a moderate wind was able to collapse a large structure.

3.6. Chapter 3. Summary

In this chapter, we considered second-order, constant coefficient problems. The basic method was to assume a solution of the form $y = \exp(\lambda t)$ and get a characteristic equation for λ. Three cases were found:

$$b^2 - 4ac > 0 \quad \rightarrow \quad \lambda_{\pm} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \rightarrow \quad y = c_1 e^{\lambda_+ t} + c_2 e^{\lambda_- t}$$

$$b^2 - 4ac < 0 \quad \rightarrow \quad \lambda_{\pm} = \beta \pm i\mu \quad \rightarrow \quad y = c_1 e^{\beta t} \cos \mu t + c_2 e^{\beta t} \sin \mu t$$

$$b^2 - 4ac = 0 \quad \rightarrow \quad \lambda_{\pm} = -\frac{b}{2a} \quad \rightarrow \quad y = c_1 e^{\lambda t} + c_2 t e^{\lambda t}$$

Thus we can classify all solutions of the second-order, homogeneous differential equation simply from the roots of the characteristic equation. We also learned some fundamental concepts concerning the second order equations.

Linear Independence and the Wronskian

Solutions to second order equations can always be written as a sum of two *linearly independent* solutions y_1 and y_2. The way we check for this is by calculating the Wronskian:

$$W(y_1, y_2) = y_1 y_2' - y_2 y_1' \neq 0 .$$

If the Wronskian is zero, then the solutions are linearly dependent, and they no longer represent two solutions for the general solution.

Reduction of Order

If we have one solution y_1 of a second-order equation, then we can calculate the second linearly independent solution by letting:

$$y_2 = v(t) y_1$$

and solving for $v(t)$. This then gives our second solution.

Method of Undetermined Coefficients

If the nonhomogeneous forcing is of the form

$$g(t) = g(e^{\alpha t}, e^{-\alpha t}, \cos \alpha t, \sin \alpha t, \alpha t^n)$$

then to find the particular solution we guess

$$y_p = A \left\{ e^{\alpha t}, e^{-\alpha t}, \cos \alpha t, \sin \alpha t, \alpha t^n \right\}$$

where we chose the bracketed terms to be same type as $g(t)$ and we solve for A, i.e. if $g(t) = \sin \alpha t$, then we chose $y_p = A \cos \alpha t + B \sin \alpha t$. If this doesn't work, then we try

$$y_p = At \left\{ e^{\alpha t}, e^{-\alpha t}, \cos \alpha t, \sin \alpha t, \alpha t^n \right\}$$

as above and determine A. Finally if this doesn't work, we can try

$$y_p = At^2 \left\{ e^{\alpha t}, e^{-\alpha t}, \cos \alpha t, \sin \alpha t, \alpha t^n \right\} .$$

We only try these last two if the forcing corresponds to a homogeneous solution of the problem.

Variation of Parameters

We end the section by giving a very general derivation of how to find a particular solution for the nonhomogeneous case by considering $y = u_1(t) y_1(t) + u_2(t) y_2(t)$. This gives the particular solution

$$y_p = -y_1 \int \frac{y_2 g(t)}{W(y_1, y_2)} dt + y_2 \int \frac{y_1 g(t)}{W(y_1, y_2)} dt .$$

With this method we can consider very general $g(t)$.

4. Chapter 4. Series Solutions of Second Order Equations

We learned in the last chapter how to solve constant coefficient differential equations of the form

$$ay'' + by' + cy = g(t)\,.$$

However, in general we would like to solve more complicated equations of the form

$$y'' + p(t)y' + q(t)y = g(t)\,.$$

The methods developed in the last chapter do not give us techniques for solving this since we can no longer rely on trying solutions of the form $\exp(\lambda t)$ and getting a characteristic equation. Thus we turn to power series solutions which take the general form

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + \cdots$$

where determining the a_n coefficients is at the heart of the solution technique.

4.1. Lec. 1. Review of Power and Taylor Series

We begin by reviewing some of the more important properties of power series. More detailed explanations can be found in any Calculus book. One of the most important concepts is that of *convergence*. That is, we are going to add an infinite number of terms and we want it to add up to a finite number. At first, this may seem counter-intuitive since if you add an infinite number of things you might naturally expect it to add up to infinity. As an example, consider the common power series:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad \exp(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad \frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 + \cdots \quad (x < 1)\,.$$

Notice that each of the above is a relatively common function which we know has a finite value for a given x. These power series expansions of the above functions are known as *Taylor series*. And if we have a function $f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$, then

$$\begin{aligned}
f(x) &= a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + \cdots & f(x_0) &= a_0 \\
f'(x) &= a_1 + 2a_2(x - x_0) + 3a_3(x - x_0)^2 + 4a_4(x - x_0)^3 + \cdots & f'(x_0) &= a_1 \\
f''(x) &= 2a_2 + 3 \cdot 2 \cdot a_3(x - x_0) + 4 \cdot 3 \cdot a_4(x - x_0)^2 + \cdots & f''(x_0) &= 2a_2 \\
f'''(x) &= 3 \cdot 2 \cdot a_3 + 4 \cdot 3 \cdot 2 \cdot a_4(x - x_0) + \cdots & f'''(x_0) &= 3 \cdot 2 \cdot a_3
\end{aligned}$$

Solving for each a_n then gives the Taylor series expansion about some point x_0

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3 + \cdots$$

so that each $a_n = f^{(n)}(x_0)/n!$ where $f^{(n)}$ is the n^{th} derivative of the function.

To properly discuss the idea of convergence, we introduce the notion of *absolute convergence*. We have absolute convergence if

$$\lim_{N\to\infty} \sum_{n=0}^{N} |a_n(x-x_0)^n| \quad \to \quad \text{converges}.$$

An excellent way to check for this convergence is by the *ratio test*:

$$\text{ratio test: } \lim_{n\to\infty} \left| \frac{a_{n+1}(x-x_0)^{n+1}}{a_n(x-x_0)^n} \right| = |x-x_0| \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L$$

Once we determine L, convergence or divergence follows from the following:

$L < 1$	series converges for that x
$L = 1$	convergence/divergence can not be determined
$L > 1$	series diverges for that x

We note that we can find the *radius of convergence* by finding all x for which $L < 1$. Thus if $L < 1$ for some $|x - x_0| < \rho$, this then gives the *interval of convergence* and the *radius of converges* which is $\rho/2$.

Example: Find the convergence of $y = \sum_{n=0}^{\infty} \frac{(x+1)^n}{n2^n}$.

We begin by noting in this example that $x_0 = -1$ and further that $a_n = 1/n2^n$. We can then apply the ratio test:

$$\begin{aligned} |x-x_0| \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= |x+1| \lim_{n\to\infty} \frac{1/(n+1)2^{n+1}}{1/n2^n} \\ &= \frac{|x+1|}{2} \lim_{n\to\infty} \frac{1}{1+1/n} = \frac{|x+1|}{2} = L \end{aligned}$$

which then gives

$$\begin{array}{llll} L < 1 & \text{for} & -3 < x < 1 & \to \quad \text{convergence} \\ L = 1 & \text{at} & x = -3, x = 1 & \to \quad \text{undetermined} \\ L > 1 & \text{for} & x < -3, x > 1 & \to \quad \text{diverges}. \end{array}$$

All that remains is to determine the behavior at $x = 1$ and $x = -3$. At $x = 1$, the powers series reduces to $y = \sum 1/n$ which is known to diverge. For $x = -3$, we then have $y = \sum (-1)^n/n$ which converges, but not absolutely. Thus our interval of convergence is $-3 \le x < 1$ and our radius of convergence about $x_0 = -1$ is 2.

Other properties of power series you should be aware of include those associated with division, multiplication, subtraction and addition. If $f(x) = \sum a_n(x-x_0)^n$ and $g(x) = \sum b_n(x-x_0)^n$ then

$$f(x) \pm g(x) = \sum_{n=0}^{\infty} (a_n \pm b_n)(x-x_0)^n$$

$$f(x)g(x) = \left(\sum_{n=0}^{\infty} a_n(x-x_0)^n\right)\left(\sum_{n=0}^{\infty} b_n(x-x_0)^n\right) = \sum_{n=0}^{\infty} c_n(x-x_0)^n$$

$$f(x)/g(x) = \left(\sum_{n=0}^{\infty} a_n(x-x_0)^n\right)/\left(\sum_{n=0}^{\infty} b_n(x-x_0)^n\right) = \sum_{n=0}^{\infty} d_n(x-x_0)^n$$

where the c_n and d_n can be difficult to determine, but they can be determined nonetheless. It should also be noted that the summation index n above is a dummy index, i.e. we can call it whatever we would like. Thus we have that

$$\sum_{n=0}^{\infty} \frac{2^n x^n}{n!} = \sum_{j=0}^{\infty} \frac{2^j x^j}{j!} .$$

This is an obvious result which is very useful for manipulating power series. For instance, suppose we have the power series

$$\sum_{n=2}^{\infty} a_n x^n$$

whose sum begins with the index $n = 2$. We can rewrite this in a more convenient form by letting $m = n - 2$ so that $n = m + 2$ and we have

$$\sum_{n=2}^{\infty} a_n x^n = \sum_{m=0}^{\infty} a_{m+2} x^{m+2} .$$

This kind of manipulation of power series is important for solving the differential equations of this chapter.

Example: Derive the Taylor series for $f(x) = \exp(x)$ about $x = 0$.

This calculation is relatively straight forward if we recall that

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3 + \cdots$$

where x_0 is the point about which we are expanding. In this example $x_0 = 0$ so that the above equation becomes

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

which leaves us to evaluate $f(0)$ and its derivatives. But since $f(x) = \exp(x)$, all its derivatives are also $\exp(x)$ which then gives $f(0) = f'(0) = f''(0) = \cdots = 1$. Thus the power series reduces to

$$f(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots$$

which is the Taylor series for $f(x) = \exp(x)$ about $x = 0$. The same procedure can be utilized to derive power series expansions for both $\sin(x)$ and $\cos(x)$.

Example: What are the a_n if $f(x) = g(x)$ where $f(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$ and $g(x) = \sum_{n=0}^{\infty} a_n x^n$.

In this case, since $f(x) = g(x)$ we have that

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} a_n x^n$$

But we can rewrite the left-hand side by letting $m = n - 1$ so that

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{m=0}^{\infty} (m+1) a_{m+1} x^m .$$

Our equality can then be written (noting that m is just a dummy variable and can be replaced by n):

$$\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n = \sum_{n=0}^{\infty} a_n x^n .$$

Equating like powers of x^n then gives

$$(n+1) a_{n+1} = a_n \quad \rightarrow \quad a_{n+1} = \frac{a_n}{n+1}$$

so that $a_1 = a_0$, $a_2 = a_1/2 = a_0/2$, $a_3 = a_2/3 = a_0/3!$, and finally culminating in $a_n = a_0/n!$. Thus we have

$$f(x) = g(x) = \sum_{n=0}^{\infty} \frac{a_0}{n!} x^n = a_0 e^x$$

where we have used our definition of the Taylor series of e^x.

4.2. Lec. 2. Power Series Solution Techniques

Now that we have reviewed some basic concepts in power series, we will move on to utilizing them in solving differential equations. As in the previous chapter, we will once again "guess" a solutions of the form

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

to the differential equation:

$$P(x) y'' + Q(x) y' + R(x) y = 0 .$$

where $P(x), Q(x)$ and $R(x)$ are polynomials. In addition, in this section we assume $P(x) \neq 0$. This means x is an *ordinary point.* If at some point $P(x) = 0$, then that point is called a *singular point.* This will be discussed further in the following section.

To begin the solution method utilizing the power series method, we first note that the derivatives of the power series guess are given by ($x_0 = 0$):

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + \cdots = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = 2a_2 + 2 \cdot 3 a_3 x + 2 \cdot 3 \cdot 4 a_4 x^2 + \cdots = \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n .$$

At this point, we can plug in our guess and its derivatives. We then proceed to collect all terms of like powers of x^n. We then equate them to zero in order to satisfy the differential equation. This procedure then gives us a recursive way of getting our power series solution. This method is best illustrated through an example.

Example: Solve $y'' - xy' - y = 0$ about $x_0 = 0$.

We first note that every point is an ordinary point since $P(x) = 1$. Next, we plug in the power series expansion above along with its derivatives. This gives

$$\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^n - x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n - \sum_{n=0}^{\infty}a_n x^n = 0$$

We first note that we must rewrite the middle term since it has a factor of x multiplying it. Rewriting gives

$$x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n = \sum_{n=0}^{\infty}(n+1)a_{n+1}x^{n+1} = \sum_{n=0}^{\infty}na_n x^n .$$

This then gives the modified equation

$$\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^n - \sum_{n=0}^{\infty}na_n x^n - \sum_{n=0}^{\infty}a_n x^n = 0 .$$

Or upon combining the sums we find

$$\sum_{n=0}^{\infty}[(n+1)(n+2)a_{n+2} - na_n - a_n]x^n = 0 .$$

But in order to satisfy the above equality for arbitrary x, the coefficient of each x^n must be zero. Thus we find the *recursion relation*

$$(n+1)(n+2)a_{n+2} - (n+1)a_n = 0 \quad \to \quad a_{n+2} = \frac{a_n}{n+2} .$$

The recursion relation gives us the important information needed to solve the problem. In particular, we notice that from the above equation we can determine all the *even* powers given a_0. Likewise, we can determine all the *odd* powers given a_1. To be more precise, we find

$$a_2 = \frac{a_0}{2}, a_4 = \frac{a_2}{4} = \frac{a_0}{2\cdot 4}, a_6 = \frac{a_4}{6} = \frac{a_0}{2\cdot 4\cdot 6}, \ldots .$$

and

$$a_3 = \frac{a_1}{3}, a_5 = \frac{a_3}{5} = \frac{a_1}{3\cdot 5}, a_7 = \frac{a_5}{7} = \frac{a_1}{3\cdot 5\cdot 7}, \ldots .$$

We can then group our a_0 terms and a_1 terms to get

$$y = a_0\left(1 + \frac{x^2}{2} + \frac{x^4}{2\cdot 4} + \frac{x^6}{2\cdot 4\cdot 6} + \cdots\right) + a_1\left(x + \frac{x^3}{3} + \frac{x^5}{3\cdot 5} + \frac{x^7}{3\cdot 5\cdot 7} + \cdots\right)$$

Or by simplifying

$$y = a_0\sum_{n=0}^{\infty}\frac{x^{2n}}{2^n\cdot n!} + a_1\sum_{n=0}^{\infty}\frac{2^n\cdot n!x^{2n+1}}{(2n+1)!} .$$

It is interesting to note that since this is a second order equation we expect two linearly-independent solutions with two arbitrary constants c_1 and c_2, i.e. we expect $y = c_1 y_1 + c_2 y_2$. But notice that this is exactly what we have with $a_0 = c_1$ and $a_1 = c_2$ and the two solutions y_1 and y_2 given by the two power series respectively.

We can also specify initial conditions. Suppose we take $y(0) = A$ and $y'(0) = B$. Then since our power series is given by

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

it is relatively easy to show that

$$a_0 = A \qquad \text{and} \qquad a_1 = B\,.$$

Our solution in total then is given by

$$y = A\sum_{n=0}^{\infty}\frac{x^{2n}}{2^n\cdot n!} + B\sum_{n=0}^{\infty}\frac{2^n\cdot n! x^{2n+1}}{(2n+1)!}$$

which is a unique solution. We can even go on to calculate the radius of convergence of each power series using the ratio test. We find for the first series that

$$\lim_{n\to\infty}\left|\frac{a_{n+1}x^{n+1}}{a_n x^n}\right| = \lim_{n\to\infty}\left|\frac{1}{2(n+1)}\right| x^2 = 0$$

so that $L = 0$ and the series converges for all x. As for the second series we have

$$\lim_{n\to\infty}\left|\frac{a_{n+1}x^{n+1}}{a_n x^n}\right| = \lim_{n\to\infty}\left|\frac{2(n+1)}{(2n+3)(2n+2)}x^2\right| = \lim_{n\to\infty}\left|\frac{x^2}{2n+3}\right| = 0$$

so that $L = 0$ and the series converges for all x.

4.3. Lec. 3. Euler Equations and Frobenius Method

In contrast to the last section, we now consider equations

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

for which $P(x_0) = 0$ at some point x_0, i.e. x_0 is a *singular point*. In order to utilize analytic techniques, the singularity must be classified as a *regular singular point*. This means that

$$\lim_{x\to x_0}(x - x_0)\frac{Q(x)}{P(x)} = A \qquad \text{and} \qquad \lim_{x\to x_0}(x - x_0)^2\frac{R(x)}{P(x)} = B$$

where A and B are some finite numbers. If these conditions do not hold, then the singular point is called *irregular*.

As a simple example, we consider the *Euler equation*

$$x^2 y'' + \alpha x y' + \beta y = 0$$

which has a singular point at $x = 0$. We see that it is a regular singular point by noting that

$$\lim_{x\to 0} x\frac{\alpha x}{x^2} = \alpha \qquad \text{and} \qquad \lim_{x\to 0} x^2\frac{\beta}{x^2} = \beta$$

where both α and β are assumed to be finite. Solving this equation is particularly simple. We simply try a solution of the form

$$y = x^r \, .$$

By noting that $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$ we can plug into our Euler equation to find

$$x^2(r(r-1)x^{r-2}) + \alpha x(rx^{r-1}) + \beta x^r = x^r[r(r-1) + \alpha r + \beta] = 0$$

which can only be solved provided

$$r^2 + (\alpha - 1)r + \beta = 0 \, .$$

This is analogous to our characteristic equation of the last chapter. In particular, we find that

$$r_\pm = \frac{1}{2}\left(-(\alpha-1) \pm \sqrt{(\alpha-1)^2 - 4\beta}\right)$$

so that

$$y = c_1 x^{r+} + c_2 x^{r-} \qquad (\alpha-1)^2 - 4\beta > 0$$

and the constants c_1 and c_2 are determined from initial conditions.

Just as in the previous chapter, there are three cases of interest to us. In the case that $(\alpha-1)^2 - 4\beta > 0$, then we can simply write the solution as above since we have two distinct roots as expected. If $(\alpha-1)^2 - 4\beta < 0$, then our roots have an imaginary part as well. Thus our roots can be expressed as

$$r_\pm = \lambda \pm i\mu$$

which upon noting that

$$x^{\lambda \pm i\mu} = x^\lambda x^{\pm i\mu} = x^\lambda e^{\ln x^{\pm i\mu}} = x^\lambda e^{\pm i\mu \ln x}$$

we find the general solution

$$y = c_1 x^\lambda \cos(\mu \ln x) + c_2 x^\lambda \sin(\mu \ln x) \qquad (\alpha-1)^2 - 4\beta < 0 \, .$$

We note that in this case we have once again made use of cosine and sine solutions instead of $\exp(\pm ix)$ type solutions. The final case is when $(\alpha-1)^2 - 4\beta = 0$ and we have a double root. Just as in the last chapter, we can use the method of reduction of order (i.e. let $y = v(x)x^r$ and solve for v) to get the second solution. Thus we find the general solution:

$$y = c_1 x^{r+} + c_2 x^{r+} \ln x \qquad (\alpha-1)^2 - 4\beta = 0$$

where $r_+ = -(\alpha-1)/2$.

Example: Solve $x^2 y'' - 2y = 0$.

We try a solution of the form $y = x^r$ and derive the equation:

$$r^2 - r - 2 = (r-2)(r+1) = 0$$

so that our solution is

$$y = c_1 x^2 + c_2 \frac{1}{x} .$$

Notice that the solution blows-up at $x = 0$. This is expected as $x = 0$ is our singular point.

Example: Solve $x^2 y'' - 2xy' + 2y = 0$.

Again we try $y = x^r$ and we find:

$$r^2 - 3r + 2 = (r - 2)(r - 1) = 0$$

so that our solution is

$$y = c_1 x^2 + c_2 x .$$

Note that in this case y is well behaved at $x = 0$. However, there is still a problem since if we consider some initial condition $y(0) = C$, the above solution can never satisfy it since $y(0) = 0$ regardless of the value of c_1 and c_2. Thus this solution once again reflects the singular nature of the solution at $x = 0$.

The Euler equation is the simplest case of a singular differential equation. And although we can feel proud that we know how to solve it, we would actually like to solve much more complicated problems. So let's consider the differential equation

$$y'' + p(x)y' + q(x)y = 0 .$$

We can rewrite this to reflect the Euler form by multiplying through by x^2. This then gives

$$x^2 y'' + x[xp(x)]y' + [x^2 q(x)]y = 0$$

for which we have a regular singular point if we can write the $xp(x)$ and $x^2 q(x)$ as power series

$$xp(x) = \sum_{n=0}^{\infty} p_n x^n \qquad \text{and} \qquad x^2 q(x) = \sum_{n=0}^{\infty} q_n x^n .$$

Since it looks like the Euler equation, we can then try a power series solution of the form

$$y = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r} .$$

Three things are needed to determine a solution: the values of r, the recursion relation giving us the a_ns and the radius of convergence of the resulting power series. Note that the r will take care of the singular behavior for us. This method is known as the *Frobenius Method*. To best illustrate its use, we apply it to an example.

Example: Solve $2x^2 y'' - xy' + (1 + x)y = 0$.

In this case, we note that $xp(x) = -1/2$ and $x^2 q(x) = (1+x)/2$. Thus we have a regular singular point. This allows us to try an expansion of the form $y = \sum_{n=0}^{\infty} a_n x^{n+r}$. Before we do this we note that

$$xy' = x\sum_{n=0}^{\infty}(r+n)a_n x^{r+n-1} = \sum_{n=0}^{\infty}(r+n)a_n x^{r+n}$$
$$x^2 y'' = x^2\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n x^{r+n-2} = \sum_{n=0}^{\infty}(r+n)(r+n-1)a_n x^{r+n}\,.$$

This can then be plugged into our governing differential equation and terms can be collected. Thus we find

$$\sum_{n=0}^{\infty}2(r+n)(r+n-1)a_n x^{r+n} - \sum_{n=0}^{\infty}(r+n)a_n x^{r+n} + \sum_{n=0}^{\infty}a_n x^{r+n} + x\sum_{n=0}^{\infty}a_n x^{r+n} = 0\,.$$

Rewriting gives

$$\sum_{n=0}^{\infty}[2(r+n)(r+n-1)a_n - (r+n)a_n + a_n]x^{r+n} + \sum_{n=0}^{\infty}a_n x^{r+n+1} = 0\,.$$

To do things properly, we need to rewrite the last term so that the sum goes in powers of x^{r+n}. We do this by letting $m = n+1$ and changing over from n to m. Thus we find

$$\sum_{n=0}^{\infty}[2(r+n)(r+n-1)a_n - (r+n)a_n + a_n]x^{r+n} + \sum_{n=1}^{\infty}a_{n-1}x^{r+n} = 0\,.$$

We now need to equate like powers of x^{r+n}. We notice that the first term on the left starts from $n = 0$ whereas the second term starts from $n = 1$. This allows us to rewrite things as:

$$a_0 x^r[2r(r-1) - r + 1] + \sum_{n=1}^{\infty}[(2(r+n)(r+n-1) - (r+n-1))a_n + a_{n-1}]x^{r+n} = 0$$

where the first term arises from the $n = 0$ term of the preceding expression. Each coefficient of the power series expansion must be zero, so we find immediately that

$$2r(r-1) - r + 1 = 0 \quad \rightarrow \quad 2r^2 - 3r + 1 = (2r-1)(r-1) = 0 \quad \rightarrow \quad r = 1/2, 1.$$

This equation is known as the *indicial equation* and is responsible for capturing the correct behavior due to the singularity. It is analogous to the Euler method for finding the behavior of the singular differential equation. We also find the recursion relation:

$$[2(r+n)(r+n-1) - (r+n-1)]a_n + a_{n-1} = 0 \quad \rightarrow \quad a_n = \frac{-a_{n-1}}{(r+n-1)(2(r+n)-1)}\,.$$

This allows us to determine each of the coefficients in the power series expansion for a given value of r.

For $r = 1$, the recursion relation reduces to

$$a_n = -\frac{a_{n-1}}{(2n+1)n}$$

which gives the power series solution which is valid for $x > 0$:

$$y_1 = x\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{[3 \cdot 5 \cdot 7 \cdots (2n+1)]n!}\right].$$

The exact range of validity can be determined by the ratio test for absolute convergence. Note that we have arbitrarily taken $a_0 = 1$. Recall that the general solution is $y = c_1 y_1 + c_2 y_2$ where the c_1 and c_2 account for the arbitrary value of a_0.

For $r = 1/2$, the recursion relation reduces to

$$a_n = -\frac{a_{n-1}}{(2n-1)n}$$

which gives the power series solution

$$y_2 = x^{1/2}\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{[1 \cdot 3 \cdot 5 \cdot 7 \cdots (2n-1)]n!}\right]$$

which is valid for $x > 0$. The exact range of validity can again be determined by performing a ratio test for absolute convergence.

4.4. Chapter 4. Summary

The aim of this chapter was to present a method by which we could solve a broader class of differential equations than those studied in Chapter 3. In particular, we aimed to study the non-constant coefficient case of

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

by assuming a solution of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

provided that $P(x) \neq 0$ at any x. By equating coefficients, we found the recursion relation which determined the a_ns.

If at some point $P(x) = 0$, then the problem became singular. Under certain conditions, the singularity is a regular singularity which allows us to try a solution of the form

$$y = \sum_{n=0}^{\infty} a_n x^{r+n}$$

where the r captures the correct singular behavior. Again we collect terms to get both an indicial equation which determines r and a recursion relation which determines the a_n.

The simplest regular singularity gives the Euler equation

$$x^2 y'' + \alpha x y' + \beta y = 0$$

for which we can look for solutions of the from $y = x^r$. Three cases arise:

$$(\alpha - 1)^2 - 4\beta > 0 \quad \rightarrow \quad r = r_\pm \quad \rightarrow \quad y = c_1 x^{r_+} + c_2 x^{r_-}$$

$$(\alpha-1)^2-4\beta=0 \quad \to \quad \text{(double root) } r=r_+ \quad \to \quad y=c_1x^{r_+}+c_2x^{r_+}\ln x$$

$$(\alpha-1)^2-4\beta<0 \quad \to \quad r=\lambda\pm i\mu \quad \to \quad y=c_1x^{\lambda}\cos(\mu\ln x)+c_2x^{\lambda}\sin(\mu\ln x)\,.$$

This is analogous to our solution method and characteristic equation of the last section.

Finally, the absolute convergence of the power series can be tested by using the ratio test:

$$\text{ratio test: } \lim_{n\to\infty}\left|\frac{a_{n+1}(x-x_0)^{n+1}}{a_n(x-x_0)^n}\right| = |x-x_0|\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L$$

Once we determine L, convergence or divergence follows from the following:

$L<1$	series converges for that x
$L=1$	convergence/divergence can not be determined
$L>1$	series diverges for that x

We note that we can find the *radius of convergence* by finding all x for which $L<1$. Thus if $L<1$ for some $|x-x_0|<\rho$, this then gives the *interval of convergence* and the *radius of converges* which is $\rho/2$.

5. Chapter 5. The Laplace Transform

All of our attempts at solving differential equations thus far have involved turning our equation into algebra by judiciously guessing a solution. In each case, we have transformed our equations into something more manageable. In keeping with this idea, we introduce an interesting transformation which can help turn many differential equations to algebra.

5.1. Lec. 1. Introduction to the Laplace Transform

The Laplace transform, unlike our previous transformation techniques, is an integral transformation. Integral transformations are generally defined as

$$F(s)=\int_{\alpha}^{\beta}K(s,t)f(t)dt$$

where $F(s)$ is the transform of $f(t)$ and $K(s,t)$ is the *Kernel*. The Laplace transform has a very specific Kernel, namely:

$$\mathcal{L}\{f(t)\}=F(s)=\int_0^{\infty}e^{-st}f(t)dt$$

There $\mathcal{L}\{f(t)\}$ denotes the Laplace transform and the Kernel is e^{-st}.

Before getting too far, we have to address the fact that we have an *improper integral*, i.e. the limit of integration goes to ∞. From calculus, you should recall that

$$\int_a^{\infty}f(t)dt=\lim_{A\to\infty}\int_a^{A}f(t)dt\,.$$

This integral can converge or diverge depending upon $f(t)$. Further, in this section we will consider *piecewise continuous* functions only. That is, functions that are continuous everywhere except at perhaps some jumps in solution at fixed locations. We are concerned with these issues precisely because we are going to be looking at the Laplace transform of various functions. So the convergence and piecewise continuity are very important.

In order to ensure that our Laplace transform integral converges, the following condition must hold:

$$|f(t)| \leq Ke^{at} \qquad \text{when} \qquad t \geq M$$

where M, a, and K are constants. What this says is that if I have a Laplace transform $f(t)$, then

$$F(s) = \int_0^\infty e^{-st} f(t) dt = \int_0^M e^{-st} f(t) dt + \int_M^\infty e^{-st} f(t) dt \,.$$

The only troublesome term is the second term on the right-hand side. But since we assumed the above inequality, we find

$$\left| \int_M^\infty e^{-st} f(t) dt \right| = \int_M^\infty e^{-st} |f(t)| dt \leq \int_M^\infty K e^{-(s-a)t} dt \,.$$

which converges for $s > a$ since the integrand decays exponentially. In all else that follows in this chapter, we will assume that $f(t)$ satisfies the exponential inequality so that the Laplace transform exists.

Example: Calculate the Laplace transform of $f(t) = 1$, $f(t) = e^{at}$, and $f(t) = \sin at$ for $t \geq 0$.

We begin with $f(t) = 1$ and the definition of the Laplace transform:

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st} \cdot 1 \cdot dt = \int_0^\infty e^{-st} dt = -\frac{1}{s} e^{-st}|_0^\infty = \frac{1}{s}$$

Similarly for $f(t) = e^{at}$:

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st} e^{at} dt = \int_0^\infty e^{-(s-a)t} dt = -\frac{1}{s-a} e^{-(s-a)t}|_0^\infty = \frac{1}{s-a} \quad (s > a)$$

And finally $f(t) = \sin(at)$:

$$\begin{aligned}
\mathcal{L}\{\sin at\} &= \int_0^\infty e^{-st} \sin at dt \qquad \text{(integrate by parts twice)} \\
&= -\frac{1}{a} e^{-st} \cos at|_0^\infty - \frac{s}{a} \int_0^\infty e^{-st} \cos at dt \\
&= \frac{1}{a} - \frac{s}{a^2} e^{-st} \sin at|_0^\infty - \frac{s^2}{a^2} \int_0^\infty e^{-st} \sin at dt \\
&= \frac{1}{a} - \frac{s^2}{a^2} \mathcal{L}\{\sin at\}
\end{aligned}$$

which can be rearranged to give

$$\left(1+\frac{s^2}{a^2}\right)\mathcal{L}\{\sin at\} = \frac{1}{a} \quad \rightarrow \quad \mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2} \quad (s>0)\,.$$

The Laplace transform for $\cos at$ can be found in a similar way.

We now move on to some important properties of the Laplace transforms. The first establishes the fact that the Laplace transform is a *linear operator*. This is tremendously important in utilizing the transform. So let's consider the following transform:

$$\begin{aligned}\mathcal{L}\{c_1f_1(t)+c_2f_2(t)\} &= \int_0^\infty e^{-st}(c_1f_1(t)+c_2f_2(t))dt\\ &= c_1\int_0^\infty e^{-st}f_1(t)dt + c_2\int_0^\infty e^{-st}f_2(t)dt\\ &= c_1\mathcal{L}\{f_1(t)\}+c_2\mathcal{L}\{f_2(t)\,.\}\end{aligned}$$

This allows us to transform a sum of functions in a fairly easy manner.

Thus far, it is not very clear how we will be able to use these methods to solve a given differential equation. This becomes clear once we consider the following:

$$\begin{aligned}\mathcal{L}\{f'\} &= \int_0^\infty e^{-st}f'dt \qquad\qquad \text{(integrate by parts)}\\ &= f(t)e^{-st}|_0^\infty + \int_0^\infty se^{-st}f(t)dt\\ &= -f(0)+s\int_0^\infty e^{-st}f(t)dt\\ &= s\mathcal{L}\{f\}-f(0)\,.\end{aligned}$$

Thus we can replace the Laplace transform of the derivative of a function by the Laplace transform of the function itself. We can also find by using the above twice that:

$$\begin{aligned}\mathcal{L}\{f''\} &= s\mathcal{L}\{f'\}-f'(0)\\ &= s\left(s\mathcal{L}\{f\}-f(0)\right)-f'(0)\\ &= s^2\mathcal{L}\{f\}-sf(0)-f'(0)\,.\end{aligned}$$

We can continue this procedure to find

$$\mathcal{L}\{f^{(n)}(t)\} = s^n\mathcal{L}\{f\}-s^{n-1}f(0)-s^{n-2}f'(0)-\cdots-sf^{(n-2)}(0)-f^{(n-1)}(0)$$

so that all derivatives can be expressed as the Laplace transform of the original function itself. The next example will illustrate how to use this in solving a differential equation.

Example: Solve $y''+y=\sin 2t$ with $y(0)=2$, $y'(0)=1$.

We take the Laplace transform of both sides of the equation:

$$\mathcal{L}\{y''+y\} = \mathcal{L}\{\sin 2t\}\,.$$

Using the fact that the Laplace operator is a linear operator gives

$$\mathcal{L}\{y''\} + \mathcal{L}\{y\} = \mathcal{L}\{\sin 2t\}$$

which gives

$$s^2 Y - sy(0) - y'(0) + Y = s^2 Y - 2s - 1 + Y = \frac{2}{s^2+4}\,.$$

Solving for $Y(s)$ gives

$$Y(s) = \frac{(2s+1)(s^2+4)+2}{(s^2+4)(s^2+1)} = \frac{2s}{s^2+1} + \frac{5}{3}\frac{1}{s^2+1} - \frac{2}{3}\frac{1}{s^2+4}$$

where we have used a partial fraction decomposition. At this point, we need to get back to our solution $y(t)$ which is the inverse Laplace transform of $Y(s)$. We do this by noting that all the terms on the right hand side are simple Laplace transforms of cosine and sine terms. Thus working from a table of Laplace transforms we find

$$y(t) = 2\cos t + \frac{5}{3}\sin t - \frac{1}{3}\sin 2t\,.$$

Although we do not prove it, it should be noted that there is a one-to-one correspondence between the Laplace transform and its inverse. Thus the above solution is the unique solution to the problem.

5.2. Lec. 2. The Laplace Transform and Heaviside Function

The Laplace transform method is powerful because it reduces to algebra and tables, i.e. you solve many differential equations with the aid of Laplace transform tables after your equation has been reduced to an algebraic expression. As a second example of this consider the following two problems.

Example: Solve $y'' - y' - 2y = 0$ with $y(0) = 1$ and $y'(0) = 0$.

We Laplace transform the equation and find

$$\mathcal{L}\{y'' - y' - 2y\} = \mathcal{L}\{y''\} - \mathcal{L}\{y'\} - 2\mathcal{L}\{y\} = 0$$

which gives

$$s^2 Y - sy(0) - y'(0) - [sY - y(0)] - 2Y = 0\,.$$

Inserting our initial conditions and solving for $Y(s)$ gives

$$Y(s) = \frac{s-1}{s^2-s-2} = \frac{s-1}{(s-2)(s+1)} = \frac{1}{3}\frac{1}{s-2} + \frac{2}{3}\frac{1}{s+1}\,.$$

But we recall from the last lecture that if $f(t) = e^{at}$ then $F(s) = 1/(s-a)$ which implies that

$$y(t) = \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t}$$

which is our unique solution.

Example: Solve $y'' - 2y' + 2y = \exp(-t)$ with $y(0) = 0$ and $y'(0) = 1$.

We Laplace transform the equation and find

$$\mathcal{L}\{y'' - 2y' + 2y\} = \mathcal{L}\{y''\} - 2\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \mathcal{L}\{\exp(-t)\}$$

which gives

$$s^2 Y - sy(0) - y'(0) - 2[sY - y(0)] + 2Y = \frac{1}{s+1}.$$

Inserting our initial conditions and solving for $Y(s)$ gives

$$Y(s) = \frac{1}{(s-1)^2+1} + \frac{1}{(s+1)((s-1)^2+1)}.$$

The one difficulty involved in Laplace transforms is getting the transform into the correct form so that we can utilize Laplace transform tables for inverting them. In this case, we have to perform a partial fraction decomposition which allows us to solve the problem. Thus we note that

$$\frac{1}{(s+1)[(s-1)^2+1]} = \frac{A}{s+1} + \frac{Bs+C}{(s-1)^2+1},$$

and we proceed to determine that $A = 1/5$, $B = -1/5$, and $C = 3/5$. In total then we have that

$$Y(s) = \frac{1}{(s-1)^2+1} + \frac{1}{5}\frac{1}{s+1} - \frac{1}{5}\frac{s}{(s-1)^2+1} + \frac{3}{5}\frac{1}{(s-1)^2+1}.$$

Combining terms into the Laplace table format then gives

$$Y(s) = \frac{7}{5}\frac{1}{(s-1)^2+1} + \frac{1}{5}\frac{1}{s+1} - \frac{1}{5}\frac{s-1}{(s-1)^2+1}.$$

Each of the individual fractions is now ready for transformation back into the time domain. Performing the transformation gives

$$y(t) = \frac{7}{5}\exp(t)\sin t - \frac{1}{5}\exp(t)\cos t + \frac{1}{5}\exp(-t)$$

which is our unique solution.

But this Laplace transform method is no better than our previous techniques of the last chapter. In fact, it seems a little more unwieldy. As you will see, the real power of the Laplace transform is in its ability to handle all kinds of piecewise-continuous functions very nicely. To begin with, we consider the *Heaviside function* which is defined as

$$u_c(t) = \begin{cases} 0 & t < c \\ 1 & t \geq c \end{cases}$$

So this function jumps from zero to one at $t = c$, i.e. it is a unit jump centered at $t = c$. These kind of functions are important to consider in various systems that arise in the physical, biological, and engineering sciences. The model systems that "come on" or "go off" at very specific times. A system that goes on and off may, for instance, be modeled by the following function:

$$y = u_\pi(t) - u_{2\pi}(t) = \begin{cases} 0 & t < \pi \\ 1 & \pi \leq t < 2\pi \\ 0 & t \geq 2\pi \end{cases}$$

which is one in the interval $\pi \leq t < 2\pi$. We can also calculate the Laplace transform of the Heaviside function:

$$\mathcal{L}\{u_c(t)\} = \int_0^\infty e^{-st} u_c(t) = \int_c^\infty e^{-st} dt = \frac{e^{-sc}}{s} \quad (s > 0) .$$

In addition to this, we can consider the Laplace transform of the Heaviside function times another arbitrary function. Thus we consider

$$g(t) = u_c(t) f(t-c)$$

whose Laplace transform is given by

$$\begin{aligned} \mathcal{L}\{u_c(t) f(t-c)\} &= \int_0^\infty e^{-st} u_c(t) f(t-c) dt \\ &= \int_c^\infty e^{-st} f(t-c) dt \qquad \text{let } \xi = t - c \, , d\xi = dt \\ &= \int_0^\infty e^{-s(\xi+c)} f(\xi) d\xi = e^{-sc} \int_0^\infty e^{-s\xi} f(\xi) d\xi \\ &= e^{-sc} \mathcal{L}\{f(t)\} \end{aligned}$$

And due to our one-to-one correspondence between the function and its transform we have

$$\mathcal{L}^{-1}\{e^{-sc} \mathcal{L}\{f(t)\}\} = u_c(t) f(t-c) .$$

Likewise we can find the transform for

$$g(t) = e^{ct} f(t)$$

which is

$$\mathcal{L}\{e^{ct} f(t)\} = \int_0^\infty e^{-st} e^{ct} f(t) dt = \int_0^\infty e^{-(s-c)t} f(t) dt = F(s-c)$$

by the definition of the Laplace transform. Conversely we have

$$\mathcal{L}^{-1}\{F(s-c)\} = e^{ct} f(t) .$$

Thus we see that the exponential and Heaviside function serve to translate the function by c. This will be important in the applications that follow.

Example: Solve $y'' + 4y = g(t)$ with $y(0) = 0$ and $y'(0) = 0$ where $g(t)$ is given by

$$g(t) = \begin{cases} 0 & 0 \leq t < 5 \\ \frac{(t-5)}{5} & 5 \leq t < 10 \\ 1 & t \geq 10 \end{cases}$$

We begin by noting that we can rewrite $g(t)$ using the Heaviside function:

$$g(t) = \frac{1}{5}\left(u_5(t)\cdot(t-5) - u_{10}(t)\cdot(t-10)\right) .$$

This forcing describes a forcing which is zero until $t = 5$ and then steadily ramps up to a value of one at $t = 10$. This is called *ramp loading* and describes many practical systems of interest in which we turn on slowly (ramp up) some parameter in the problem. We now take the Laplace transform of the equation

$$\mathcal{L}\{y'' + 4y\} = \mathcal{L}\{g(t)\}$$

which results in

$$\mathcal{L}\{y''\} + 4\mathcal{L}\{y\} = \frac{1}{5}\mathcal{L}\{u_5(t)(t-5) - u_{10}(t)(t-10)\} .$$

Using the derived results of this and the previous lecture this results in the equation for $Y(s)$:

$$s^2Y - sy(0) - y'(0) + 4Y = \frac{1}{5}\left(e^{-5s}\mathcal{L}\{t\} - e^{-10s}\mathcal{L}\{t\}\right)$$

which can be reduced to

$$Y(s) = \frac{1}{5}\frac{1}{s^2(s^2+4)}(e^{-5s} - e^{-10s}) .$$

To solve this, we note that the exponential terms simply give us shifts in time whereas the factor $1/(s^2(s^2+4))$ gives us the fundamental behavior. We note that

$$\frac{1}{s^2(s^2+4)} = \frac{1}{4}\frac{1}{s^2} - \frac{1}{4}\frac{1}{s^2+4} \quad \rightarrow \quad \frac{t}{4} - \frac{1}{8}\sin 2t .$$

where the second part is derived from the inverse Laplace transform. Since the exponential terms only shift the time, we then find the solution

$$y = \frac{1}{5}\left[u_5(t)\left(\frac{t-5}{4} - \frac{\sin 2(t-5)}{8}\right) - u_{10}(t)\left(\frac{t-10}{4} - \frac{\sin 2(t-10)}{8}\right)\right]$$

So although this problem is fairly complicated, it is nicely done with the Laplace transform method. We can compare this with how we would solve this with the Chapter 3 methods of Boyce & DiPrima, i.e. we would have to solve the equations from time 0 to 5, get the final condition, then use these as the initial condition for the differential equation from 5 to 10, then use these final conditions for the initial conditions for time 10 and beyond. This is a rather complicated formula for getting the solution and so we can think of the Laplace method as a better way to do such a problem.

5.3. Lec. 3. Impulse Functions: The Dirac Delta Function

Continuing on with this idea of forces acting on the differential equations over only finite periods of time leads to the concept of an *Impulse Function.* For instance, we can consider the differential equation

$$ay'' + by' + cy = g(t)$$

where

$$g(t) = \begin{cases} g_0(t) & t_0 - \tau < t < t_0 + \tau \\ 0 & \text{elsewhere} \end{cases} .$$

Thus $g(t)$ acts only over the finite time of 2τ. The strength of the function $g(t)$ acting on the system can be given by the *Impulse* of the function:

$$I(\tau) = \int_{-\infty}^{\infty} g(t)dt = \int_{t_0-\tau}^{t_0+\tau} g_0(t)dt.$$

As a specific example of an impulse function, consider the following defined function

$$g(t) = \begin{cases} \frac{1}{2\tau} & -\tau < t < \tau \\ 0 & t < -\tau, \ t > \tau \end{cases}$$

which gives an impulse of

$$I(\tau) = \int_{-\infty}^{\infty} g(t)dt = \frac{1}{2\tau}\int_{-\tau}^{\tau} dt = \frac{1}{2\tau}2\tau = 1 \, .$$

Thus the Impulse of our defined function is always unity independent of the value of τ. What if we were to let $\tau \to 0$. Then we would have:

$$\lim_{\tau\to 0} g(t) = 0 \quad \text{for} \quad t \neq 0 \quad \to \quad \lim_{\tau \to 0} I(t) = 1 \, .$$

We define this unit impulse function as:

$$\delta(t) = 0 \quad \text{for} \quad t \neq 0 \quad \to \quad \int_{-\infty}^{\infty} \delta(t) = 1 \, .$$

This $\delta(t)$ is known as a generalized function and is called the *Dirac Delta Function*. So we notice from this function that it essentially is zero everywhere and goes to infinity at $t = 0$ in such a way that if we integrate over it we get unity. We note that a unit impulse acting at t_0 can be represented by $\delta(t - t_0)$.

One of the important properties of this integral is its effect on an integrated function. Therefore we consider doing the integration:

$$\begin{aligned} \int_{-\infty}^{\infty} \delta(t-t_0)f(t)dt &= \lim_{\tau\to 0} \frac{1}{2\tau}\int_{t_0-\tau}^{t_0+\tau} f(t)dt \\ &= \lim_{\tau\to 0} \frac{1}{2\tau}2\tau f(t^*) \quad (\text{mean-value theorem} : t_0 - \tau = t^* = t_0 + \tau) \\ &= \lim_{\tau \to 0} = f(t_0) \, . \end{aligned}$$

Thus the integration *sifts* out the value of the function at t_0. This makes it trivial to calculate the Laplace transform:

$$\mathcal{L}\{\delta(t - t_0)\} = \int_0^{\infty} e^{-st}\delta(t - t_0)dt = e^{-st_0}$$

where we have made use of the sifting property and we note that its value is unity at $t_0 = 0$.

Example: Solve $2y'' + y' + 2y = \delta(t-5)$ with $y(0) = 0$ and $y'(0) = 0$.

We Laplace transform the equation to find

$$2[s^2Y - sy(0) - y'(0)] + [sY - y(0)] + 2Y = e^{-5s}$$

which results in

$$(2s^2 + s + 2)Y = e^{-5s}$$

when the initial conditions are implemented. We now solve for $Y(s)$:

$$Y(s) = \frac{e^{-5s}}{2} \cdot \frac{1}{s^2 + s/2 + 1} = \frac{e^{-5s}}{2} \cdot \frac{1}{(s+1/4)^2 + 15/16}\,.$$

The manipulation of the equation into this form arises from our desire to put the right-hand side into a form which is listed in the Laplace transform tables. In particular, we have that

$$\frac{1}{(s+1/4)^2 + 15/16} \quad \to \quad \frac{4}{\sqrt{15}} e^{-t/4} \sin\left[\frac{\sqrt{15}}{4} t\right]\,.$$

We then recall that the exponential term e^{-5s} only serves to shift the function. Thus the total solution is given by

$$y(t) = u_5(t) e^{-(t-5)/4} \cdot \frac{2}{\sqrt{15}} \sin\left[\frac{\sqrt{15}}{4}(t-5)\right]$$

which corresponds to a solution which is zero until acted on by the delta function at $t = 5$. After that time, the system has damped oscillations due to the forcing.

There remains one problem with Laplace transforms. It always requires us to perform a partial fraction decomposition. Since most people have grown to loath such things, it would be nice to find an alternative way of solving some of the Laplace transform problems. There does in fact exist a way to overcome this issue. Consider for the moment that when solving the Laplace transform problem you end up with

$$H(s) = F(s)G(s)$$

where you recognize that your answer is clearly a product of two Laplace transforms

$$F(s) = \int_0^\infty e^{-s\xi} f(\xi) d\xi \qquad \text{and} \qquad G(s) = \int_0^\infty e^{-s\eta} g(\eta) d\eta\,.$$

The question then arises: can I find $h(t)$ by knowing that it is a product of two known transforms of $f(t)$ and $g(t)$?

The answer will of course be yes. We will use the method of *convolution* to solve this problem. We begin by constructing $H(s)$.

$$H(s) = F(s)G(s) = \int_0^\infty e^{-s\xi} f(\xi) d\xi \cdot \int_0^\infty e^{-s\eta} g(\eta) d\eta\,.$$

By doing the ξ integral first, we can rewrite things

$$\begin{aligned} F(s)G(s) &= \int_0^\infty g(\eta)d\eta \int_0^\infty e^{-s(\xi+\eta)} f(\xi)d\xi \\ &= \int_0^\infty g(\eta)d\eta \int_\eta^\infty e^{-st} f(t-\eta)dt \qquad\qquad (t=\xi+\eta \text{ for fixed } \eta) \\ &= \int_0^\infty e^{-st}dt \int_0^t f(t-\eta)g(\eta)d\eta \\ &= \int_0^\infty h(t)dt \end{aligned}$$

where in the second to last step we interchanged the order of integration (see a calculus textbook), and we define

$$h(t) = \int_0^t f(t-\eta)g(\eta)d\eta\,.$$

This then gives us a way to determine the $h(t)$ in terms of the functions $f(t)$ and $g(t)$. What this effectively does is replaces partial fraction decomposition with integration.

Example: Find the inverse Laplace transform of $H(s) = a/(s^2(s^2+a^2))$.

We note that

$$H(s) = \frac{a}{s^2(s^2+a^2)} = \frac{1}{s^2}\cdot\frac{a}{s^2+a^2} = F(s)G(s)$$

where we know that

$$F(s) = \frac{1}{s^2} \quad \rightarrow \quad f(t) = t$$

$$G(s) = \frac{a}{s^2+a^2} \quad \rightarrow \quad g(t) = \sin at\,.$$

From the preceding arguments, we then have

$$h(t) = \int_0^t f(t-\tau)g(\tau)d\tau = \int_0^t (t-\tau)\sin a\tau d\tau = \frac{at-\sin at}{a^2}\,.$$

Thus we found the solution without the need for partial fractions.

5.4. Chapter 5. Summary

The whole idea of this chapter, as in the last chapter, is to find a way to transform the differential equation into algebra. The Laplace transform provides just such a method by defining

$$Y(s) = \int_0^\infty e^{-st}y(t)dt\,.$$

The most important properties of this integral transform are that

$$\mathcal{L}\{y''\} = Y'' = s^2 Y - sy(0) - y'(0)$$
$$\mathcal{L}\{y'\} = Y' = sY - y(0)\,.$$

It is these two properties which are of greatest use to us in solving our differential equations. In particular, we can Laplace transform our differential equation and solve for $Y(s)$. The recipe for a solution is then:

- Laplace transform the equation
- Use the initial conditions and solve for $Y(s)$
- Use Laplace transform tables to get the solution
- If applicable, use the Convolution theorem

Although many of the problems considered could have been done with previous methods, the Laplace transform provides an elegant way to handle jumps and impulses in the forcing. This is something that the previous methods do not handle well.

6. Chapter 6. Systems of Linear Differential Equations

A very large number of applications arise from systems which are coupled in some manner. For instance, two masses which are attached together by a spring may interact since the motion of one mass effects the other through the spring. The key concepts in this chapter are much like those of Chapter. 3, except now must introduce a more general way of treating systems by developing methods of Linear Algebra.

6.1. Lec. 1. Introduction to Systems and Matrices

Although we have discovered many techniques for solving differential equations, there are many relatively simple problems which our methods have thus far not addressed. As a simple example, we consider a two mass-spring system where the masses m_1 and m_2 are acted on by forces $F_1(t)$ and $F_2(t)$. A schematic of this situation is depicted in Fig. 6. For each mass, we can write down Newton's Law:

$$\sum F_1 = m_1 \frac{d^2 x_1}{dt^2} \qquad \text{and} \qquad \sum F_2 = m_2 \frac{d^2 x_2}{dt^2}$$

where $\sum F_1$ and $\sum F_2$ are the sum of the forces on m_1 and m_2 respectively. Note that the equations for $x_1(t)$ and $x_2(t)$ are coupled because of the spring with spring constant k_2. The resulting governing equations are then of the form:

$$m_1 \frac{d^2 x_1}{dt^2} = -k_1 x_1 + k_2(x_2 - x_1) + F_1 = -(k_1 + k_2)x_1 + k_2 x_2 + F_1$$
$$m_2 \frac{d^2 x_2}{dt^2} = -k_3 x_2 - k_2(x_2 - x_1) + F_2 = -(k_2 + k_3)x_2 + k_2 x_1 + F_2$$

which are not in a form that we have learned to solve via the methods of the previous chapters. And yet, this seems to be a fairly straightforward example of something that goes on in many physical systems.

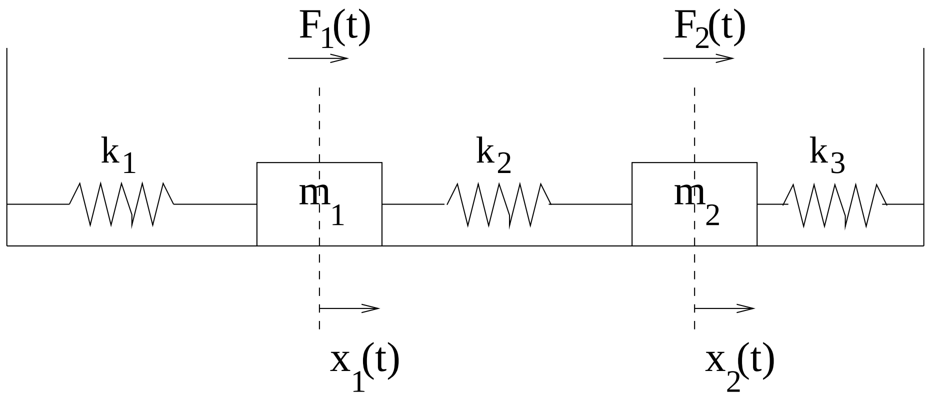

FIG. 6. *Two masses which are coupled together through the spring with constant k_2.*

As a second example, let's consider a system which we do know how to solve and which we can write in a systems form. In particular, let's consider:

$$mu'' + \gamma u' + ku = F(t)$$

which models, for instance, a pendulum with damping and a forcing. By defining

$$x = u \qquad \text{and} \qquad y = u'$$

we can rewrite this system as

$$\begin{aligned} x' &= y \\ my' &= -\gamma y - kx + F(t) . \end{aligned}$$

Thus we turned our second-order differential equation into a two by two system of first order differential equations. But we still need to know how to solve such systems.

In general, we would actually like to consider an n by n coupled system of first order equations:

$$\begin{aligned} x_1' &= F_1(x_1, x_2, x_3, ..., x_n, t) \\ x_2' &= F_2(x_1, x_2, x_3, ..., x_n, t) \\ &\vdots \\ x_n' &= F_n(x_1, x_2, x_3, ..., x_n, t) \end{aligned}$$

In this chapter, we will be only interested in *linear* systems where, for instance:

$$F_j = p_{j1}(t)x_1 + p_{j2}(t)x_2 + \cdots + p_{jn}(t)x_n + g_j(t)$$

where $j = 1, 2, ..., n$. Thus there are no *nonlinear* terms such as x_j^2, $\cos x_j$, $x_1 \cdot x_2$, etc. Nonlinear terms will be considered only in the next chapter. Further, just as in our previous chapters, if the $g_j(t) = 0$, then we have a *homogeneous* equation while if $g_j(t) \neq 0$, the equation is non-homogeneous. For such equations, there is a theorem which states that we have a unique solution to the above equation provided we can find some $x_j(t) = \phi_j(t)$ which satisfy the governing equations and some initial conditions $x_j(t_0) = \phi_j(t_0)$ so that the solution exists around the t_0 provided the F_j and

$\partial F_j/\partial x_j$ are continuous on some interval around t_0.

Before continuing further with this chapter, we need to lay down some preliminaries of *Matrix Theory* and *Linear Algebra*. A matrix is a mathematical object which allows us to manipulate systems in an elegant and efficient fashion: it is the filing cabinet of the math world. A matrix is of the form:

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = (a_{ij})$$

which denotes an $m \times n$ matrix where the m refers to the number of rows and n to the number of columns. Some important concepts which we introduce are the following:

Transpose: $\mathbf{A}^T = (a_{ij})^T = (a_{ji}) \quad \rightarrow \quad \text{if } \mathbf{A} = \begin{pmatrix} 1 & 5 \\ 2 & 3 \end{pmatrix} \text{ then } \mathbf{A}^T = \begin{pmatrix} 1 & 2 \\ 5 & 3 \end{pmatrix}$

Complex Conjugate: $\overline{\mathbf{A}} = \overline{(a_{ij})} \quad \rightarrow \quad \text{if } \mathbf{A} = \begin{pmatrix} i & 5 \\ 3+i & 6 \end{pmatrix} \text{ then } \overline{\mathbf{A}} = \begin{pmatrix} -i & 5 \\ 3-i & 6 \end{pmatrix}$

Adjoint: $\overline{\mathbf{A}}^T = \mathbf{A}^* \quad \rightarrow \quad \text{if } \mathbf{A} = \begin{pmatrix} i & 5 \\ 3+i & 6 \end{pmatrix} \text{ then } \mathbf{A}^* = \begin{pmatrix} -i & 3-i \\ 5 & 6 \end{pmatrix}.$

These three concepts will be important in utilizing the matrix approach to solving differential equations. Finally, we also introduce the concept of a *square matrix*: $n \times n$, and a *vector*: $n \times 1$ or $1 \times n$. We also have the following important properties of matrices:

1. $\mathbf{A} = \mathbf{B}$ if $a_{ij} = b_{ij}$ for each i and j.

2. Zero matrix $\mathbf{0}$ for which $a_{ij} = 0$ for each i and j.

3. Addition and Subtraction: $\mathbf{A} \pm \mathbf{B} = (a_{ij}) \pm (b_{ij}) = (a_{ij} + b_{ij})$.

   - Commutative: $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$
   - Associative: $\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$

4. Multiply by a number: $\alpha\mathbf{A} = \alpha(a_{ij}) = (\alpha a_{ij})$.

5. Matrix multiply: $\mathbf{AB} = \mathbf{C}$ where $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$.

$$\begin{pmatrix} 3 & 2 & 1 \\ 6 & 5 & 0 \\ 1 & 8 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 3\cdot 1 + 2\cdot 0 + 1\cdot 2 \\ 6\cdot 1 + 5\cdot 0 + 0\cdot 2 \\ 1\cdot 1 + 8\cdot 0 + 3\cdot 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix}$$

   - Distributive: $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$
   - Associative: $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$
   - NOT Commutative: $\mathbf{AB} \neq \mathbf{BA}$

6. Vectors: $\vec{u}^T\vec{v} = \sum_{i=1}^n u_i v_i$

$$\vec{u}^T\vec{v} = (u_1 \ u_2 \ \cdots \ u_n)\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = (u_1v_1 + u_2v_2 + u_3v_3 + \cdots + u_nv_n)$$

- $\vec{u}^T\vec{v} = \vec{v}^T\vec{u}$
- $\vec{u}^T(\vec{v}+\vec{w}) = \vec{u}^T\vec{v} + \vec{u}^T\vec{w}$
- $(\alpha\vec{u})^T\vec{v} = \alpha(\vec{u}^T\vec{v}) = \vec{u}^T(\alpha\vec{v})$

Inner Products: $(\vec{u},\vec{v}) = \sum_{i=1}^n u_i\overline{v_i} = \vec{u}^T\overline{\vec{v}}$

- $(\vec{u},\vec{v}) = \overline{(\vec{v},\vec{u})}$
- $(\alpha\vec{u},\vec{v}) = \alpha(\vec{u},\vec{v})$
- $(\vec{u},\alpha\vec{v}) = \overline{\alpha}(\vec{u},\vec{v})$
- $(\vec{u},\vec{v}+\vec{w}) = (\vec{u},\vec{v}) + (\vec{u},\vec{w})$

Vector Magnitudes: $(\vec{u},\vec{u})^{1/2} = \sum_{i=1}^n u_i\overline{u_i} = \sum_{i=1}^n |u_i|$

Orthogonality: $(\vec{u},\vec{v}) = 0$

7. Identity: $\mathbf{I} = (\delta_{ij})$ ($\delta_{ij} = 1$ for $i = j$ and 0 otherwise) $\rightarrow$ $\mathbf{AI} = \mathbf{IA} = \mathbf{A}$

8. Inverse: $\mathbf{AB} = \mathbf{I}$ if $\mathbf{B} = \mathbf{A}^{-1}$ which exists for det($\mathbf{A}$)$\neq$0 (nonsingular).

These properties and identities will be important in manipulating and solving systems of differential equations. They should be thought of as a guide to understanding the systems approach to differential equations.

6.2. Lec. 2. Eigenvalues, Eigenvectors, and Linear Independence

This is the second lecture in which a large number of definitions are discussed. The most important of these ideas relates to what are called *eigenvalues* and *eigenvectors*. But before getting to these, we consider an $n \times n$ system of algebraic equations:

$$\begin{aligned} &a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ &a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ &\vdots \\ &a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{aligned}$$

which can be solved for the x_i since we have n equations and n unknowns. We can rewrite this in the matrix formalism of the last chapter:

$$\mathbf{A}\vec{x} = \vec{b} \quad \rightarrow \quad \mathbf{A} = (a_{ij}),\ \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix},\ \vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

In such an equation, if $\vec{b} = 0$ the equation is called *homogeneous* and if $\vec{b} \neq 0$ the equation is *non-homogeneous.* This terminology is exactly as in our previous chapters.

There are two interesting cases to consider in solving such an equation as $\mathbf{A}\vec{x} = \vec{b}$. The first case is when

$$\det(\mathbf{A}) \neq 0$$

which from the last section implies that we have an inverse to $\mathbf{A}$ given by $\mathbf{A}^{-1}$. Thus we find upon multiplying our equation $\mathbf{A}\vec{x} = \vec{b}$ through by $\mathbf{A}^{-1}$ on the left that

$$\vec{x} = \mathbf{A}^{-1}\vec{b}$$

since $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$. Thus we can find a solution once we find the inverse. We also note that:

$$\text{if } \vec{b} = 0 \text{ then } \vec{x} = 0$$

giving a trivial solution for the homogeneous case. The second case of interest arises when

$$\det(\mathbf{A}) = 0$$

which implies we have no inverse. This case will be of considerable interest to us in what follows.

A couple of things should be pointed out before proceeding. First, the equation $\mathbf{A}\vec{x} = \vec{b}$ does not have a solution for generic $\vec{b}$. In particular, consider the two equations:

$$\mathbf{A}\vec{x} = \vec{b} \qquad \text{and} \qquad \mathbf{A}^*\vec{y} = 0$$

where we recall that $\mathbf{A}^*$ is the adjoint of $\mathbf{A}$. Taking the inner product of the first equation with respect to $\vec{y}$ gives:

$$(\mathbf{A}\vec{x}, \vec{y}) = (\vec{b}, \vec{y}).$$

By noting that $(\mathbf{A}\vec{x}, \vec{y}) = (\vec{x}, \mathbf{A}^*\vec{y})$, which is the definition of the adjoint, and that $\mathbf{A}^*\vec{y} = 0$ by definition, we find

$$(\vec{b}, \vec{y}) = 0.$$

This is known as the *Fredholm Alternative Theorem* and is commonly called a *solvability condition.* It states that $\vec{y}$ must be orthogonal to $\vec{b}$ in order for the equation to make sense. To calculate determinants, we note that

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

$$\det\begin{pmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{pmatrix} = A_1(B_2C_3 - C_2B_3) - A_2(B_1C_3 - C_1B_3) + A_3(B_1C_2 - C_1B_2).$$

Since we will mostly be working with 2×2 and 3×3 systems, these are important to know.

Example: Solve

$$x_1 - 2x_2 + 3x_3 = 7$$
$$-x_1 + x_2 - 2x_3 = -5$$
$$2x_1 - x_2 - x_3 = 4$$

To purpose of this example is to illustrate how to manipulate such systems using linear algebra techniques. We begin be rewriting the equations in *augmented matrix form*:

$$\left(\begin{array}{rrr|r} 1 & -2 & 3 & 7 \\ -1 & 1 & -2 & -5 \\ 2 & -1 & -1 & 4 \end{array}\right)$$

which corresponds to our original equations above. The trick now is to eliminate and solve for the x_i. Acting on each equation of the original system is equivalent to acting on a row of the augmented matrix. So we begin by adding the first and second equations together to generate a new second equation, and we also multiply the first equation by -2 and add it to the third equation to generate a new third equation. This gives

$$\left(\begin{array}{rrr|r} 1 & -2 & 3 & 7 \\ 0 & -1 & 1 & 2 \\ 0 & 3 & -7 & -10 \end{array}\right)$$

which can be manipulated further by multiplying the second equation by 3 and adding it to the third

$$\left(\begin{array}{rrr|r} 1 & -2 & 3 & 7 \\ 0 & -1 & 1 & 2 \\ 0 & 0 & -4 & -4 \end{array}\right) .$$

We can then simplify again by dividing the last equation by -4:

$$\left(\begin{array}{rrr|r} 1 & -2 & 3 & 7 \\ 0 & -1 & 1 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right) .$$

This then results in $x_3 = 1$ from the last equation and

$$-x_2 + x_3 = 2 \quad \rightarrow \quad x_2 = -1$$

from the second equation and results in

$$x_1 - 2x_2 + 3x_3 = 7 \quad \rightarrow \quad x_1 = 2$$

from the first equation. In total then, our solution is given by

$$\vec{x} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

which is in our convenient vector notation.

We now move on to a concept which is already familiar to us, that of linear dependence and independence. If we consider a set of vectors added together:

$$c_1\vec{x}_1 + c_2\vec{x}_2 + \cdots + c_n\vec{x}_n = 0$$

Then

- if there exists $c_i \neq 0$ which satisfy this: *linear dependent*
- if the c_i can only be zero: *linear independence*

There is a very simple way to determine whether a set of vectors is linearly independent or not. We can rewrite our equation as:

$$\mathbf{X}\vec{c} = 0 \qquad \text{where} \qquad \mathbf{X} = (\vec{x}_1\ \vec{x}_2\ \cdots\ \vec{x}_n)\ , \begin{pmatrix} c_1 \\ c_2 \\ \cdots \\ c_n \end{pmatrix}$$

But this is an equation of the form $\mathbf{A}\vec{x} = \vec{b}$. And we know that in the homogeneous case ($\vec{b} = 0$), that if the determinant of $\mathbf{X}$ is not zero that $\mathbf{X}$ has an inverse and $\vec{c}$ must be zero. Alternatively, if the determinant is zero, then the c_i are not necessarily zero. This gives us:

- if $\det(\mathbf{X}) = 0$: then $\vec{c} \neq 0$ and *linear dependence*
- if $\det(\mathbf{X}) \neq 0$: then $\vec{c} = 0$ and *linear independence*

Thus all we have to do is calculate the determinant to determine linear dependence or independence.

We now reconsider the equation

$$\mathbf{A}\vec{x} = \vec{b}\,.$$

Suppose that the vector $\vec{b}$ is actually $\vec{x}$ times some constant. That is, what if $\vec{b} = \lambda\vec{x}$, then

$$\mathbf{A}\vec{x} = \lambda\vec{x}$$

which is called an *eigenvalue problem.* This is very interesting as it implies that we act on the vector $\vec{x}$ with some matrix $\mathbf{A}$ and it simply makes $\vec{x}$ shorter or longer depending on λ. The $\vec{x}$ for which this holds is called an *eigenvector* and the corresponding λ is called the *eigenvalue.* We can rearrange the equation to read

$$(\mathbf{A} - \lambda\mathbf{I})\vec{x} = 0$$

for which we know that $\vec{x} = 0$ if the $\det(\mathbf{A} - \lambda\mathbf{I}) \neq 0$. However, we are interested in solutions $\vec{x}$ which are not zero. And the only way this can happen is if

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0\,.$$

For an $n \times n$ system, this condition yields a polynomial of degree n in λ whose roots are the eigenvalues. Recall that a second-order differential equation can be rewritten as two first order equations. Thus the resulting polynomial is of degree two. In fact, the resulting polynomial is the *characteristic equation* we derived in Chapter 3. The important thing then to determine is whether the eigenvalues are real, complex, or perhaps double roots.

Example: Find the eigenvalues and eigenvectors of

$$\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}$$

The eigenvalues are determined from

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0 \quad \rightarrow \quad \det\begin{pmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{pmatrix}$$

which gives the polynomial

$$(1-\lambda)(3-\lambda) + 1 = 0 \quad \rightarrow \quad (\lambda - 2)^2 = 0 \quad \rightarrow \quad \lambda = 2$$

which is a double root. The eigenvectors can be found be recalling that $(\mathbf{A} - \lambda\mathbf{I})\vec{x} = 0$ which gives

$$\begin{pmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{pmatrix}\vec{x} = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}\vec{x} = 0$$

which gives $x_1 = -x_2$. So if $x_1 = c$ where c is a constant then

$$\vec{x}_1 = c\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

is the eigenvector.

Finally, we close by introducing a matrix for which the adjoint is exactly the same as the original matrix:

$$\mathbf{A}^* = \mathbf{A}\,.$$

In this case, the matrix $\mathbf{A}$ is said to be *self-adjoint* or *Hermitian*. The properties of the special matrix are that the eigenvalues are all real, there exist n linearly independent eigenvectors which are orthogonal and for a repeated root with multiplicity m, there are m orthogonal eigenvectors which result. Hermitian matrices are very important and arise in a variety of phenomena such as quantum mechanics and electrodynamics.

6.3. Lec. 3. Systems of Differential Equations

To utilize the linear algebra techniques of the first two lectures, we consider the system of differential equations:

$$\vec{x}' = \mathbf{P}(t)\vec{x} + \vec{g}(t)$$

where $\mathbf{P}(t)$ and $\vec{g}(t)$ are continuous on some interval I. As in previous chapters, $\vec{g}(t) \neq 0$ is the non-homogeneous case and $\vec{g}(t) = 0$ is the homogeneous case. The following theorem can be found for the homogeneous solution.

Theorem: If the vectors $\vec{x}^{(1)}, \vec{x}^{(2)}, ..., \vec{x}^{(n)}$ are linearly independent solutions of the homogeneous problem for all points in I, then each solution

$$\vec{x} = \vec{\phi}(t)$$

can be expressed as a linear combination

$$\vec{\phi} = c_1\vec{x}^{(1)} + c_2\vec{x}^{(2)} + \cdots + c_n\vec{x}^{(n)}$$

in exactly one way.

The $\vec{x}^{(1)}, \vec{x}^{(2)}, ..., \vec{x}^{(n)}$ form the fundamental solution set which make up the general solution $\vec{\phi}(t)$. A couple of things to note about this fundamental solution set. First, we once again define the *Wronskian*:

$$W[\vec{x}^{(1)}, \vec{x}^{(2)}, ..., \vec{x}^{(n)}] = \det \mathbf{X} \neq 0$$

where $\mathbf{X}$ is a matrix whose columns are made up of the $\vec{x}^{(1)}, \vec{x}^{(2)}, ..., \vec{x}^{(n)}$ and whose determinant is not zero for a linearly independent set of solutions. Second, the solution

$$\vec{\phi}(t_0) = c_1\vec{x}^{(1)} + c_2\vec{x}^{(2)} + \cdots + c_n\vec{x}^{(n)} = \mathbf{X}\vec{c} = \vec{b}$$

where t_0 and $\vec{b}$ are some initial conditions that can be inverted to determine the c_i. Since the determinant of $\mathbf{X}$ is not zero, we can find $\mathbf{X}^{-1}$ so that

$$\vec{c} = \mathbf{X}^{-1}\vec{b}$$

and the $\vec{c}$ is then uniquely determined as stated in the above theorem.

We now turn to actually solving such a differential equations system. To simplify, we consider the constant coefficient matrix $\mathbf{A}$ and the governing equation:

$$\vec{x}' = \mathbf{A}\vec{x}\,.$$

We re-introduce the concept of *equilibrium* by noting that this occurs when $\vec{x}' = 0$. This then gives

$$\mathbf{A}\vec{x} = 0 \quad \rightarrow \quad \vec{x} = \mathbf{A}^{-1}0 = 0$$

since $\det \mathbf{A} \neq 0$. Thus the origin is an equilibrium point of the constant coefficient, linear system of equations. The simplest system we can consider is that corresponding to a 1×1 system:

$$(n = 1) \quad x' = ax \quad \rightarrow \quad x = ce^{at}\,.$$

This case is trivial and was extensively considered in our beginning lectures of this class. A slightly more difficult case which leads to nontrivial behavior is that given by a 2×2 system. In this case, we will show that there is a convenient way to explore the dynamics by using *phase-portraits* in a *phase-plane analysis*. We begin, however, by recalling that the 2×2 first-order system can be written as a second order equation. This motivates us to consider guessing solutions of the following form

$$\vec{x}' = \mathbf{A}\vec{x} \quad \rightarrow \quad \vec{x} = \vec{v}e^{\lambda t}$$

so that when using the fact that $\vec{x}' = \lambda\vec{v}e^{\lambda t}$ we find

$$\mathbf{A}\vec{v} = \lambda\vec{v}$$

which is an *eigenvalue problem.* The only way to insure that $\vec{v} \neq 0$ is to require

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0\,.$$

Otherwise, $(\mathbf{A} - \lambda\mathbf{I})$ can be inverted and we find $\vec{v} = 0$. The actual solution techniques for the system of differential equations is more clearly exhibited through some examples.

Example: Solve $\vec{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}\vec{x}$.

This problem is solved by first trying the solution

$$\vec{x} = \vec{v}e^{\lambda t}$$

which yields the eigenvalue problem

$$\begin{pmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{pmatrix}\vec{v} = 0\,.$$

In order to have a nontrivial solution $\vec{v}$, we require that the determinant of the matrix be zero. Thus we find

$$\det\begin{vmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{vmatrix} = (1-\lambda)(1-\lambda) - 4 = \lambda^2 - 2\lambda - 3 = (\lambda-3)(\lambda+1) = 0$$

so that the eigenvalues are

$$\lambda = 3 \qquad \text{and} \qquad \lambda = -1\,.$$

The eigenvectors can then be calculated from the above equations. Thus we find

$$\lambda = 3: \quad \begin{pmatrix} 1-3 & 1 \\ 4 & 1-3 \end{pmatrix}\vec{v} = \begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$

which upon solving gives $-2v_1 + v_2 = 0$ and results in the eigenvector

$$\vec{v}^{(1)} = c_1\begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

The second eigenvector is determined from

$$\lambda = -1: \quad \begin{pmatrix} 1+1 & 1 \\ 4 & 1+1 \end{pmatrix}\vec{v} = \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$

which upon solving gives $2v_1 + v_2 = 0$ and results in the eigenvector

$$\vec{v}^{(2)} = c_2\begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

The general solution is then given by

$$\vec{x} = c_1\begin{pmatrix} 1 \\ 2 \end{pmatrix}e^{3t} + c_2\begin{pmatrix} 1 \\ -2 \end{pmatrix}e^{-t}$$

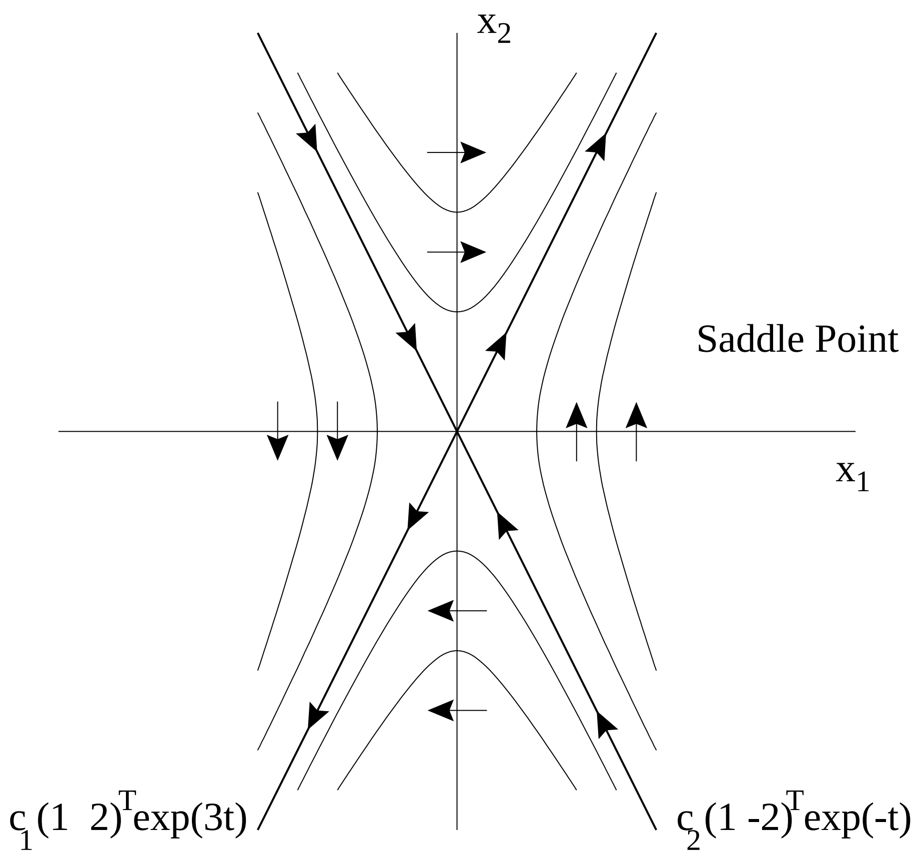

FIG. 7. *Saddle point behavior of solutions when the eigenvalues are real and of opposite sign. Note that the eigenvectors are fundamental in determining the location of growth and decay directions.*

where the constants c_1 and c_2 are determined from the initial conditions. We verify that the Wronskian is not zero by constructing it explicitly for this case. Thus we have

$$W[\vec{x}^{(1)}, \vec{x}^{(2)}] = \begin{pmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{pmatrix} \quad \rightarrow \quad \det W = -4e^{2t} \neq 0$$

and $\vec{x}^{(1)}$ and $\vec{x}^{(2)}$ form a fundamental set of solutions. The behavior of the solution can be conveniently analyzed from the phase-plane portraits which are a graph of x_1 versus x_2 as the solution develops in time. Figure 7 depicts the behavior of the solution and shows the characteristic behavior of the solution along the eigenvectors. Thus along $\vec{v}^{(1)}$ the solution grows like $\exp(3t)$ while along $\vec{v}^{(2)}$ the solution decays like $\exp(-t)$. This behavior, for which there are two real and opposite signed eigenvalues, always generates a *saddle* behavior as depicted in Fig. 7.

Example: Solve $\vec{x}' = \begin{pmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{pmatrix} \vec{x}$.

We once again begin with assuming a solution $\vec{x} = \vec{v}\exp(\lambda t)$ which gives the

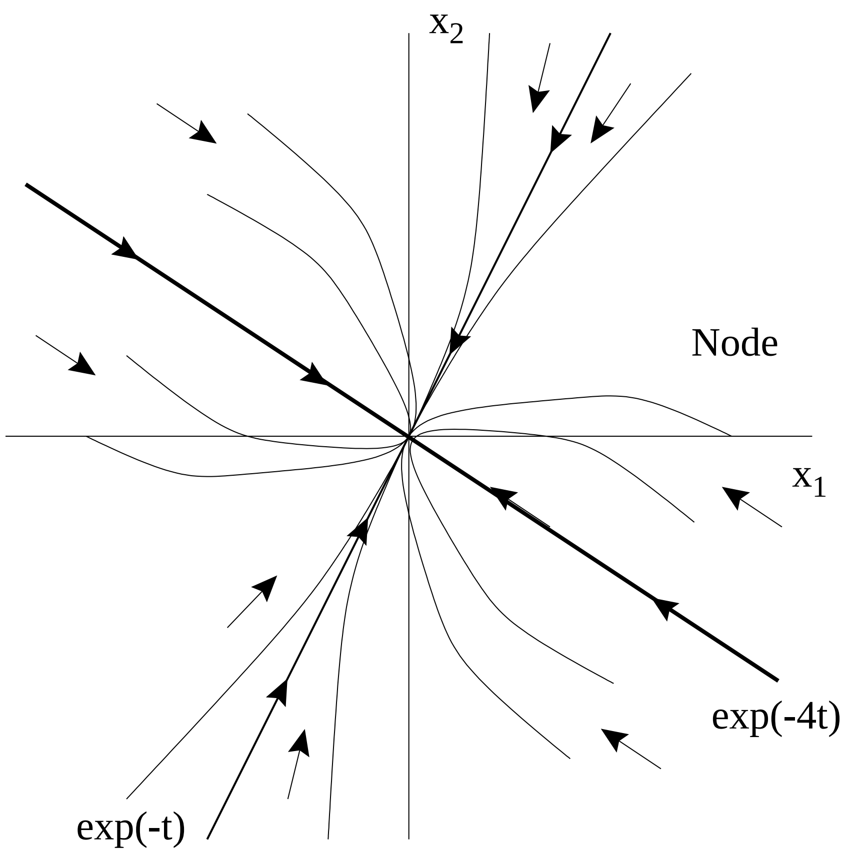

FIG. 8. *Node point behavior of solutions when the eigenvalues are real and of the same sign. Note that the eigenvectors are fundamental in determining the location of growth (decay) directions.*

eigenvalue problem

$$\begin{pmatrix} -3-\lambda & \sqrt{2} \\ \sqrt{2} & -2-\lambda \end{pmatrix} \vec{v} = 0\,.$$

We find the eigenvalues from the determinant of this equation

$$\det \begin{vmatrix} -3-\lambda & \sqrt{2} \\ \sqrt{2} & -2-\lambda \end{vmatrix} = (-3-\lambda)(-2-\lambda) - 2 = \lambda^2 + 5\lambda + 4 = (\lambda+4)(\lambda+1) = 0$$

which gives

$$\lambda = -4 \qquad \text{and} \qquad \lambda = -1\,.$$

The eigenvectors are then found to be

$$\lambda = -4: \qquad \begin{pmatrix} -3+4 & \sqrt{2} \\ \sqrt{2} & -2+4 \end{pmatrix} \vec{v} = \begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$

which upon solving gives $v_1 = -\sqrt{2}$ and $v_2 = 1$ so that the eigenvector is

$$\vec{v}^{(1)} = c_1 \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix}\,.$$

The second eigenvector is determined from

$$\lambda = -1: \qquad \begin{pmatrix} -3+1 & \sqrt{2} \\ \sqrt{2} & -2+1 \end{pmatrix} \vec{v} = \begin{pmatrix} -2 & \sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$

which upon solving gives $v_1 = 1$ and $v_2 = \sqrt{2}$ so that the eigenvector is

$$\vec{v}^{(2)} = c_2 \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}.$$

The general solution is then given by

$$\vec{x} = c_1 \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} e^{-4t} + c_2 \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} e^{-t}$$

where the constants c_1 and c_2 are determined from the initial conditions. So unlike the previous example, the eigenvalues here are both real and both negative. This results in a *node* point at the equilibrium point at the origin. This behavior is depicted in Fig. 8. As before the eigenvectors and eigenvalues are the essential pieces in determining the phase-plane behavior.

6.4. Lec. 4. Complex and Repeated Roots

The examples of the last lecture showed some interesting behavior. And the concept of the phase-plane was important in determining the fundamental nature of the solutions. But just as in the series of lectures of Chapter 3, we would like to know what happens when the roots are complex or double roots. We begin by considering

$$\vec{x}' = \mathbf{A}\vec{x}.$$

Recall that we determine the eigenvalues by letting $\vec{x} = \vec{v}\exp(\lambda t)$ so that

$$\mathbf{A}\vec{v} = \lambda\vec{v} \quad \rightarrow \quad (\mathbf{A} - \lambda\mathbf{I})\vec{v} = 0$$

which gives us the eigenvalues by requiring that det$(\mathbf{A} - \lambda\mathbf{I})$=0.

Suppose that we find an eigenvalue and eigenvector λ_1 and $\vec{v}^{(1)}$ respectively. In addition, let's assume that $\mathbf{A}$ is a *real* matrix so that each component is some real number (i.e. there are no complex numbers). Then we have

$$(\mathbf{A} - \lambda_1\mathbf{I})\vec{v}^{(1)} = 0.$$

Taking the complex conjugate of this equation and recalling that $\mathbf{A}$ was assumed to be real we have

$$(\mathbf{A} - \overline{\lambda_1}\mathbf{I})\overline{\vec{v}^{(1)}} = 0.$$

If we call $\lambda_2 = \overline{\lambda_1}$ and $\vec{v}^{(2)} = \overline{\vec{v}^{(1)}}$, then it is clear that the eigenvalues and corresponding eigenvectors are complex conjugates of each other. Thus we only need to find one eigenvalue and one eigenvector and the second is simply the complex conjugate.

Example: Solve $\vec{x}' = \begin{pmatrix} -1/2 & 1 \\ -1 & -1/2 \end{pmatrix} \vec{x}$.

We begin by letting $\vec{x} = \vec{v}e^{\lambda t}$ so that

$$\begin{pmatrix} -1/2-\lambda & 1 \\ -1 & -1/2-\lambda \end{pmatrix} \vec{v} = 0 \, .$$

Taking the determinant of the above to be zero

$$\det \begin{vmatrix} -1/2-\lambda & 1 \\ -1 & -1/2-\lambda \end{vmatrix} = (-1/2-\lambda)(-1/2-\lambda)+1 = \lambda^2+\lambda+5/4 = 0$$

results in the eigenvalues

$$\lambda_{\pm} = \frac{-1 \pm \sqrt{1-4\cdot(5/4)}}{2} = -\frac{1}{2} \pm i$$

which are a complex conjugate pair. We can find the eigenvector of $\lambda_- = -1/2 - i$ by substituting back in above so that

$$\begin{pmatrix} -1/2-(-1/2-i) & 1 \\ -1 & -1/2-(-1/2-i) \end{pmatrix} \vec{v} = \begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix} \vec{v} = 0 \, .$$

which results in

$$iv_1 + v_2 = 0 \, .$$

If we take $v_1 = 1$, then $v_2 = -i$ so that our eigenvectors are then

$$\vec{v}^{(1)} = \begin{pmatrix} 1 \\ -i \end{pmatrix} \quad \rightarrow \quad \vec{v}^{(2)} = \begin{pmatrix} 1 \\ i \end{pmatrix}$$

so that our general solution is

$$\vec{x}(t) = c_1 \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{(-1/2-i)t} + c_2 \begin{pmatrix} 1 \\ i \end{pmatrix} e^{(-1/2+i)t}$$

where c_1 and c_2 are arbitrary constants.

But recall from Chapter 3 that we decided to write our solution in terms of sines and cosines instead of things like $e^{\mu \pm i\beta}$. Thus we consider one of our fundamental sets of solutions:

$$\vec{x}^{(2)} = \begin{pmatrix} 1 \\ i \end{pmatrix} e^{(-1/2+i)t} = \begin{pmatrix} 1 \\ i \end{pmatrix} e^{-t/2}(\cos t + i \sin t) = e^{-t/2} \left[\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + i \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} \right]$$

which is written in terms of a real and imaginary part. Recall that since the second eigenvector is just a complex conjugate, the only thing that changes is the sign in front of i. Thus we can choose a new basis for our fundamental set of solutions by adding and subtracting the fundamental sets $\vec{x}^{(1)}$ and $\vec{x}^{(2)}$:

$$\vec{X}^{(1)} = \frac{\vec{x}^{(1)} + \vec{x}^{(2)}}{2} = e^{-t/2} \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}$$
$$\vec{X}^{(2)} = \frac{\vec{x}^{(1)} - \vec{x}^{(2)}}{2i} = e^{-t/2} \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} \, .$$

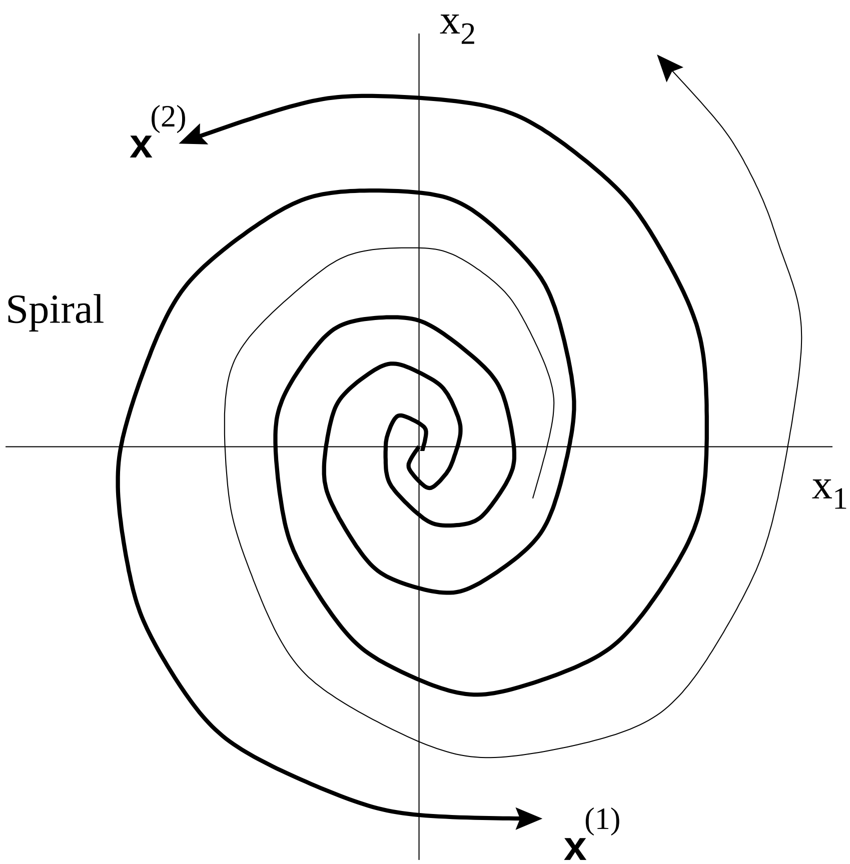

FIG. 9. *Spiral point behavior of solutions when the eigenvalues are complex conjugates.*

We note that we could have gotten this just as easily by simply taking the real and imaginary parts of one of the eigenvectors and using these as the new set of solutions. Notice now that the the solution is purely real as desired:

$$\vec{x} = c_1 \vec{X}^{(1)} + c_2 \vec{X}^{(2)} = c_1 e^{-t/2} \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + c_2 e^{-t/2} \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} .$$

The behavior of the solution represents a *spiral* (also known as a *sink*, or as a *source* if the real part had has a positive part). The characteristic behavior is depicted in Fig. 9 which demonstrates the behavior in the phase-plane.

We also consider in this lecture the case of double roots (repeated roots) for the eigenvalue problem. In this case, we often need to consider what are called *generalized eigenvectors*. This is not always necessary since there are times that a double root will still produce two linearly independent eigenvectors. But in the case that it only produces one, then special care must be taken. An example will serve to illustrate the point.

Example: Solve $\vec{x}' = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \vec{x}$.

We begin by letting $\vec{x} = \vec{v}e^{\lambda t}$ so that

$$\begin{pmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{pmatrix} \vec{v} = 0\,.$$

Taking the determinant of the above to be zero

$$\det \begin{vmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda)+1 = \lambda^2 - 4\lambda + 4 = (\lambda-2)^2 = 0$$

results in the double eigenvalue: $\lambda = 2$. We can find the eigenvector of $\lambda = 2$ by substituting back in above so that

$$\begin{pmatrix} 1-2 & -1 \\ 1 & 3-2 \end{pmatrix} \vec{v} = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \vec{v} = 0\,.$$

which results in

$$v_1 + v_2 = 0\,.$$

If we take $v_1 = 1$, then $v_2 = -1$ so that one of our eigenvectors is

$$\vec{v}^{(1)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}\,.$$

The question arises now about finding a second eigenvector. By following the methods of Chapter 3, we guess a solution of the form

$$\vec{x}^{(2)} = \vec{v}te^{2t} + \vec{\eta}e^{2t}$$

where the vectors $\vec{v}$ and $\vec{\eta}$ are to be determined by plugging into our governing equation. Plugging in we find:

$$2\vec{v}te^{2t} + \vec{v}e^{2t} + 2\vec{\eta}e^{2t} = \mathbf{A}\vec{v}te^{2t} + \mathbf{A}\vec{\eta}e^{2t}$$

which can be simplified to

$$te^{2t}\left(2\vec{v} - \mathbf{A}\vec{v}\right) + e^{2t}\left(\vec{v} + 2\vec{\eta} - \mathbf{A}\vec{\eta}\right) = 0\,.$$

To satisfy the equation, both expressions in parenthesis must be zero. Thus we have from the first expression

$$2\vec{v} - \mathbf{A}\vec{v} = 0 \quad \rightarrow \quad \vec{v} = \vec{v}^{(1)}$$

which gives for the second expression

$$(\mathbf{A} - 2\mathbf{I})\vec{\eta} = \vec{v}^{(1)} \quad \rightarrow \quad \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \vec{\eta} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

whose solution is found by letting

$$\eta_1 + \eta_2 = -1\,.$$

If we let $\eta_1 = k$ then $\eta_2 = -1 - k$ and

$$\vec{\eta} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} + k \begin{pmatrix} 1 \\ -1 \end{pmatrix}\,.$$

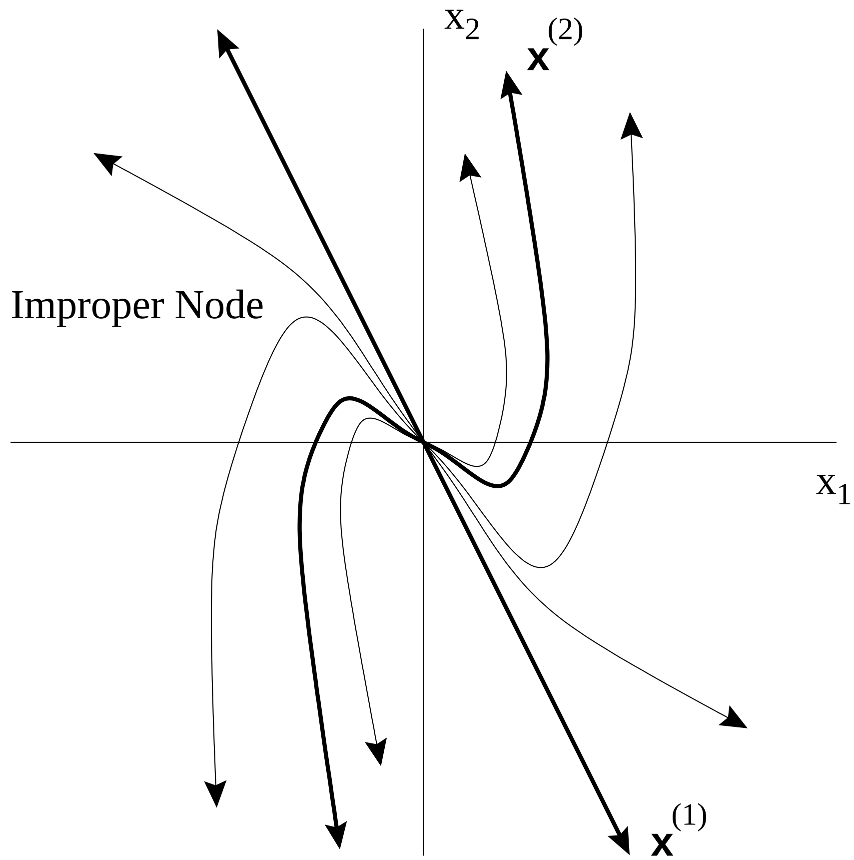

FIG. 10. *Improper node point behavior of solutions when the eigenvalues are repeated (double) roots.*

Since the constant k is arbitrary, we chose it to be zero so that

$$\vec{x}^{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} te^{2t} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{2t}.$$

Our general solution is then

$$\vec{x} = c_1\vec{x}^{(1)} + c_2\vec{x}^{(2)} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t} + c_2 \left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} te^{2t} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{2t} \right]$$

where c_1 and c_2 are arbitrary constants. The corresponding phase-plane behavior generated by this repeated root is shown in Fig. 10. It is called an *improper node*.

6.5. Lec. 5. The Fundamental Matrix and Miscellany

In this lecture, a variety of miscellaneous issues are addressed. We begin by defining the *fundamental matrix*:

$$\psi(t) = \left(\vec{x}^{(1)} \;\; \vec{x}^{(2)} \;\; \cdots \;\; \vec{x}^{(n)}\right)$$

whose columns are simply the fundamental set of solutions of our governing equation so that:

$$\vec{x}' = \mathbf{P}(t)\vec{x} \quad \rightarrow \quad \vec{x} = c_1\vec{x}^{(1)} + c_2\vec{x}^{(2)} + \cdots + c_n\vec{x}^{(n)} .$$

As an example, we can consider one of the examples of Lecture 3:

$$\vec{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \vec{x}$$

which yielded the fundamental set of solutions

$$\vec{x}^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} \qquad \text{and} \qquad \vec{x}^{(2)} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}$$

and which results in the fundamental matrix:

$$\psi(t) = \begin{pmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-2t} \end{pmatrix} .$$

Thus the general solution to a problem can be more compactly written as

$$\vec{x} = c_1\vec{x}^{(1)} + c_2\vec{x}^{(2)} + \cdots + c_n\vec{x}^{(n)} = \psi(t)\vec{c}$$

where $\vec{c}$ contains the arbitrary constants $c_1, c_2, ..., c_n$.

When considering the initial value problem for which

$$\vec{x}(t_0) = \vec{x}_0 ,$$

we find then that

$$\psi(t_0)\vec{c} = \vec{x}_0 .$$

But since the fundamental matrix ψ is composed of n linearly-independent vectors, it has an inverse and we find that the vector $\vec{c}$ is determined to be

$$\vec{c} = \psi^{-1}(t_0)\vec{x}_0 .$$

Thus our solution to the initial problem is simply

$$\vec{x}(t) = \psi(t)\psi^{-1}(t_0)\vec{x}_0 .$$

This solution form indicates that the solution $\vec{x}(t)$ is simply a transformation of the initial conditions $\vec{x}_0$ via the matrix $\psi(t)\psi^{-1}(t_0)$.

Another interesting piece of information concerns the idea of *diagonalization* of a matrix. To do this, we define the matrix:

$$\mathbf{T} = \begin{pmatrix} \vec{v}^{(1)} & \vec{v}^{(2)} & \cdots \vec{v}^{(n)} \end{pmatrix}$$

whose columns are the eigenvectors of a constant matrix **A**. We further let

$$\vec{x} = \mathbf{T}\vec{y}$$

which when inserted into our equation $\vec{x}' = \mathbf{A}\vec{x}$ yields

$$\mathbf{T}\vec{y}' = \mathbf{AT}\vec{y}.$$

Since the matrix $\mathbf{T}$ is composed of n linearly independent columns, it must have an inverse so that we then find

$$\vec{y}' = \mathbf{T}^{-1}\mathbf{AT}\vec{y} \quad \rightarrow \quad \vec{y}' = \mathbf{D}\vec{y}.$$

Since $\mathbf{T}$ is the matrix composed of the eigenvectors, the matrix $\mathbf{D}$ becomes a *diagonal* matrix whose diagonal elements are simply the eigenvalues of $\mathbf{A}$:

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}.$$

The beauty of such a transformation is that the resulting system for $\vec{y}$ is now separable so that each term y_i can be calculated independently of the rest:

$$\begin{aligned} y_1' &= \lambda_1 y_1 \\ y_2' &= \lambda_2 y_2 \\ &\cdots \\ y_n' &= \lambda_n y_n \end{aligned}$$

This is in contrast to our original system $\vec{x}' = \mathbf{A}\vec{x}$ for which all the x_i where coupled together through the matrix $\mathbf{A}$. Once a solution for the y_i are found, then we can find $\vec{x}$ by noting that $\vec{x} = \mathbf{T}\vec{y}$.

To conclude with the miscellany, we note that we can also define exponentiation with matrices. Therefore we let

$$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \frac{1}{2!}\mathbf{A}^2t^2 + \frac{1}{3!}\mathbf{A}^3t^3 + \cdots$$

which is the same definition of the Taylor series of e^{at} where a is some constant. By differentiation, we can then conclude that

$$\frac{d}{dt}e^{\mathbf{A}t} = \mathbf{A}e^{\mathbf{A}t}$$

and further that at $t = 0$, $e^{\mathbf{A}t} = \mathbf{I}$. Thus when considering the differential equation $\vec{x}' = \mathbf{A}\vec{x}$, we can write the general solution as

$$\vec{x} = e^{\mathbf{A}t}\vec{x}_0$$

where $\vec{x}(0) = \vec{x}_0$.

6.6. Lec. 6. Nonhomogeneous Equations

Just as in Chapter 3, we would like to now use our knowledge of solving homogeneous equations to solve the more general nonhomogeneous equation

$$\vec{x}' = \mathbf{P}(t)\vec{x} + \vec{g}(t)$$

where $\vec{g}(t)$ and $\mathbf{P}(t)$ are continuous on some interval I. Just as before, we can write our solution as

$$\vec{x}(t) = c_1\vec{x}^{(1)} + c_2\vec{x}^{(2)} + \cdots + c_n\vec{x}^{(n)} + \vec{x}_p$$

where $\vec{x}_p$ is a particular solution and the $\vec{x}^{(i)}$ are the homogeneous solutions. The question we must then address is how to find the particular solution $\vec{x}_p$.

Method 1: Diagonalization

Recall the matrix transform $\mathbf{T}$ of the last section which we defined as

$$\mathbf{T} = \left(\vec{v}^{(1)} \;\; \vec{v}^{(2)} \;\; \cdots \vec{v}^{(n)}\right)$$

where the $\vec{v}^{(i)}$ are the eigenvectors of $\mathbf{A}$. Defining

$$\vec{x} = \mathbf{T}\vec{y}$$

gives

$$\mathbf{T}\vec{y}' = \mathbf{A}\mathbf{T}\vec{y} + \vec{g}(t)\,.$$

We recall that since $\mathbf{T}$ is made up of the linearly independent eigenvectors of $\mathbf{A}$, it has an inverse. Thus we find the system

$$\vec{y}' = \mathbf{D}\vec{y} + \vec{h}(t)$$

where

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} \qquad \text{and} \qquad \vec{h}(t) = \mathbf{T}^{-1}\vec{g}(t)\,.$$

So then if we write down each term component by component we find

$$\begin{aligned} y_1' &= \lambda_1 y_1 + h_1(t) \\ y_2' &= \lambda_1 y_2 + h_2(t) \\ &\vdots \\ y_n' &= \lambda_1 y_n + h_n(t) \end{aligned}$$

which can be solved component by component with the integrating factor method of Chapter 2. This yields

$$y_i = e^{\lambda_i t}\int_0^t e^{-\lambda_i s} h_i(s) ds + c_i e^{\lambda_i t}$$

which can be used to calculate our solution $\vec{x} = \mathbf{T}\vec{y}$. Note that this technique yields both the particular solution and the homogeneous solutions, i.e. the terms multiplied by the c_is.

Method 2: Undetermined Coefficients

Just as in Chapter 3, this method allows us to guess a solution of the form of the nonhomogeneous term $\vec{g}(t)$ with some arbitrary constant. We plug this guess in and determine the constant to determine our particular solution.

Method 3: Variation of Parameters

Another technique familiar to us from Chapter 3 is the variation of parameters. Recall that we can write the homogeneous solution as a product of the fundamental matrix and a constant matrix:

$$\vec{x} = \psi(t)\vec{c}$$

Just as in the variation of parameters method of Chapter 3, we can attempt a solution of the form

$$\vec{x} = \psi(t)\vec{u}(t)$$

where we replace the constant $\vec{c}$ by some time-dependent vector $\vec{u}(t)$. Plugging this into our original equation $\vec{x}' = \mathbf{P}(t)\vec{x} + \vec{g}(t)$ yields

$$\psi'(t)\vec{u}(t) + \psi(t)\vec{u}'(t) = \mathbf{P}(t)\psi(t)\vec{u}(t) + \vec{g}(t)\,.$$

But since ψ is the fundamental matrix, it satisfies $\psi' = \mathbf{P}(t)\psi$ since every column is a solution of the homogeneous equation. We are then left with

$$\psi(t)\vec{u}'(t) = \vec{g}(t)$$

which can be solved for the vector $\vec{u}'(t)$. Or alternatively, since ψ has an inverse, we have

$$\vec{u}' = \psi^{-1}\vec{g} \quad \rightarrow \quad \vec{u}(t) = \int \psi^{-1}(t)\vec{g}(t)dt + \vec{c}\,.$$

Finally, we find the general solution by recalling

$$\vec{x} = \psi\vec{u} = \psi\vec{c} + \psi\int \psi^{-1}(t)\vec{g}(t)dt$$

which gives the particular and homogeneous solution together. The use of the three above methods is demonstrated in the following example.

Example: Solve $\vec{x}' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}\vec{x} + \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix}$.

We begin by solving the homogeneous problem for which $\vec{g}(t) = 0$. Letting $\vec{x} = \vec{v}e^{\lambda t}$ gives

$$\begin{pmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{pmatrix}\vec{v} = 0\,.$$

Taking the determinant of the above to be zero

$$\det\begin{vmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{vmatrix} = (-2-\lambda)(-2-\lambda) - 1 = \lambda^2 + 4\lambda + 3 = (\lambda+3)(\lambda+1) = 0$$

results in the eigenvalues $\lambda_1 = -3$ and $\lambda_2 = -1$. The eigenvector corresponding to $\lambda = -3$ is found from

$$\begin{pmatrix} -2+3 & 1 \\ 1 & -2+3 \end{pmatrix} \vec{v} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \vec{v} = 0 \, .$$

which results in

$$v_1 + v_2 = 0 \, .$$

If we take $v_1 = 1$, then $v_2 = -1$ so that the eigenvector is

$$\vec{v}^{(1)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} .$$

The eigenvector corresponding to $\lambda = -1$ is found from

$$\begin{pmatrix} -2+1 & 1 \\ 1 & -2+1 \end{pmatrix} \vec{v} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \vec{v} = 0 \, .$$

which results in

$$v_1 - v_2 = 0 \, .$$

If we take $v_1 = 1$, then $v_2 = 1$ so that the eigenvector is

$$\vec{v}^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} .$$

Our homogeneous solution is then

$$\vec{x}(t) = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}$$

where c_1 and c_2 are the arbitrary constants.

The particular solution to the nonhomogeneous equation is then found by each of the three methods discussed above. First, we consider the *diagonalization method.* In this case, we can construct the transform matrix is given by

$$\mathbf{T} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

where the eigenvectors $\vec{v}^{(1)}$ and $\vec{v}^{(2)}$ have been normalized to unity, i.e. the magnitudes are $(\vec{v}^{(i)}, \vec{v}^{(i)}) = 1$. Since the matrix $\mathbf{A}$ is Hermitian, that is, it is real and symmetric, then we have that

$$\mathbf{T}^{-1} = \mathbf{T}^{*} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} .$$

Applying the transformation $\vec{x} = \mathbf{T}\vec{y}$ yields

$$\vec{y}' = \begin{pmatrix} -3 & 0 \\ 0 & -1 \end{pmatrix} \vec{y} + \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix}$$

which can be easily solved for y_1 and y_2:

$$y_1' + 3y_1 = \frac{2}{\sqrt{2}}e^{-t} - \frac{3}{\sqrt{2}}t$$
$$y_2' + y_2 = \frac{2}{\sqrt{2}}e^{-t} + \frac{3}{\sqrt{2}}t\,.$$

Both equations can be solved using the integrating factor method of Chapter 2. This yields

$$y_1 = \frac{\sqrt{2}}{2}e^{-t} - \frac{3}{\sqrt{2}}\left(\frac{t}{3} - \frac{1}{9}\right) + c_1 e^{-3t}$$
$$y_2 = \sqrt{2}te^{-t} + \frac{3}{\sqrt{2}}(t-1) + c_2 e^{-t}\,.$$

Our solution $\vec{x} = \mathbf{T}\vec{y}$ can then be calculated

$$\vec{x}(t) = k_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t} + k_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{-t} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} t - \frac{1}{3}\begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

where k_1 and k_2 are arbitrary constants.

We can also use the *method of undetermined coefficients.* We begin by noting that

$$\vec{g}(t) = \begin{pmatrix} 2 \\ 0 \end{pmatrix} e^{-t} + \begin{pmatrix} 0 \\ 3 \end{pmatrix} t\,.$$

This motivates us to guess a solution of the form

$$\vec{x}_p = \vec{a}te^{-t} + \vec{b}e^{-t} + \vec{c}t + \vec{d}$$

where the first two terms are guessed since e^{-t} is already a solution to the homogeneous solution. Thus we need to consider the more general form above just as in Chapter 3. We plug this into our governing equations and solve for the four constant vectors $\vec{a}, \vec{b}, \vec{c}$, and $\vec{d}$. We thus find

$$\vec{x_p} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{-t} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-t} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} t - \frac{1}{3}\begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

which we note is slightly different than the previous particular solution. That is okay, remember that the particular solution is *not unique*, only the final solution with a set of given initial conditions is.

Finally we solve for the particular solution using *variation of parameters.* Our fundamental matrix is simply given by

$$\psi(t) = \begin{pmatrix} \vec{x}^{(1)} & \vec{x}^{(2)} \end{pmatrix} \begin{pmatrix} e^{-3t} & e^{-t} \\ -e^{-3t} & e^{-t} \end{pmatrix}$$

which when inserted into the expression for $\vec{u}'$ gives

$$\psi(t)\vec{u}' = \vec{g} \quad \rightarrow \quad \begin{pmatrix} e^{-3t} & e^{-t} \\ -e^{-3t} & e^{-t} \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix}\,.$$

The individual components can then be solved by noting

$$u_1' = e^{2t} - \frac{3}{2}te^{3t}$$
$$u_2' = 1 + \frac{3}{2}te^{t}$$

which can be solved to give

$$u_1 = \frac{1}{2}e^{2t} - \frac{1}{2}te^{3t} + \frac{1}{6}e^{3t} + c_1$$
$$u_2 = t + \frac{3}{2}te^{t} - \frac{3}{2}e^{t} + c_2$$

and since $\vec{x} = \psi\vec{u}$ we have

$$\vec{x}(t) = c_1\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{-3t} + c_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{-t} + \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{-t} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}te^{-t} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}t - \frac{1}{3}\begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

where c_1 and c_2 are arbitrary constants.

7. Chapter 7. Nonlinear Systems of Equations

We are finally at a point where we can start to study systems which are much more realistic in nature: *Nonlinear Systems*. The methods we will utilize in describing such systems, which take the form:

$$\frac{dx}{dt} = F(x, y, t)$$
$$\frac{dy}{dt} = G(x, y, t)$$

with $F(x, y, t)$ and $G(x, y, t)$ being some nonlinear functions of x and y, are qualitative rather than quantitative. Further, they rely on understanding two primary parts of the problem:

- Equilibrium Points (also known as *Critical Points*)
- Stability of Critical Points

Provided these are two issues are understood, we can develop a variety of approaches to solving very complicated problems. It should be noted that although our analytic techniques provide a tremendous amount of insight into the problem, quantitative understanding comes largely through numerical simulations of the governing equations. However, this is beyond the scope of this course.

7.1. Lec. 1. Introduction to Nonlinear Systems

We begin this Chapter by reviewing the primary conclusions of the last Chapter concerning eigenvalues and eigenvectors. Thus we consider the linear system:

$$\vec{x}' = \mathbf{A}\vec{x}\,.$$

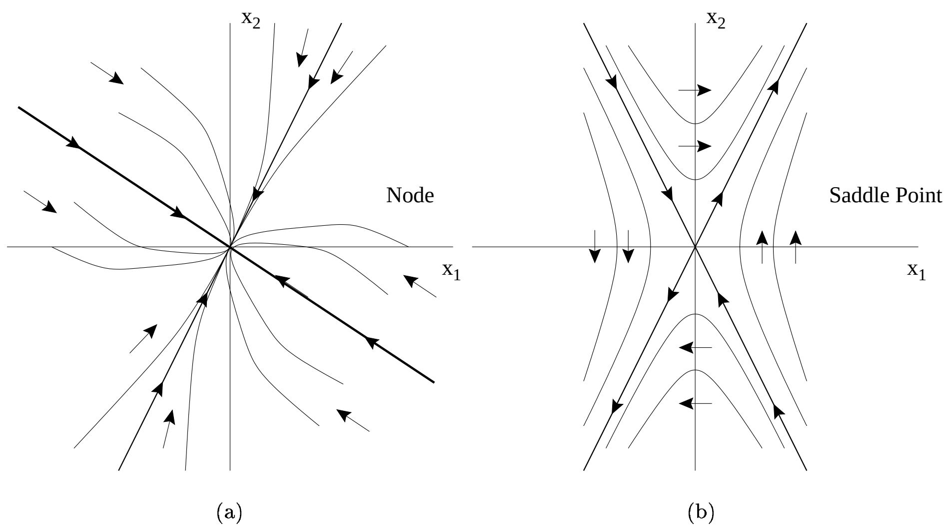

FIG. 11. *Behavior of Node (a) and Saddle (b) which are determined by having real distinct eigenvalues which are of the same sign (a) or of opposite sign (b). Changing the signs of the eigenvalues simply changes the directions of the arrows. Thus the Node can be stable or unstable whereas the Saddle is always unstable.*

The equilibrium points of this system are determined by setting $\vec{x}' = 0$. This then yields

$$\mathbf{A}\vec{x} = 0 \quad \rightarrow \quad \vec{x} = 0$$

since we want to assume that $\mathbf{A}$ has an inverse. The dynamics about the equilibrium point $\vec{x} = 0$ can then be found by solving for the eigenvalues and eigenvectors:

$$\vec{x} = \vec{v}e^{\lambda t} \quad \rightarrow \quad (\mathbf{A} - \lambda\mathbf{I})\vec{v} = 0\,.$$

Thus once the eigenvalues and eigenvectors are determined, the phase-plane trajectories can be drawn and the solution behavior understood. There are actually five distinct cases of behavior which may arise. They are as follows:

Case 1: eigenvalues – real, unequal, same sign

In this case, once the eigenvalues and eigenvectors are found we can express the solution as

$$\vec{x} = c_1\vec{v}^{(1)}e^{\lambda_1 t} + c_2\vec{v}^{(2)}e^{\lambda_2 t}$$

where we have assumed that λ_1 and λ_2 are different and real. The prototypical behavior of this case is depicted in Fig. 11a. In this figure, we assumed that $\lambda_1 < \lambda_2 < 0$ so that decay occurs most rapidly along the eigenvector $\vec{v}^{(1)}$. This is also called a *node* or *nodal sink*. Note that all trajectories go to zero so that the critical point is *stable*. If instead we found that $\lambda_1 > \lambda_2 > 0$, then the direction of the arrow in Fig. 11a would be reversed and the critical point (called a *nodal source*) would be *unstable*.

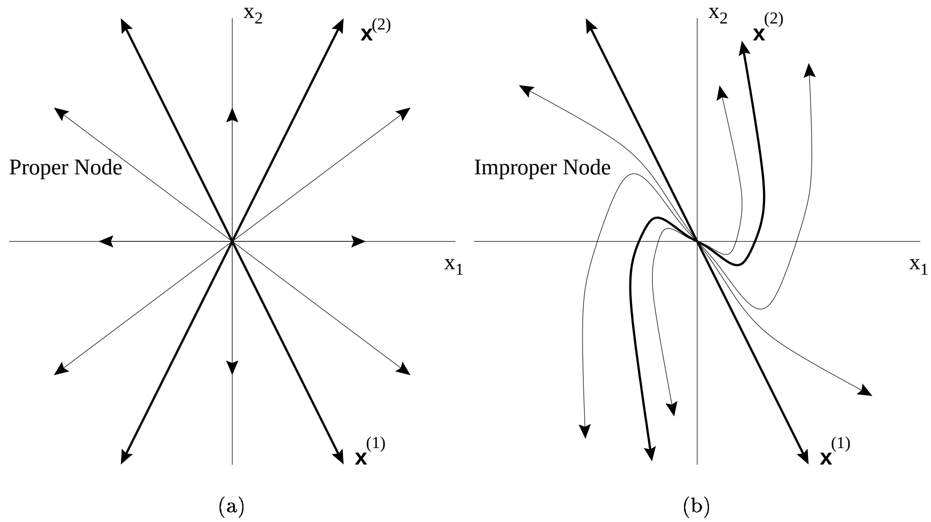

FIG. 12. *Proper (a) and Improper (b) Nodes corresponding to the double root case with two independent eigenvectors (a) or one eigenvector and a second generalized eigenvector (b). The nodes can be stable or unstable depending on the sign of the eigenvalue, i.e. the arrows are switched with a switch of the sign of the eigenvalue.*

Case 2: eigenvalues – real, opposite sign

If the eigenvalues are of opposite sign and real we can write the general solution

$$\vec{x} = c_1 \vec{v}^{(1)} e^{\lambda_1 t} + c_2 \vec{v}^{(2)} e^{-\lambda_2 t}$$

where we have assumed that λ_1 and λ_2 are real and of the same sign. The prototypical behavior of this case is depicted in Fig. 11b. In this figure, we assumed that $\lambda_1, \lambda_2 > 0$ so that growth occurs along the eigenvector $\vec{v}^{(1)}$ and decay along $\vec{v}^{(2)}$. This is called a *saddle point*. Note that all trajectories eventually go out to infinity along the eigenvector $\vec{v}^{(1)}$. This implies that saddle points are always unstable. If instead we found that $\lambda_1, \lambda_2 < 0$, then the direction of the arrow in Fig. 11b would be reversed and solutions would go unstable along $\vec{v}^{(2)}$.

Case 3: eigenvalues – real and equal (double root)

For the case of a double root, two possibilities exist: either we can find two linearly independent eigenvectors so that our solution is

$$\vec{x} = c_1 \vec{v}^{(1)} e^{\lambda_1 t} + c_2 \vec{v}^{(2)} e^{\lambda_1 t}$$

or, there is only one eigenvector, and we must generate a generalized eigenvector via the methods of the last Chapter so that our solution takes the form

$$\vec{x} = c_1 \vec{v}^{(1)} e^{\lambda_1 t} + c_2 \left[\vec{v}^{(1)} t e^{\lambda_1 t} + \vec{\eta} e^{\lambda_1 t} \right] .$$

In the first case, a *proper node* (or *star point*) is generated as depicted in Fig. 12a for $\lambda_1 > 0$. The second case generates an *improper node* which is depicted in Fig. 12b,

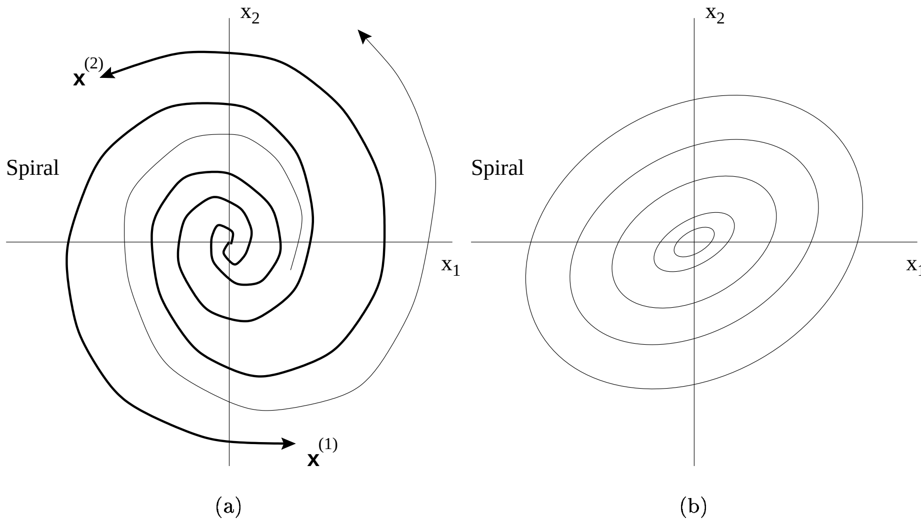

FIG. 13. *Spiral (a) and Center (b) behavior when the eigenvalues are complex conjugates. The spiral has a nontrivial real part which determines whether the trajectories spiral in or out in (a) whereas the center has strictly periodic behavior.*

also for the case of $\lambda_1 > 0$. Both points are *stable* for $\lambda_1 < 0$, whereas for $\lambda_1 > 0$, the arrows are reversed in Figs. 12a and 12b and the critical point is *unstable*.

Case 4: eigenvalues – complex eigenvalues

For complex roots, we know that the eigenvalues and eigenvectors come in complex pairs. In particular, the eigenvalues are given by

$$\lambda_{\pm} = \beta \pm i\mu \, .$$

The resulting behavior is a *spiral* where the stability is strictly determined from the real part β. For $\beta > 0$ the solutions spiral outward as in Fig. 13a so that the equilibrium is unstable. For $\beta < 0$, the solutions spiral inward (reverse the arrows in Fig. 13a) so that the the equilibrium point is stable.

Case 5: eigenvalues – purely imaginary

In the case of purely imaginary eigenvalues:

$$\lambda_{\pm} = \pm i\mu$$

our solutions are completely oscillatory and we have a *neutrally stable* situation with a *center* critical point. This behavior is demonstrated in Fig. 13b where generically, the solution trajectories are *ellipses*. Thus solutions neither grow or decay: they simply display periodic motion.

Now that the behavior in linear systems is completely categorized, let's move on to consider a classical problem which is actually *nonlinear*. Such is the case of the pendulum. The schematic of the pendulum is shown in Fig. 14. By conservation of

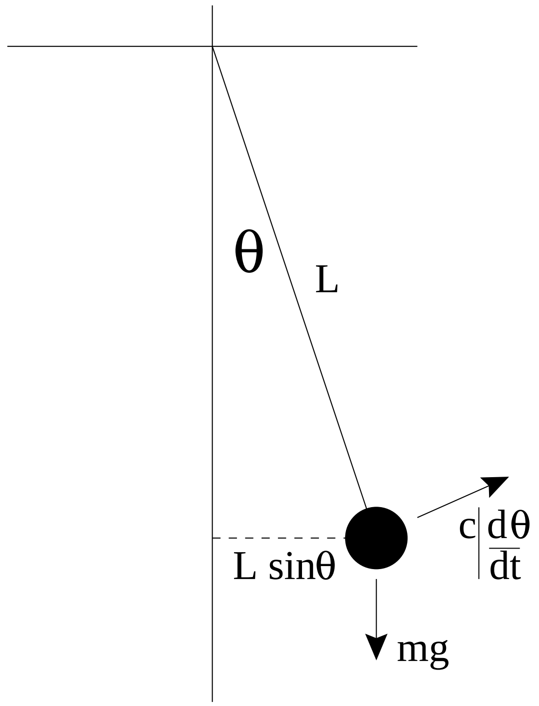

FIG. 14. *Schematic of pendulum oscillating from a fixed support subject to forces of gravity and damping.*

angular momentum, we find that the swing of the pendulum can be described by the equation:

$$mL^2\frac{d^2\Theta}{dt^2} = -cL\frac{d\Theta}{dt} - mgL\sin\Theta$$

where m is the pendulum mass, L is the length, g is the acceleration due to gravity, and c measures the frictional/damping forces acting on the pendulum. We can rewrite this equation as

$$\Theta'' + \gamma\Theta' + \omega^2\sin\Theta = 0$$

where $\gamma = c/mL$ and $\omega^2 = g/L$.

To convert this into a system of equations, we define

$$x = \Theta \qquad \text{and} \qquad y = \frac{d\Theta}{dt}$$

which then results in the *nonlinear* system

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} y \\ -\omega^2\sin x - \gamma y \end{pmatrix}.$$

Equilibrium solutions are found by letting $x' = y' = 0$ which yields

$$y = 0 \qquad \text{and} \qquad \sin x = 0$$

so that the critical points are

$$y = 0 \qquad \text{and} \qquad x = \pm n\pi \quad n = 0, 1, 2, \ldots .$$

Thus unlike our linear systems above, we have more than one equilibrium point. In fact, we have an infinity of them which lie at multiples of π along the the x-axis. We will learn how to deal with this in the next lecture.

7.2. Lec. 2. The Pendulum and Perturbation Theory

In this section, we consider one of the classical problems of physics: the pendulum. Up to now, you have probably understood the pendulum problem as a case of simple harmonic motion which was introduced in physics. However, you may recall that in this behavior was only *approximate*, i.e. it relied on the pendulum swings being rather small. Here we consider the fully *nonlinear* pendulum dynamics. Our equations of motion were given in the last lecture as:

$$\begin{aligned} x' &= y \\ y' &= -\omega^2 \sin x - \gamma y \,. \end{aligned}$$

To simplify the analysis, we begin by considering the case in which there is no damping so that $\gamma = 0$ and our governing equations are

$$\begin{aligned} x' &= y \\ y' &= -\omega^2 \sin x \,. \end{aligned}$$

As in the last lecture, the key now is to find the critical points and their stability. The critical points (equilibrium) are determined for $x' = y' = 0$ so that

$$y = 0 \qquad \text{and} \qquad x = \pm n\pi \;\; n = 0, 1, 2, \ldots$$

which were given in the last lecture. The idea now is to look very close to one of the equilibrium points using the ideas of *perturbation theory.* Therefore we let

$$\begin{aligned} x &= \pm n\pi + \tilde{x} \\ y &= 0 + \tilde{y} \end{aligned}$$

where $\tilde{x}$ and $\tilde{y}$ are both very small. Thus this implies we are very near one of the fixed points. Plugging this into our governing equations for no damping yields the system:

$$\begin{aligned} \tilde{x}' &= \tilde{y} \\ \tilde{y}' &= -\omega^2 \sin(\pm n\pi + \tilde{x}) \,. \end{aligned}$$

To begin, we consider the well known example of the pendulum which oscillates about the equilibrium $x = 0$ (which corresponds to $\Theta = 0$). In this case $n = 0$ so that we have

$$\sin(\pm n\pi + \tilde{x}) = \sin \tilde{x} = \tilde{x} - \frac{\tilde{x}^3}{3!} + \frac{\tilde{x}^5}{5!} + \cdots \approx \tilde{x}$$

where we have used the Taylor series representation of sine and approximated everything by $\tilde{x}$ since all the other terms are much smaller (provided, of course, that $\tilde{x}$ is very small). This is the standard trick that is used in introductory physics in order to turn the nonlinear system into a linear one. In particular, if we plug this result into the above equation we preceding equation we find.

$$\begin{aligned}\tilde{x}' &= \tilde{y}\\ \tilde{y}' &= -\omega^2\tilde{x}\,.\end{aligned}$$

which results in the *linear* system:

$$\vec{x}' = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}\vec{x}$$

where $\vec{x} = (\tilde{x}\ \ \tilde{y})^T$. We learned how to solve this system in the last Chapter and in the first lecture of this Chapter. Thus we let $\vec{x} = \vec{v}e^{\lambda t}$ which yields the eigenvalue problem

$$\begin{pmatrix} -\lambda & 1 \\ -\omega^2 & -\lambda \end{pmatrix}\vec{v} = 0\,.$$

The eigenvalues are found by taking the determinant of the above matrix to be zero. This then yields

$$\lambda^2 + \omega^2 = 0 \quad \rightarrow \quad \lambda = \pm i\omega$$

which are purely imaginary eigenvalues. Thus the equilibrium point for $n = 0$, i.e. $(x, y) = (0, 0)$ is a center. Thus solutions near the critical point are all elliptic trajectories.

More generally, we can consider all the equilibrium points that are multiples of 2π away from the origin $(x, y) = (0, 0)$. Thus we consider the perturbation theory for these points:

$$\begin{aligned}x &= \pm 2n\pi + \tilde{x}\\ y &= 0 + \tilde{y}\end{aligned}$$

where $\tilde{x}$ and $\tilde{y}$ are both very small. This implies we are very near one of the fixed points located at multiples of 2π from the origin. Plugging this into our governing equations for no damping now yields the system:

$$\begin{aligned}\tilde{x}' &= \tilde{y}\\ \tilde{y}' &= -\omega^2\sin(\pm 2n\pi + \tilde{x})\,.\end{aligned}$$

But since

$$\sin(\pm 2n\pi + \tilde{x}) = \sin\tilde{x} \approx \tilde{x}\,,$$

we then arrive at exactly the same linearized equations as before. Therefore, we can conclude that *all* the equilibrium points in multiples of 2π from the origin are centers with periodic solutions near each critical points (see Fig. 15).

This is not the case for critical points which are odd multiples of π from the origin. For these we can perturb around each critical point by letting

$$\begin{aligned}x &= \pm 2n\pi + \pi + \tilde{x}\\ y &= 0 + \tilde{y}\end{aligned}$$

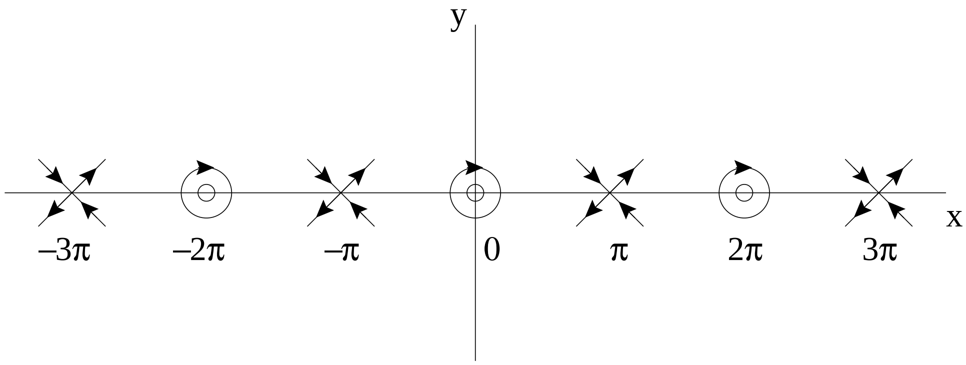

FIG. 15. *Behavior of solutions near each of the fixed points which are multiples of* π *from the origin. Note that multiples of* 2π *produce centers while multiples of odd* π *are saddles.*

where $\tilde{x}$ and $\tilde{y}$ are both very small. This implies we are very near one of the fixed points located at odd multiples of π from the origin (i.e. $\pm\pi, \pm 3\pi, ...$). Plugging this into our governing equations for no damping now yields the system:

$$\begin{aligned} \tilde{x}' &= \tilde{y} \\ \tilde{y}' &= -\omega^2 \sin(\pm 2n\pi + \pi + \tilde{x}) . \end{aligned}$$

But since

$$\sin(\pm 2n\pi + \pi + \tilde{x}) = \sin(\pi + \tilde{x}) = \sin\pi \cos\tilde{x} + \cos\pi \sin\tilde{x} = -\sin\tilde{x} \approx -\tilde{x} ,$$

we then arrive at a slightly different set of linearized equations. In matrix form, this can be written as

$$\vec{x}' = \begin{pmatrix} 0 & 1 \\ \omega^2 & 0 \end{pmatrix} \vec{x}$$

where the only difference now is in the sign of ω^2. Letting $\vec{x} = \vec{v}e^{\lambda t}$ yields the eigenvalue problem

$$\begin{pmatrix} -\lambda & 1 \\ \omega^2 & -\lambda \end{pmatrix} \vec{v} = 0 .$$

whose eigenvalues are

$$\lambda^2 - \omega^2 = 0 \quad \rightarrow \quad \lambda = \pm\omega$$

which are purely real eigenvalues or opposite sign. Thus the equilibrium point for odd multiples of π are saddles. The eigenvectors can then be found:

$$\begin{aligned} \lambda = \omega : &\quad \begin{pmatrix} -\omega & 1 \\ \omega^2 & -\omega \end{pmatrix} \vec{v} = 0 \;\rightarrow\; -\omega v_1 + v_2 = 0 \;\rightarrow\; \vec{v}^{(1)} = \begin{pmatrix} 1 \\ \omega \end{pmatrix} \\ \lambda = -\omega : &\quad \begin{pmatrix} \omega & 1 \\ \omega^2 & \omega \end{pmatrix} \vec{v} = 0 \;\rightarrow\; \omega v_1 + v_2 = 0 \;\rightarrow\; \vec{v}^{(2)} = \begin{pmatrix} 1 \\ -\omega \end{pmatrix} . \end{aligned}$$

Thus a complete description of the saddle is given near each critical point in odd multiples of π from the origin. The resulting dynamics is depicted in Fig. 15 which shows the results of our perturbation calculations locally near each of the fixed points.

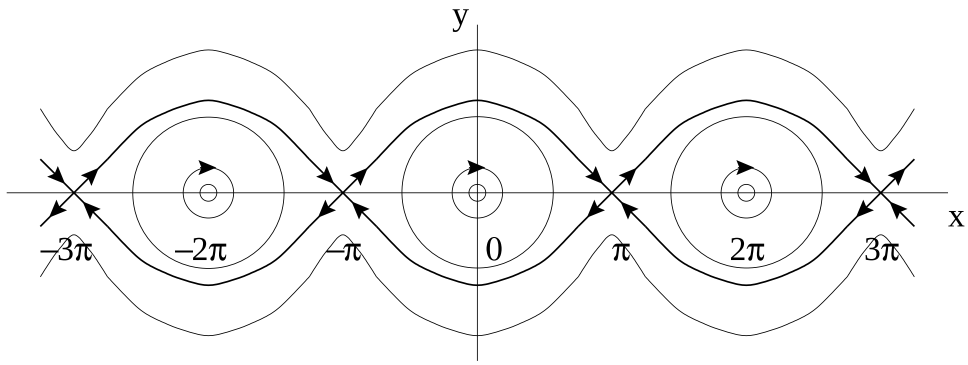

FIG. 16. *Full nonlinear behavior of solutions near each of the fixed points and beyond. This behavior is deducted directly from Fig. 15 and is the only consistent behavior since all the critical points are on the line $y = 0$. The bolded lines are the* separatrix *which separate the oscillatory behavior from the behavior above it corresponding to a pendulum which continuously rotates around.*

Although the perturbation results are insightful, their full power is not realized until we generalize our thinking of Fig. 15. In particular, since trajectories cannot intersect, we can utilize the picture in Fig. 15 to develop a full qualitative understanding of the dynamics. Specifically, we can describe the behavior far from the critical points by their behavior near the critical points. In Fig. 16, we develop the full nonlinear qualitative behavior by simply taking the phase-plane picture and generalizing it in the only way possible. This results in a dynamical picture which makes a great deal of sense. Note that near the critical points in multiples of 2π (which corresponds to the rest position of the pendulum), the behavior is exactly as expected: oscillatory. Whereas for odd π values (which corresponds to a pendulum sticking straight up), the behavior is given by a saddle and is unstable. Note that the trajectory separating the oscillatory behavior from the trajectories above it is called the *separatrix*. The separatrix projects along the unstable eigenvector of one saddle into the stable eigenvector of a neighboring eigenvector. The behavior above this corresponds to the undamped pendulum swinging around and around its support. This is the case if we give it a strong enough initial speed. And since there is no damping in this model, it will continue to circle around and around and will never fall into the oscillatory back and forth motion predicted near the center equilibrium.

We now generalize our treatment in order to treat the case of the damped pendulum. Recall that the system in this case is given by

$$\begin{aligned} x' &= y \\ y' &= -\omega^2 \sin x - \gamma y \,. \end{aligned}$$

As in the undamped case, the key now is to find the critical points and their stability. The critical points (equilibrium) are determined for $x' = y' = 0$ so that

$$y = 0 \qquad \text{and} \qquad x = \pm n\pi \;\; n = 0, 1, 2, ...$$

exactly as in the undamped case. We once again perturb about these equilibrium points to determine stability. Therefore we let

$$\begin{aligned} x &= \pm n\pi + \tilde{x} \\ y &= 0 + \tilde{y} \end{aligned}$$

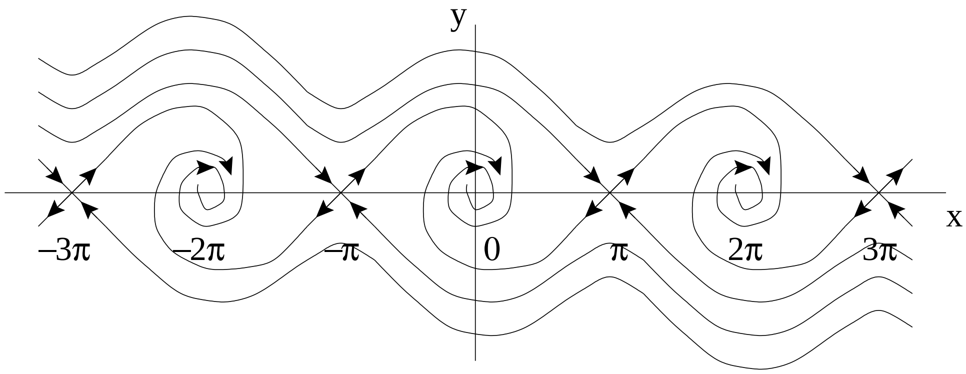

FIG. 17. *Full nonlinear behavior of solutions near each of the fixed points when under-damping is applied to the pendulum. There are no separatrix in this case.*

where $\tilde{x}$ and $\tilde{y}$ are both very small. Plugging this into our damped equations yields the system:

$$\tilde{x}' = \tilde{y}$$
$$\tilde{y}' = -\omega^2 \sin(\pm n\pi + \tilde{x}) - \gamma\tilde{y}\,.$$

As in the undamped pendulum case, there are two interesting cases to consider. The first is when the critical point is at the origin or at multiples of 2π from it. Thus we have

$$\sin(\pm 2n\pi + \tilde{x}) = \sin\tilde{x} \approx \tilde{x}$$

where we have again approximated sine by $\tilde{x}$ since it is small. Plugging this result into the linear damped equation above results in the system:

$$\vec{x}' = \begin{pmatrix} 0 & 1 \\ -\omega^2 & -\gamma \end{pmatrix} \vec{x}$$

where $\vec{x} = (\tilde{x}\ \ \tilde{y})^T$. Letting $\vec{x} = \vec{v}e^{\lambda t}$ yields the eigenvalue problem

$$\begin{pmatrix} -\lambda & 1 \\ -\omega^2 & -\gamma - \lambda \end{pmatrix} \vec{v} = 0\,.$$

The eigenvalues are found from the determinant to be

$$-\lambda(-\gamma - \lambda) + \omega^2 = \lambda^2 + \gamma\lambda + \omega^2 = 0 \quad \rightarrow \quad \lambda = -\frac{\gamma}{2} \pm i\sqrt{\omega^2 - \frac{\gamma^2}{4}}\,.$$

So depending on the quantity $\omega^2 - \gamma^2/4$, the equilibrium is either a spiral ($\omega^2 > \gamma^2/4$), an improper node ($\omega^2 = \gamma^2/4$), or a node ($\omega^2 < \gamma^2/4$). The three different cases are referred to as underdamped, critically damped, and overdamped respectively. In any case, the real part of the eigenvalue is negative so that the equilibrium point is asymptotically stable. In what we depict in Fig. 17, we assume that $\omega^2 > \gamma^2/4$ so that the equilibrium points at multiples of 2π are all spirals.

When the critical points are at odd multiples of π from the origin, we once again have

$$\sin(\pm 2n\pi + \pi + \tilde{x}) = \sin(\pi + \tilde{x}) = -\sin\tilde{x} \approx -\tilde{x}$$

Plugging this result into the linear damped equation results in the system:

$$\vec{x}' = \begin{pmatrix} 0 & 1 \\ \omega^2 & -\gamma \end{pmatrix} \vec{x}$$

where $\vec{x} = (\tilde{x} \;\; \tilde{y})^T$. Letting $\vec{x} = \vec{v}e^{\lambda t}$ yields the eigenvalue problem

$$\begin{pmatrix} -\lambda & 1 \\ \omega^2 & -\gamma - \lambda \end{pmatrix} \vec{v} = 0 \, .$$

whose eigenvalues are found from the determinant to be

$$-\lambda(-\gamma - \lambda) - \omega^2 = \lambda^2 + \gamma\lambda - \omega^2 = 0 \quad \rightarrow \quad \lambda = -\frac{\gamma}{2} \pm \sqrt{\omega^2 + \frac{\gamma^2}{4}} \, .$$

This yields two real eigenvalues which are of opposite sign. Thus a saddle is once again generated for all equilibrium points which are odd multiples of π from the origin. The complete nonlinear dynamics is depicted in Fig. 17 where the interaction of the spirals and saddle nodes is shown. Note how in this case, the solutions eventually end up in one of the spiral points.

7.3. Lec. 3. Predator-Prey Models

We now have enough background material to develop a more general theory and understanding of nonlinear systems. There are two key concepts in nonlinear systems that determine all the resulting dynamics. The two primary issues are:

- Equilibrium (*Critical Points*)
- Stability

Both of these concepts, which were mentioned at the introduction of this chapter, are rather intuitive in nature and have been illustrated in the previous two lectures.

We begin by considering the following general system of equations

$$\begin{aligned} x' &= F(x, y, t) \\ y' &= G(x, y, t) \, , \end{aligned}$$

where $F(x, y, t)$ and $G(x, y, t)$ are some general functions of x, y, and time t. We will simplify this for the present by considering the *autonomous* system:

$$\begin{aligned} x' &= F(x, y) \\ y' &= G(x, y) \, , \end{aligned}$$

where F and G are not explicitly time dependent.

We begin to analyze this system by considering the concept of equilibrium. Equilibrium occurs when there is no "motion" in the system, i.e. when both $x' = 0$ and $y' = 0$. The point at which this occurs is the equilibrium point (x_0, y_0) which satisfies:

$$\begin{aligned} F(x_0, y_0) &= 0 \\ G(x_0, y_0) &= 0 \, , \end{aligned}$$

since $x' = y' = 0$. This is all there is to equilibrium. We simply find the points (there may be more than one) which satisfy the above equations simultaneously. Once this is done, the behavior of the system can be determined entirely from the stability of each equilibrium (critical) point.

The stability of each critical point may be determined by looking very near each individual point. Thus we assume that

$$x = x_0 + \tilde{x}$$
$$y = y_0 + \tilde{y}\,,$$

where $\tilde{x}$ and $\tilde{y}$ are both very small so that they can be considered to be in a very small neighborhood of the critical point. Plugging this into our original equations gives us

$$\tilde{x}' = F(x_0 + \tilde{x}, y_0 + \tilde{y})$$
$$\tilde{y}' = G(x_0 + \tilde{x}, y_0 + \tilde{y})\,,$$

where we recall that since x_0 and y_0 are constants then $x_0' = y_0' = 0$. The key now is to remember our Taylor expansion formula from the series chapter. Thus to expand about some point, we have

$$f(x_0 + \tilde{x}) = f(x_0) + \tilde{x}f'(x_0) + \frac{\tilde{x}^2}{2!}f''(x_0) + \frac{\tilde{x}^3}{3!}f'''(x_0) + \cdots\,.$$

Keeping only the first few terms in this approximation is good provided $\tilde{x}$ is small.

In the full problem, we now have to expand about both x_0 and y_0. Doing so yields the following:

$$\tilde{x}' = F(x_0, y_0) + \tilde{x}F_x(x_0, y_0) + \tilde{y}F_y(x_0, y_0) + \cdots$$
$$\tilde{y}' = G(x_0, y_0) + \tilde{x}G_x(x_0, y_0) + \tilde{y}G_y(x_0, y_0) + \cdots$$

where we have neglected all terms which are smaller than $\tilde{x}^2$, $\tilde{y}^2$, and $\tilde{x}\tilde{y}$. In matrix form, this *linearized* system can be written as:

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix}' = \begin{pmatrix} F_x(x_0, y_0) & F_y(x_0, y_0) \\ G_x(x_0, y_0) & G_y(x_0, y_0) \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix}.$$

We call this a *linearized* system since we turned the original *nonlinear* system into a *linear* system near the critical points. The methods developed in the previous chapter and reviewed in the introductory lecture of this chapter are now applicable. Thus we simply need to determine the eigenvalues of the above system and the resulting global, i.e. nonlinear, dynamics can be understood qualitatively.

To make use of these ideas, we turn to some specific examples to help illustrate the key ideas. We begin by considering what are called *predator–prey models*. These models consider the interaction of two species: predators and their prey. It should be obvious that such species will have significant impact on one another. In particular, if there is an abundance of prey, then the predator population will grow due to the surplus of food. Alternatively, if the prey population is low, then the predators may die off due to starvation.

To model the interaction between these species, we begin by considering the predators and prey in the absence of any interaction. Thus the prey population (denoted by $x(t)$) is governed by

$$\frac{dx}{dt} = ax$$

where $a > 0$ is a net growth constant. The solution to this simple differential equation is $x(t) = x(0)\exp(at)$ so that the population grows without bound. We have assumed here that the food supply is essentially unlimited for the prey so that the unlimited growth makes sense since there is nothing to kill off the population.

Likewise, the predators can be modeled in the absence of their prey. In this case, the population (denoted by $y(t)$) is governed by

$$\frac{dy}{dt} = -cy$$

where $c > 0$ is a net decay constant. The reason for the decay is that the population basically starves off since there is no food (prey) to eat.

We now try to model the interaction. Essentially, the interaction must account for the fact the the predators eat the prey. Such an interaction term can result in the following system:

$$\begin{aligned} \frac{dx}{dt} &= ax - \alpha xy \\ \frac{dy}{dt} &= -cx + \alpha xy \end{aligned}$$

where $\alpha > 0$ is the interaction constant. Note that *alpha* acts as a decay to the prey population since the predators will eat them, and as a growth term to the predators since they now have a food supply. These nonlinear and autonomous equations are known as the *Lotka–Volterra equations.*

We rely on the methods introduced in this lecture to study this system. In particular, we consider the equilibrium points and their associated stability in order to determine the qualitative dynamics of the Lotka-Volterra equations.

The critical points are determined by setting $x' = y' = 0$ which gives

$$\begin{aligned} &ax - \alpha xy = x(a - \alpha y) = 0 \\ &-cy + \alpha xy = y(\alpha x - c) = 0\,. \end{aligned}$$

This gives two possible fixed points

$$\begin{aligned} &\text{I. } x = 0 \text{ and } y = 0 \\ &\text{II. } x = c/\alpha \text{ and } y = a/\alpha\,. \end{aligned}$$

Each of these fixed points needs to be investigated separately in order to determine the full (qualitative) nonlinear dynamics.

We begin with the critical point I. $(x_0, y_0) = (0, 0)$. Following the methods outlined above we calculate the following:

$$\begin{array}{llll} F(x,y) = ax - \alpha xy & \longrightarrow & F_x = a - \alpha y & \longrightarrow & F_x(0,0) = a \\ & & F_y = -\alpha x & & F_y(0,0) = 0 \\ \\ G(x,y) = -cy + \alpha xy & \longrightarrow & G_x = \alpha y & \longrightarrow & G_x(0,0) = 0 \\ & & G_y = -c + \alpha x & & G_y(0,0) = -c\,. \end{array}$$

The resulting linearized system is

$$\vec{w}' = \begin{pmatrix} a & 0 \\ 0 & -c \end{pmatrix} \vec{w} = 0$$

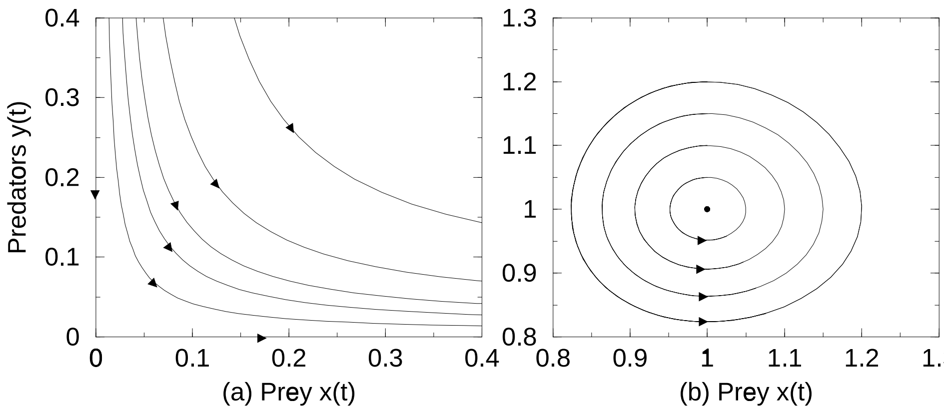

FIG. 18. *Behavior near each of the fixed points. Here we have assumed that* $a = c = \alpha = 1$ *so that the two critical points are the saddle at* $(0,0)$ *and the center at* $(1,1)$.

where $\vec{w} = (\tilde{x}\ \tilde{y})^T$. As with all previous linear systems, we make the substitution $\vec{w} = \vec{v}\exp(\lambda t)$ in order to yield the eigenvalue problem:

$$\begin{pmatrix} a-\lambda & 0 \\ 0 & -c-\lambda \end{pmatrix} \vec{v} = 0\,.$$

Setting the determinant to zero gives the characteristic equation

$$(a-\lambda)(-c-\lambda) = 0$$

whose eigenvalues are

$$\lambda = a \ \text{ and } \ \lambda = -c$$

Thus the eigenvalues are real and of opposite sign giving us a saddle at the critical point $(0,0)$.

The eigenvectors can also be easily determined for this case. They are as follows:

$$\lambda = a: \ \begin{pmatrix} a-a & 0 \\ 0 & -c-a \end{pmatrix} \vec{v} = \begin{pmatrix} 0 & 0 \\ 0 & -(c+a) \end{pmatrix} \vec{v} = 0 \ \longrightarrow \ \vec{v}^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and

$$\lambda = -c: \ \begin{pmatrix} a-c & 0 \\ 0 & -c+c \end{pmatrix} \vec{v} = \begin{pmatrix} a-c & 0 \\ 0 & 0 \end{pmatrix} \vec{v} = 0 \ \longrightarrow \ \vec{v}^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The behavior near the critical point at the origin in thus completely determined. Figure 18a depicts the saddle behavior near the origin.

We now consider critical point II. $(x_0, y_0) = (c/\alpha, a/\alpha)$. Following the previous calculation we find:

$$F_x(c/\alpha, a/\alpha) = 0,\ F_y(c/\alpha, a/\alpha) = -c,\ G_x(c/\alpha, a/\alpha) = a,\ G_y(c/\alpha, a/\alpha) = 0\,.$$

The resulting linearized system is

$$\vec{w}' = \begin{pmatrix} 0 & -c \\ a & 0 \end{pmatrix} \vec{w} = 0\,.$$

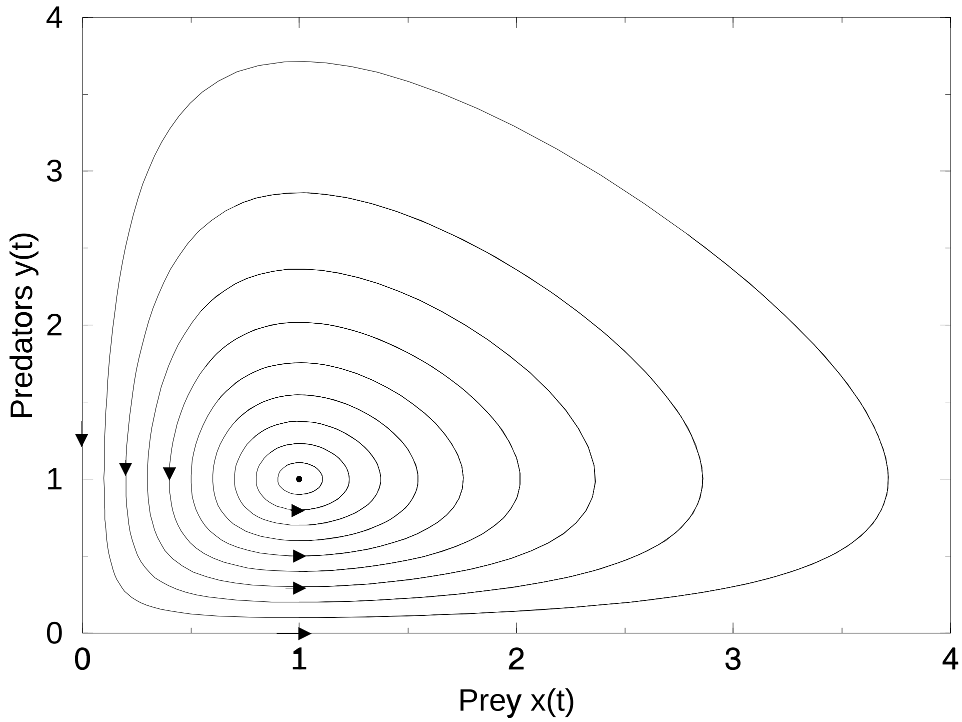

FIG. 19. *Fully nonlinear behavior of the predator–prey system. As with Fig. 18, we have assumed that $a = c = \alpha = 1$ so that the two critical points are the saddle at $(0,0)$ and the center at $(1,1)$. Note the periodic behavior between these two fixed points.*

Making the substitution $\vec{w} = \vec{v}\exp(\lambda t)$ yields the eigenvalue problem:

$$\begin{pmatrix} -\lambda & -c \\ a & -\lambda \end{pmatrix} \vec{v} = 0\,.$$

Setting the determinant to zero gives the characteristic equation

$$\lambda^2 + ac = 0$$

whose eigenvalues are

$$\lambda_\pm = \pm i\sqrt{ac}$$

Thus the eigenvalues are imaginary giving a center at the critical point $(c/\alpha, a/\alpha)$. Before calculating the eigenvectors for this case, we note that the periodic behavior goes counter–clockwise in order to be consistent with the flow of the critical point I. The behavior near the critical point is depicted in Fig. 18b. The full nonlinear dynamics is depicted in Fig. 19 which shows the saddle behavior near critical point I. and periodic motion around the critical point II.

To get a better idea of the periodic motion, we can calculate the eigenvectors associated with critical point II.

$$\lambda = i\sqrt{ac}: \begin{pmatrix} -i\sqrt{ac} & -c \\ a & -i\sqrt{ac} \end{pmatrix} \vec{v} = 0 \longrightarrow -i\sqrt{ac}v_1 - cv_2 = 0 \longrightarrow \vec{v}^{(1)} = \begin{pmatrix} \sqrt{ac} \\ -ia \end{pmatrix}.$$

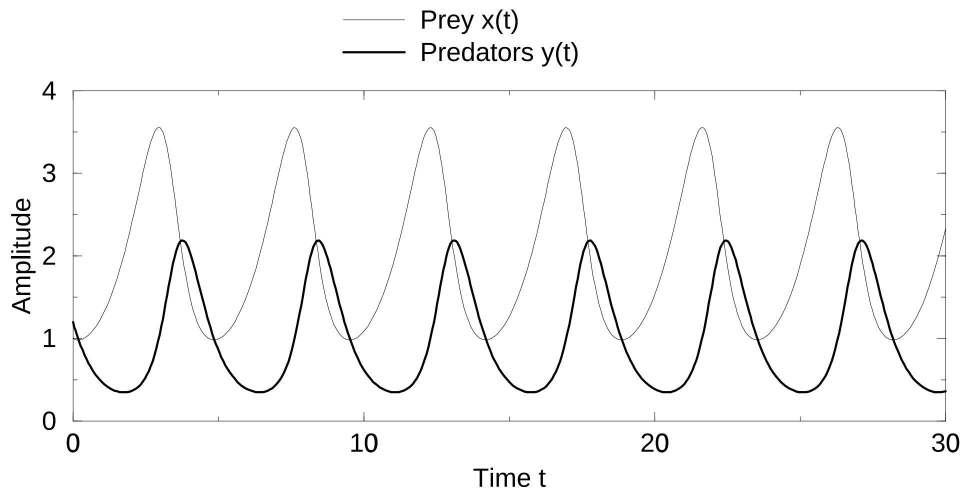

FIG. 20. *Fully nonlinear behavior of the predator–prey system as a function of time. Here* $a = \alpha = 1$ *and* $c = 2$. *Note the* $\pi/2$ *lag between the solutions.*

Rewriting the eigenvector then gives

$$\vec{w}^{(1)} = \begin{pmatrix} \sqrt{ac} \\ -ia \end{pmatrix} \exp\left(i\sqrt{ac}t\right) = \begin{pmatrix} \sqrt{ac}\cos\sqrt{ac}t \\ a\sin\sqrt{ac}t \end{pmatrix} + i \begin{pmatrix} \sqrt{ac}\sin\sqrt{ac}t \\ -a\cos\sqrt{ac}t \end{pmatrix}.$$

Rewriting our solution in terms of a purely real solution then is easily done by combining the real and imaginary parts to form

$$\vec{w} = c_1 \begin{pmatrix} \sqrt{ac}\cos\sqrt{ac}t \\ a\sin\sqrt{ac}t \end{pmatrix} + c_2 \begin{pmatrix} \sqrt{ac}\sin\sqrt{ac}t \\ -a\cos\sqrt{ac}t \end{pmatrix}.$$

Thus the population of predators ($y(t)$) and prey ($x(t)$) can be calculated explicitly:

$$\begin{aligned} x(t) &= \frac{c}{\alpha} + c_1\sqrt{ac}\cos\sqrt{ac}t + c_2\sqrt{ac}\sin\sqrt{ac}t \\ y(t) &= \frac{a}{\alpha} + c_1 a\sin\sqrt{ac}t - c_2 a\cos\sqrt{ac}t\,. \end{aligned}$$

To simplify this further, we can replace the constants c_1 and c_2 by the two new constants K and ϕ so that

$$\begin{aligned} x(t) &= \frac{c}{\alpha} + \frac{c}{\alpha}K\cos(\sqrt{ac}t + \phi) \\ y(t) &= \frac{a}{\alpha} + \frac{a}{\alpha}\sqrt{\frac{c}{a}}K\sin(\sqrt{ac}t + \phi)\,. \end{aligned}$$

This representation allows us to see explicitly the fact that the populations are $\pi/2$ out of phase. Thus the reaction to changes of population occur (near the critical point) one quarter of a cycle out of phase. This behavior is demonstrated in Fig. 20.

7.4. Lec. 4. Limit Cycles and Periodic Solutions

We now move on to consider an important class of solutions that arise in many physical systems. These are known as periodic solutions, and they arise often as a *limit cycle* of a given system of differential equations. These concepts will be made more clear as we proceed with the lecture. To begin, we consider the autonomous differential equation

$$\frac{d\vec{x}}{dt} = f(\vec{x}) \, .$$

Periodic solutions of this equation are such that

$$\vec{x}(t) = \vec{x}(t+T) = \vec{x}(t+NT)$$

where T is the period and $N = 1, 2, 3, \cdots$. From our understanding of systems of equations so far, we can conjecture two things about periodic solutions:

- They are closed curves in the phase–plane
- Their eigenvalues must be purely imaginary

The fact that the solution is a closed curve is obvious from the fact that the solution must return to itself in order to be periodic. The eigenvalues must also be purely imaginary since a real part would cause the solutions to either grow to infinity or shrink to zero. The dynamics of periodic solutions and limit cycles is best illustrated through an example.

Example: Solve $\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} y + x - x(x^2+y^2) \\ -x + y - y(x^2+y^2) \end{pmatrix}$.

We begin as in the previous sections by looking for fixed points of the right–hand side. Thus we look for roots of the system:

$$\begin{aligned} y + x - x(x^2+y^2) &= 0 \\ -x + y - y(x^2+y^2) &= 0 \end{aligned}$$

which yields a single fixed point

$$x_0 = y_0 = 0$$

at the origin. We linearize about the origin to determine its stability, therefore we calculate

$$\begin{array}{lcl} F_x = 1 - 3x^2 - y^2 & & F_x = 1 \\ F_y = 1 - 2xy & \text{at } (0,0) & F_y = 1 \\ G_x = -1 - 2xy & & G_x = -1 \\ G_y = 1 - 3y^2 - x^2 & & G_y = 1 \end{array}$$

which yields the linearized system

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix}' = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix}$$

where $x = 0 + \tilde{x}$ and $y = 0 + \tilde{y}$. We then let $\vec{w} = \vec{v}\exp(\lambda t)$ which yields the eigenvalue problem:

$$\begin{pmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{pmatrix} \vec{v} = 0\,.$$

Taking the determinant gives the characteristic equation

$$(1-\lambda)(1-\lambda) + 1 = \lambda^2 - 2\lambda + 2 = 0$$

whose roots are the complex conjugate pair:

$$\lambda_{\pm} = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i\,.$$

Thus the origin is a spiral which is unstable since the real part of the eigenvalue is real.

It is tempting to conclude that we now understand the full nonlinear dynamics from the stability near the origin. However, by looking carefully at the problem, we see some very interesting behavior which occurs when $x^2 + y^2 = 1$. We rewrite the original system to elucidate this fact:

$$\begin{aligned} x' &= y + x(1 - (x^2 + y^2)) \\ y' &= -x + y(1 - (x^2 + y^2))\,. \end{aligned}$$

There are three distinct cases we now consider.

Case 1: $x^2 + y^2 > 1$ and $1 - (x^2 + y^2) = -a$

In this case, the system reduces to

$$\begin{aligned} x' &= y - ax \\ y' &= -x - ay \end{aligned}$$

which is a linear system. Making the substitution $\vec{x} = \vec{v}\exp(\lambda t)$ yields the eigenvalue problem:

$$\begin{pmatrix} -a-\lambda & 1 \\ -1 & -a-\lambda \end{pmatrix} \vec{v} = 0$$

whose eigenvalue are given by the determinant being zero:

$$(-a-\lambda)(-a-\lambda) + 1 = 0 \;\rightarrow\; \lambda_{\pm} = -a \pm i\,.$$

Since the real part is negative ($a > 0$), the solution for $x^2 + y^2 > 1$ is a decaying spiral.

Case 2: $x^2 + y^2 < 1$ and $1 - (x^2 + y^2) = a$

In this case, the system reduces to

$$\begin{aligned} x' &= y + ax \\ y' &= -x + ay \end{aligned}$$

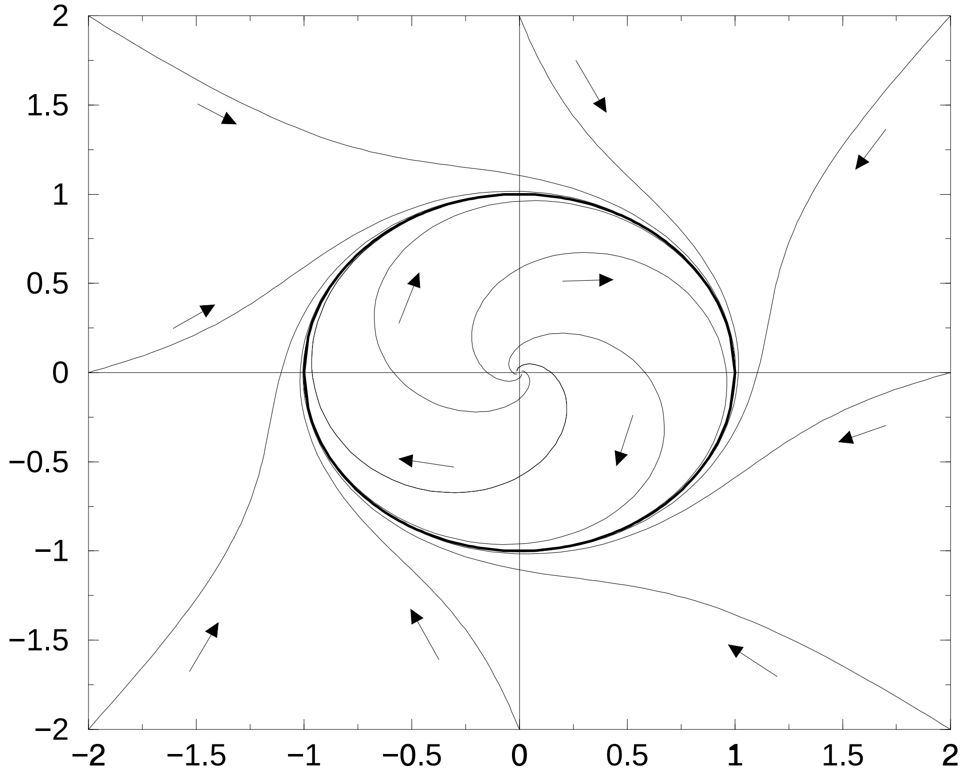

FIG. 21. *Fully nonlinear behavior of the limit cycle system as a function of time. Note that solutions inside the circle $x^2 + y^2 = 1$ spiral out, while those outside the circle spiral in. So for large time all solutions collapse to the limit cycle which is the unit circle.*

which is a linear system. Making the substitution $\vec{x} = \vec{v}\exp(\lambda t)$ yields the eigenvalue problem:

$$\begin{pmatrix} a-\lambda & 1 \\ -1 & a-\lambda \end{pmatrix} \vec{v} = 0$$

whose eigenvalue are given by the determinant being zero:

$$(a-\lambda)(a-\lambda) + 1 = 0 \;\rightarrow\; \lambda_\pm = a \pm i\,.$$

Since the real part is now positive ($a > 0$), the solution for $x^2 + y^2 < 1$ is a growing spiral.

Case 3: $x^2 + y^2 = 1$ and $1 - (x^2 + y^2) = 0$

In this case, the system reduces to

$$\begin{aligned} x' &= y \\ y' &= -x \end{aligned}$$

which is a simple linear system. Making the substitution $\vec{x} = \vec{v}\exp(\lambda t)$ yields the eigenvalue problem:

$$\begin{pmatrix} -\lambda & 1 \\ -1 & -\lambda \end{pmatrix} \vec{v} = 0$$

whose eigenvalue are given by the determinant being zero:

$$\lambda^2 + 1 = 0 \rightarrow \lambda_{\pm} = \pm i \, .$$

Since the eigenvalues are now purely imaginary, the solution for $x^2 + y^2 = 1$ is periodic.

The equation of the circle

$$x^2 + y^2 = 1$$

thus plays an important role in the full nonlinear dynamics. In particular, solutions which are inside the circle grow, and those outside decay. Those exactly on the circle are periodic solutions which neither grow or decay. The circle with radius one is known as a *limit cycle* since all solutions approach it as time goes to infinity. The periodic solution $x^2 + y^2 = 1$ is also called asymptotically, or orbitally, stable. It could have also been unstable or semi–stable as in previous chapters.

Figure 21 demonstrates the behavior of the full nonlinear system. Note that as predicted, the solutions outside of the unit circle spiral inwards, while those inside spiral outwards. Thus the unit circle is the orbitally stable (asymptotically stable) limit cycle of the system. So although there is only a single fixed point at the origin, it is not enough to determine the full nonlinear dynamics since the nonlinearity acts to create a stable, attracting periodic solution which is not a fixed point of the original system.

The question which naturally arises with this chapter is the following: How do we know if there is going to be a limit cycle? This can be a difficult question to answer in general, but here are some some theorems which can be helpful in answering this question provided F and G, i.e. the right-hand forcings, have continuous first partial derivative in some domain D.

I. A closed (periodic) trajectory must enclose one critical point which is not a saddle.

II. If $F_x + G_y$ has the same sign in a simply connected domain (i.e. no holes in it), there is no closed trajectory.

III. Given some region R that contains no critical points, if there exists some time t_0 for which if $t \geq t_0$ and $x = \phi(t)$ and $y = \psi(t)$ stays in R, then $x = \phi(t)$ and $y = \psi(t)$ is either periodic or goes to a limit cycle.

By thinking a little bit about the limit cycle idea, it is obvious that condition **I** must hold. In our example, our critical point which was enclosed by the limit cycle was an unstable spiral. Condition **II** is not quite so obvious. However, we note that from our example that

$$F_x + G_y = 1 - 3x^2 - y^2 + 1 - 3y^2 - x^2 = 2 - 4(x^2 + y^2) \, .$$

So if we chose a domain for which $x^2 + y^2 < 1/2$, then there is no limit cycle within. This is consistent with our results. If a larger domain is considered whose boundaries are both inside and outside $x^2 + y^2 = 1/2$, then the sign of $F_x + G_y$ changes sign and there is the possibility of a closed trajectory. The last condition **III** is also an intuitively obvious idea. For our example, we could have considered a region R which was around the limit cycle $x^2 + y^2 = 1$. Trajectories which entered this region would never leave, thus the existence of the periodic solution would be ensured.

As a last note, the most famous example of limit cycle behavior arises in the VanderPol equation

$$x'' - \mu(1 - x^2)x' + x = 0$$

which is a model for the current in a triode oscillator. Note that for $x^2 > 1$, the first derivative acts as a damping term, whereas for $x^2 < 1$ this same term causes growth of the solution. Thus the existence of a limit cycle can be conjectured since it is qualitatively similar to the example we have just considered.

7.5. Lec. 5. Chaos and Strange Attractors: Lorenz Equations

Up to this point, we have only been considering second–order equations for which we have a coupled set of first order equations. However, it is natural to ask what can happen in third–order and higher systems. We will consider a particular third–order system which was derived by Edward Lorenz is 1963 as a meteorological model for the motion of the earth's atmosphere. Through a variety of simplifications, Lorenz came up with the following equations (The *Lorenz Equations*):

$$\begin{aligned} x' &= \sigma(-x + y) \\ y' &= rx - y - xz \\ z' &= -bz + xy \end{aligned}$$

where x measures the intensity of the fluid motion and y and z describe the temperature variation in the horizontal and vertical directions respectively. The constants σ, r, and b are all real and positive and depend upon the material properties of the atmosphere. Reasonable values can be calculated for our atmosphere for $\sigma = 10$ and $b = 8/3$. The object then is to see how the solutions change with the parameter r which relates to the temperature difference between layers of the atmosphere.

Our standard techniques for two coupled first–order systems holds here. That is, we once again search for equilibrium solutions and consider their stability. Equilibrium is achieved when

$$x' = y' = z' = 0$$

which gives

$$\begin{aligned} 0 &= \sigma(-x + y) \\ 0 &= rx - y - xz \\ 0 &= -bz + xy\,. \end{aligned}$$

From the first relation we find that $x = y$ so that the remaining two equations are

$$\begin{aligned} &x(r - 1 - z) = 0 \\ &bz = x^2\,. \end{aligned}$$

This yields three distinct equilibrium points (x_0, y_0, z_0) which are given by

$$\begin{aligned}
I. &\quad (x_0, y_0, z_0) = (0, 0, 0) \\
II. &\quad (x_0, y_0, z_0) = (\sqrt{b(r-1)}, \sqrt{b(r-1)}, r-1) \\
III. &\quad (x_0, y_0, z_0) = (-\sqrt{b(r-1)}, -\sqrt{b(r-1)}, r-1)
\end{aligned}$$

which actually gives three points provided $r > 1$ and only the single point I for $r < 1$ since we require real solutions.

To determine the dynamics of the system, we proceed to linearize about each equilibrium point and consider the associated eigenvalues. This process is very much like the generic process introduced two lectures back, i.e. we consider the system

$$\begin{aligned}
x' &= F(x, y, z) \\
y' &= G(x, y, z) \\
z' &= H(x, y, z)
\end{aligned}$$

and linearize about an equilibrium solution (x_0, y_0, z_0) so that

$$\begin{aligned}
x &= x_0 + \tilde{x} \\
y &= y_0 + \tilde{y} \\
z &= z_0 + \tilde{z}\,.
\end{aligned}$$

The resulting linearized system is then

$$\begin{pmatrix} \tilde{x}' \\ \tilde{y}' \\ \tilde{z}' \end{pmatrix} = \begin{pmatrix} F_x(x_0, y_0, z_0) & F_y(x_0, y_0, z_0) & F_z(x_0, y_0, z_0) \\ G_x(x_0, y_0, z_0) & G_y(x_0, y_0, z_0) & G_z(x_0, y_0, z_0) \\ H_x(x_0, y_0, z_0) & H_y(x_0, y_0, z_0) & H_z(x_0, y_0, z_0) \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix}.$$

For our particular case we find that

$$\begin{aligned}
&F_x = -\sigma, F_y = \sigma, F_z = 0 \\
&G_x = r - z, G_y = -1, G_z = -x \\
&H_x = y, H_y = x, H_z = -b
\end{aligned}$$

and we can now evaluate the stability of each equilibrium point.

We begin by considering the equilibrium point at the origin where $(x_0, y_0, z_0) = (0, 0, 0)$. In this case, the linearized system becomes:

$$\vec{x}' = \begin{pmatrix} -\sigma & \sigma & 0 \\ r & -1 & 0 \\ 0 & 0 & -b \end{pmatrix} \vec{x}$$

where $\vec{x} = (\tilde{x}\ \tilde{y}\ \tilde{z})^T$. We can generate the appropriate eigenvalue problem by letting $\vec{x} = \exp(\lambda t)\vec{v}$ so that

$$\begin{pmatrix} -\sigma - \lambda & \sigma & 0 \\ r & -1-\lambda & 0 \\ 0 & 0 & -b-\lambda \end{pmatrix} \vec{v} = 0\,.$$

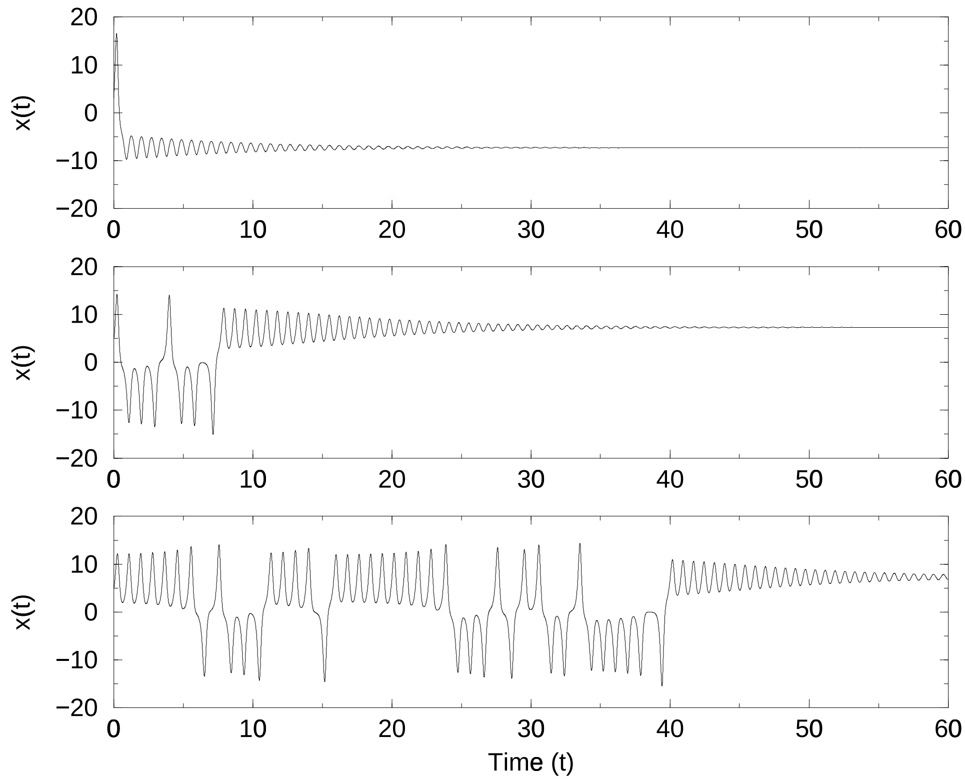

FIG. 22. *Plot of $x(t)$ versus t for $r = 21$ and three initial conditions of the Lorenz equations: (3,8,0), (5,5,5), and (5,5,10) (from top to bottom).*

Taking the determinant gives the characteristic equation

$$(-b-\lambda)\left[(-\sigma-\lambda)(-1-\lambda)-r\sigma\right]=0$$

whose eigenvalues are

$$\begin{aligned}\lambda &= -b\\ \lambda &= \frac{-(1+\sigma)+\sqrt{(1+\sigma)^2+4\sigma(r-1)}}{2}\\ \lambda &= \frac{-(1+\sigma)-\sqrt{(1+\sigma)^2+4\sigma(r-1)}}{2}\,.\end{aligned}$$

For $r < 1$, all the eigenvalues are real and negative so that the origin is asymptotically stable. However, once $r > 1$, one of the eigenvalues becomes positive and gives rise to growth and instability. Thus, all the interesting dynamics occurs once $r > 1$.

Of course, to get a complete picture of the dynamics, we must also linearize about the other two equilibrium points. Here, only a summary will be given concerning the stability of the other two points. For $1 < r < r_1 \approx 1.3456$, there are three negative eigenvalues so that the two additional fixed points are asymptotically stable. For $r_1 < r < r_2 \approx 24.737$, the other two points have have a single real and negative eigenvalue along with a pair of complex conjugate eigenvalues with a negative real part. Thus in this parameter regime the other two fixed points are again asymptotically stable.

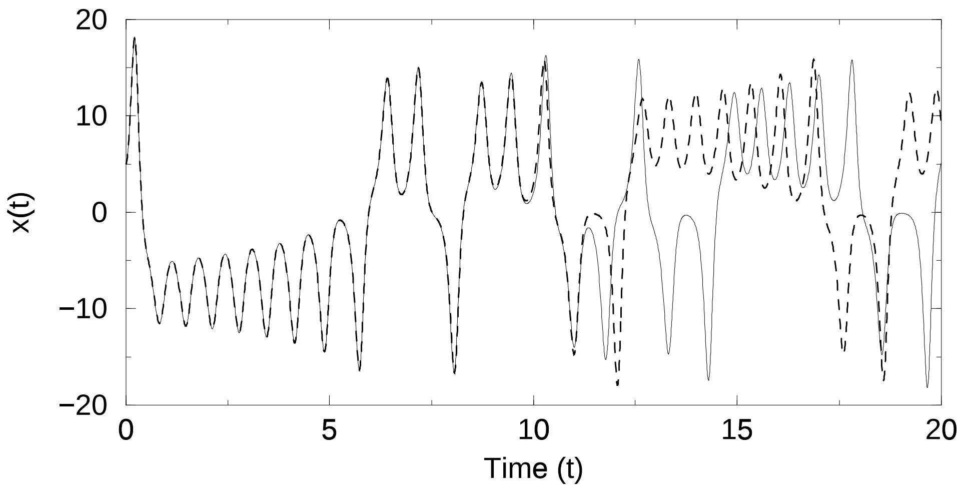

FIG. 23. *Plot of $x(t)$ versus t for $r = 28$ with the initial conditions (5,5,5) (solid line) and (5.01,5,5) (dashed line). Note that the solutions diverge sharply just after time $t = 10$ indicating the characteristic sensitivity to initial conditions of chaotic systems.*

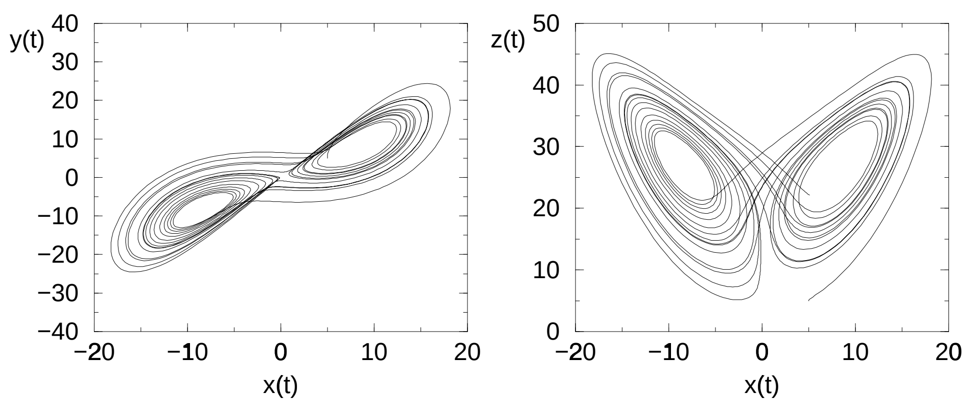

FIG. 24. *Plot of $x(t)$ versus $y(t)$ and $x(t)$ versus $z(t)$ for $r = 28$. All fixed points are unstable, yet the solution oscillates back and forth between the two fixed points in a chaotic fashion. This is known as a* strange attractor.

For $r > r_2$, the complex conjugate pair of eigenvalues have a positive real part so that solutions spiral away from these two points and the equilibrium points are unstable. For this parameter regime, all equilibria are unstable and it is difficult to determine from the linearization procedure what the long term behavior might be. We thus turn to numerical simulations of the Lorenz equations to aid in our understanding of this system.

We begin the simulations by first exploring the parameter regime for which the two nontrivial fixed points are stable. In the simulations that follow we take $\sigma = 10$ and $b = 8/3$ and vary r. For the case of stable equilibrium, we take $r = 21$ which is below the critical value of r_2. The simulations are depicted in Fig. 22 for three different initial conditions. Note that the solutions eventually settle onto one of the equilibrium points. The eventual fixed point on which the dynamics eventually settles

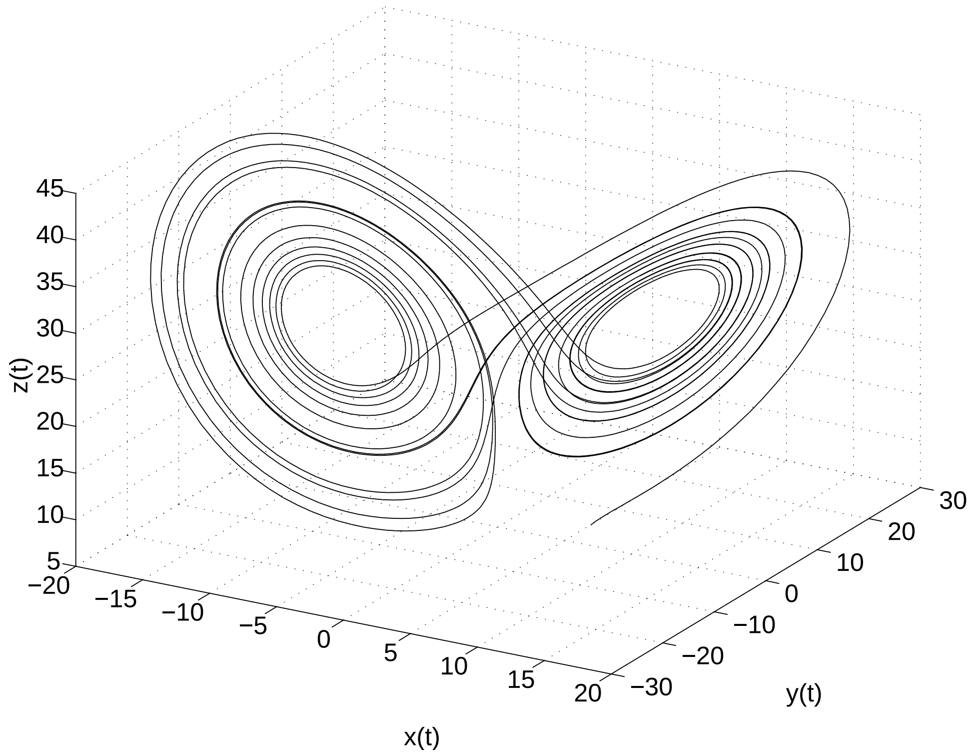

FIG. 25. *Plot of $x(t)$ versus $y(t)$ versus $z(t)$ for $r = 28$. All fixed points are unstable, yet the solution oscillates back and forth between the two fixed points in a chaotic fashion. This is known as a* strange attractor.

is determined from the initial conditions. This parameter regime is well behaved and completely understood from our theoretical linearization techniques.

Above the critical value of r_2, none of the fixed points are stable and a very interesting dynamic occurs. In Fig. 23 we show the dynamics of $x(t)$ versus t for $r = 28$ and the two initial conditions: (5,5,5) and (5.01,5,5). One would expect that such similar initial conditions would give rise to solutions which are fairly close. And indeed this is true for times just past $t = 10$. After this, the solutions diverge sharply and the solutions appear to have nothing in common. This sensitivity to initial conditions is the hallmark feature of what is called *chaotic dynamics*. In essence, it says that unless you known the initial conditions exactly, you will never be able to predict the long term behavior of your solution since all initial conditions, regardless of how close, will eventually diverge and be completely different. However, the solutions in this case have a characteristic dynamic: they circle in a random fashion the two unstable fixed points. This is depicted in Fig. 24 where we plot $x(t)$ versus both $y(t)$ and $z(t)$. A full three–dimensional rendering of the dynamics is given in Fig. 25 with the initial condition (5,5,5). The resulting dynamic is known as a *strange attractor*.

The moral of this story is that once you have a nonlinear system which is of three–degrees of freedom or higher, the resulting dynamics can be chaotic, i.e. unpredictable. Everything we have done thus far has been aimed at establishing methods for predicting the solution for all time. Here we have a fundamental limitation since we can no longer do this. The consequences: we can't predict weather except for maybe a few days out. This idea has revolutionized the way we think of scientific problems since at the very core of many physical systems lies chaotic dynamics for which we forfeit our power of prediction.

Appendix A. Useful Integrals

$\int x^n dx = \frac{x^{n+1}}{n+1} + C \;\; (n \neq 1)$

$\int \frac{dx}{x} = \ln|x| + C$

$\int \exp(x)dx = \exp(x) + C$

$\int x\exp(x)dx = x\exp(x) - \exp(x) + C$

$\int p^x dx = \frac{p^x}{\ln p} + C$

$\int x^n \exp(x)dx = x^n \exp(x) - n\int x^{n-1}\exp(x)dx$

$\int \frac{x}{x^2+1} = \frac{1}{2}\ln|x^2+1| + C$

$\int x^n \ln x dx = x^{n+1}\left[\frac{\ln x}{n+1} - \frac{1}{(n+1)^2}\right] + C$

$\int \sin x dx = -\cos x + C$

$\int \cos x dx = \sin x + C$

$\int \sin^2 x dx = \frac{1}{2}x - \frac{1}{4}\sin 2x + C$

$\int \cos^2 x dx = \frac{1}{2}x + \frac{1}{4}\sin 2x + C$

$\int \sin^n x dx = -\frac{\sin^{n-1} x\cos x}{n} + \frac{n-1}{n}\int \sin^{n-2} x dx$

$\int \cos^n x dx = \frac{\cos^{n-1} x\sin x}{n} + \frac{n-1}{n}\int \cos^{n-2} x dx$

$\int x\sin x dx = -x\cos x + \sin x + C$

$\int x\cos x dx = x\sin x + \cos x + C$

$\int x^n \sin x dx = -x^n\cos x + n\int x^{n-1}\cos x dx$

$\int x^n \cos x dx = x^n\sin x - n\int x^{n-1}\sin x dx$

$\int \exp(ax)\sin bx dx = \frac{\exp(ax)}{a^2+b^2}(a\sin bx - b\cos bx) + C$

$\int \exp(ax)\cos bx dx = \frac{\exp(ax)}{a^2+b^2}(a\cos bx + b\sin bx) + C$

$\int \tan x dx = \ln|\sec x| + C$

$\int \sec x dx = \ln|\sec x + \tan x| + C$

$\int \tan^2 x dx = \tan x - x + C$

$\int \sec^2 x dx = \tan x + C$

$\int \cot x dx = \ln|\sin x| + C$

$\int \csc x dx = \ln|\csc x + \cot x| + C$

$\int \cot^2 x dx = -\cot x - x + C$

$\int \csc^2 x dx = -\cot x + C$

$\int \sinh x dx = \cosh x + C$

$\int \cosh x dx = \sinh x + C$

$\int \text{sech}^2 x dx = \tanh x + C$

$\int \text{csch}^2 x dx = -\coth x + C$

Appendix B. Laplace Transforms

	$F(s)$	$f(t)$
1.	$\frac{1}{s}$	$H(t)$, Heaviside function
2.	$\frac{1}{s^2}$	t
3.	$\frac{1}{s^n}$	$\frac{t^{n-1}}{(n-1)!}$
4.	$\frac{1}{\sqrt{s}}$	$\frac{1}{\sqrt{\pi t}}$
5.	$s^{-3/2}$	$2\sqrt{\frac{t}{\pi}}$
6.	$s^{-[n+(1/2)]}$	$\frac{2^n t^{n-(1/2)}}{1\cdot 3\cdot 5\cdots(2n-1)\sqrt{\pi}}$
7.	$\frac{1}{s-a}$	$\exp(at)$
8.	$\frac{1}{(s-a)^2}$	$t\ \exp(at)$
9.	$\frac{1}{(s-a)^n}$	$\frac{1}{(n-1)!}t^{n-1}\exp(at)$
10.	$\frac{1}{(s-a)(s-b)}$	$\frac{1}{a-b}[\exp(at)\text{-}\exp(bt)]$
11.	$\frac{s}{(s-a)(s-b)}$	$\frac{1}{a-b}[a\ \exp(at)\text{-}\ b\ \exp(bt)]$
12.	$\frac{1}{(s-a)(s-b)(s-c)}$	$-\ \frac{(b-c)\exp(at)+(c-a)\exp(bt)+(a-b)\exp(ct)}{(a-b)(b-c)(c-a)}$
13.	$\frac{1}{s^2+a^2}$	$\frac{1}{a}\sin(at)$
14.	$\frac{s}{s^2+a^2}$	$\cos(at)$
15.	$\frac{1}{s^2-a^2}$	$\frac{1}{a}\sinh(at)$
16.	$\frac{s}{s^2-a^2}$	$\cosh(at)$
17.	$\frac{1}{s(s^2+a^2)}$	$\frac{1}{a^2}[1-\cos(at)]$
18.	$\frac{1}{s^2(s^2+a^2)}$	$\frac{1}{a^3}[at-\sin(at)]$
19.	$\frac{1}{(s^2+a^2)^2}$	$\frac{1}{2a^3}[\sin(at)-at\cos(at)]$
20.	$\frac{s}{(s^2+a^2)^2}$	$\frac{t}{2a}\sin(at)$
21.	$\frac{s^2}{(s^2+a^2)^2}$	$\frac{1}{2a}[\sin(at)+at\cos(at)]$
22.	$\frac{s^2-a^2}{(s^2+a^2)^2}$	$t\cos(at)$
23.	$\frac{s}{(s^2+a^2)(s^2+b^2)}$	$\frac{\cos(at)-\cos(bt)}{b^2-a^2}$
24.	$\frac{1}{(s-a)^2+b^2}$	$\frac{1}{b}\exp(at)\sin(bt)$
25.	$\frac{s-a}{(s-a)^2+b^2}$	$\exp(at)\cos(bt)$

Appendix C. Solving ODEs with MATLAB

To aid and complement our understanding of the dynamics associated with a given ODE, it is often instructive to solve it numerically. Numerical techniques have revolutionized the approach to many difficult problems since it provides a fast, efficient way to get an exact (although not analytic) solution to a wide variety of problems of interest. In this appendix, solving ODEs with MATLAB will be emphasized. Details of numerical solution techniques are outlined in Boyce and DiPrima, Chapter 8.

To use matlab, we require two separate matlab routines or files. These are always denoted by a .m extention to a file. For instance, in this example we will construct the files **myode.m** and **myode2.m**. The file **myode.m** will call upon **myode2.m** in order to carry out the calculations. From the MATLAB command prompt, all that needs to be done is to type:

```
>> myode
```

to start the run. In the following example, we solve the Lorenz equations. Note that all ODEs in MATLAB must be put into a system of equations form.

FILE: **myode.m**

```
clear all          % clear all previously defined values
tspan=[0 20];      % define the range of times to be solved for
y0=[5.01; 5; 5.0]; % define the initial conditions x(0), y(0), z(0)

[t,y]=ode45('myode2',tspan,y0); % solve using ode45 MATLAB routine

figure(1)     % plot each x(t), y(t), z(t) versus time t
plot(t,y(:,1))
hold on
plot(t,y(:,2))
plot(t,y(:,3))

figure(2)     % make a 3D plot of the solution
plot3(y(:,1),y(:,2),y(:,3))
view(30,30)
xlabel('x(t)'); ylabel('y(t)'); zlabel('z(t)'); grid on
```

FILE: **myode2.m**

```
function yprime=nonlinear2(t,y);

sigma=10.0; b=8.0/3.0; r=28.0;
yprime=[  sigma*(-y(1)+y(2))
          r*y(1)-y(2)-y(1)*y(3)
          -b*y(3)+y(1)*y(2)      ];
```

Experiment with this basic format and use the HELP feature of MATLAB to aid you. Specific topics you may want to find help on are: *ode45, plot, plot3, clear all.*

Appendix D. Worked Problems

Example: (B&D 2.1 #1) Solve $y' + 3y = t + \exp(-2t)$.

We begin by noting that the integrating factor method gives

$$\begin{aligned} p(t) &= 3 \\ g(t) &= t + \exp(-2t)\,. \end{aligned}$$

This then gives the integrating factor

$$\mu(t) = \exp\left[\int p(t)dt\right] = \exp\left[\int 3dt\right] = \exp(3t)$$

which yields the solution

$$\begin{aligned} y(t) &= \frac{\int \mu(t)g(t)dt + C}{\mu(t)} \\ &= \exp(-3t)\left[\int \exp(3t)(t + \exp(-2t))dt + C\right] \\ &= C\exp(-3t) + \exp(-3t)\left[\int t\exp(-3t)dt + \int \exp(t)dt\right] \\ &= C\exp(-3t) + \exp(-3t)\left[\exp(3t)(3t-1)/9 + \exp(t)\right] \\ y(t) &= C\exp(-3t) + \exp(-2t) + t/3 - 1/9\,. \end{aligned}$$

And as time $t \to \infty$, we have that

$$y(t \to \infty) \to t/3 - 1/9\,.$$

Example: (B&D 2.1 #4) Solve $y' + (1/t)y = 3\cos(t)$.

We begin by noting that the integrating factor method gives

$$\begin{aligned} p(t) &= 1/t \\ g(t) &= 3\cos(t)\,. \end{aligned}$$

This then gives the integrating factor

$$\mu(t) = \exp\left[\int p(t)dt\right] = \exp\left[\int (1/t)dt\right] = \exp(\ln t) = t$$

which yields the solution

$$\begin{aligned} y(t) &= \frac{\int \mu(t)g(t)dt + C}{\mu(t)} \\ &= (1/t)\left[\int 3t\cos(2t)dt + C\right] \\ &= (1/t)\left[(3/2)t\sin(2t) + (3/4)\cos(2t) + C\right] \\ y(t) &= \frac{3}{2}\sin(2t) + \frac{3}{4t}\cos(2t) + \frac{C}{t}. \end{aligned}$$

And as time $t \to \infty$, we have that

$$y(t \to \infty) \to \frac{3}{2}\sin(2t)\,.$$

Example: (B&D 2.2 #12) Solve $ty' + 2y = \sin(t)$ with $y(\pi) = 1/\pi$.

Putting this into standard form gives

$$y' + \frac{2}{t}y = \frac{\sin(t)}{t}.$$

We note that the integrating factor method gives

$$p(t) = 2/t$$
$$g(t) = \sin(t)/t.$$

This then gives the integrating factor

$$\mu(t) = \exp\left[\int p(t)dt\right] = \exp\left[\int (2/t)dt\right] = \exp(2\ln t) = \exp(\ln t^2) = t^2$$

which yields the solution

$$y(t) = \frac{\int \mu(t)g(t)dt + C}{\mu(t)} = (1/t^2)\left[\int t\sin(t)dt + C\right] = t^{-2}\left[-t\cos(t) + \sin(t) + C\right].$$

The initial condition $y(\pi) = 1/\pi = -\frac{-1}{\pi} + \frac{0}{\pi} + \frac{0}{\pi^2}$ so that $C = 0$ and

$$y(t) = t^{-2}\left[\sin(t) - t\cos(t)\right].$$

Example: (B&D 2.2 #16) Solve $(1-t^2)y' + ty = t(1-t^2)$ with $y(0) = 2$.

Putting this into standard form gives

$$y' - \frac{t}{1-t^2}y = t.$$

We note that the integrating factor method gives

$$p(t) = -t/(1-t^2)$$
$$g(t) = t.$$

This then gives the integrating factor

$$\mu(t) = \exp\left[\int p(t)dt\right] = \exp\left[\int \frac{-tdt}{1-t^2}\right] = \exp\left[\frac{\ln(1-t^2)}{2}\right] = \exp(\ln(1-t^2)^{1/2}) = (1-t^2)^{1/2}$$

which yields the solution

$$y(t) = \frac{\int \mu(t)g(t)dt + C}{\mu(t)}$$
$$= (1-t^2)^{-1/2}\left[\int t(1-t^2)^{1/2}dt + C\right]$$
$$y(t) = -(1/3)(1-t^2) + C(1-t^2)^{-1/2}.$$

The initial condition $y(0) = 2 = -1/3 + C$ so that $C = 7/3$ and

$$y(t) = -\frac{1}{3}(1-t^2) + \frac{7}{3}(1-t^2)^{-1/2}.$$

Example: (B&D 2.3 #12) Solve $\frac{dr}{d\theta} = \frac{r^2}{\theta}$ with $r(1) = 2$.

This equation is separable. So we can rewrite it as

$$\frac{dr}{r^2} = \frac{d\theta}{\theta}\,.$$

Integrating both sides gives

$$\int \frac{dr}{r^2} = \int \frac{d\theta}{\theta} + C$$

which upon integration yields

$$-\frac{1}{r} = \ln\theta + C\,.$$

The initial condition $r(1) = 2$ gives $-1/2 = \ln 1 + C$ so that $C = -1/2$ and

$$r = \frac{2}{1 - 2\ln\theta}\,,$$

which is valid for $0 < r < \exp(1/2)$, i.e. when r is well behaved.

Example: (B&D 2.3 #16) Solve $\frac{dy}{dx} = \frac{x(x^2+1)}{4y^3}$ with $y(0) = -1/\sqrt{2}$.

This equation is separable. So we can rewrite it as

$$4y^3 dy = x(x^2+1)dx = (x^3+x)dx\,.$$

Integrating both sides gives

$$y^4 = \frac{1}{4}x^4 + \frac{1}{2}x^2 + C\,.$$

The initial condition gives

$$\left(-\frac{1}{\sqrt{2}}\right)^4 = \frac{1}{4} = C$$

so that

$$y^4 = \frac{1}{4}x^4 + \frac{1}{2}x^2 + \frac{1}{4} = \frac{(x^2+1)^2}{4}\,.$$

Taking the square root and the positive root (since $y^2 > 0$) gives

$$y^2 = \frac{1}{2}(x^2+1)\,.$$

One more square root leads to

$$y = -\sqrt{\frac{x^2+1}{2}}$$

where we have taken the negative root in order to satisfy the initial condition.

Example: (B&D 2.6 #20) Solve $y' = \alpha y(1-y)$ with $y(0) = y_0$.

(a) Determine the equilibrium points and their stability.

Equilibrium points are those points for which the population isn't changing, i.e. it is neither growing nor decaying. Thus, we consider the points where

$$y' = 0 \, .$$

But since

$$y' = \alpha y(1-y)$$

this implies that

$$y(1-y) = 0 \, .$$

Thus our equilibrium are

$$y = 0 \quad \text{and} \quad y = 1 \, .$$

To determine stability, we first consider two different regimes: $0 < y < 1$ and $y > 1$. For $0 < y < 1$ we have the following:

$$0 < y < 1 : \quad \alpha y(1-y) > 0 \ \rightarrow \ y' > 0 \ \rightarrow \text{growth} \, .$$

Whereas for $y > 1$ we have

$$y > 1 : \quad \alpha y(1-y) < 0 \ \rightarrow \ y' < 0 \ \rightarrow \text{decay} \, .$$

Thus the population grows for values below $y = 1$ and decays for values above $y = 1$. This means that $y = 1$ is asymptotically stable since all solutions approach it as time goes to infinity. However, the equilibrium at $y = 0$ is unstable since all solutions go away from it as time goes to infinity. This reasoning is worked out in detail in my notes (see Lecture 2 of Ch. 2 and take $K = 1$).

(b) Solve the equation with the given initial condition.

The solution to the problem is straightforward since the problem is separable. In particular, the governing equation is separated to

$$\frac{dy}{y(1-y)} = \alpha dt \, .$$

Integrating both sides yields

$$\alpha t + c = \int \frac{dy}{y(1-y)} = \int \frac{dy}{y} + \int \frac{dy}{1-y} = \ln y - \ln(1-y) = \ln \frac{y}{1-y}$$

which upon exponentiation gives

$$y/(1-y) = C \exp(\alpha t) \, .$$

Insertion of the initial condition $y(0) = y_0$ gives $C = y_0/(1-y_0)$ which can then be used in the above equation to manipulate the solution form to

$$y(t) = \frac{y_0}{y_0 + (1-y_0)\exp(-\alpha t)} \, .$$

Problem: (B&D 2.6 #24) A chemical reaction is governed by the equation $x' = \alpha(p-x)(q-x)$ with $\alpha > 0$.

(a) With $x(0) = 0$, determine the limiting value of $x(t)$ as $t \to \infty$ without solving the problem. Then solve the equation.

We begin considering the limiting behavior by looking at the equilibrium solutions for which $x' = 0$. From our governing evolution we find:

$$\frac{dx}{dt} = 0 \quad \rightarrow \quad \alpha(p-x)(q-x) = 0 \quad \rightarrow \quad x = p \text{ and } x = q\,.$$

It then remains to determine the dynamic behavior of the system. We can break the dynamics into three cases:

1. x is less than both p, q : $\dfrac{dx}{dt} = \alpha(p-x)(q-x) > 0$ growth
2. x is somewhere between p, q : $\dfrac{dx}{dt} = \alpha(p-x)(q-x) < 0$ decay
3. x is greater than both p, q : $\dfrac{dx}{dt} = \alpha(p-x)(q-x) > 0$ growth.

We can represent this dynamics graphically. Figure 1a shows that the smaller of p and q, referred to as $\min\{p, q\}$, is asymptotically stable while $\max\{p, q\}$ is unstable. This follows immediately from the three cases above. Thus if $x(0) = 0$, we find

$$x(t) \to \min\{p, q\} \;\text{ as }\; x \to \infty\,.$$

To solve the problem, we use the method of separation. We thus find

$$\frac{dx}{dt} = \alpha(p-x)(q-x) \quad \rightarrow \quad \frac{dx}{(p-x)(q-x)} = \alpha dt \quad \rightarrow \quad \int \frac{dx}{(p-x)(q-x)} = \alpha t + c\,.$$

The only difficulty lies in evaluating the right-hand side integral. We perform the partial fraction decomposition to help us

$$\frac{1}{(p-x)(q-x)} = \frac{A}{p-x} + \frac{B}{q-x} \quad \rightarrow \quad A = -B = \frac{1}{q-p}\,.$$

Thus our integral becomes

$$\frac{1}{q-p}\left(\int \frac{dx}{p-x} - \int \frac{dx}{q-x}\right) = \frac{1}{q-p}\left[-\ln(p-x) + \ln(q-x)\right] = \frac{1}{q-p}\ln\frac{q-x}{p-x}\,.$$

Plugging in above yields

$$\frac{1}{q-p}\ln\frac{q-x}{p-x} = \alpha t + c \quad \rightarrow \quad \ln\frac{q-x}{p-x} = \alpha(q-p)t + C\,.$$

Exponentiating both sides then gives

$$\frac{q-x}{p-x} = C\exp\left[\alpha(q-p)t\right]\,.$$

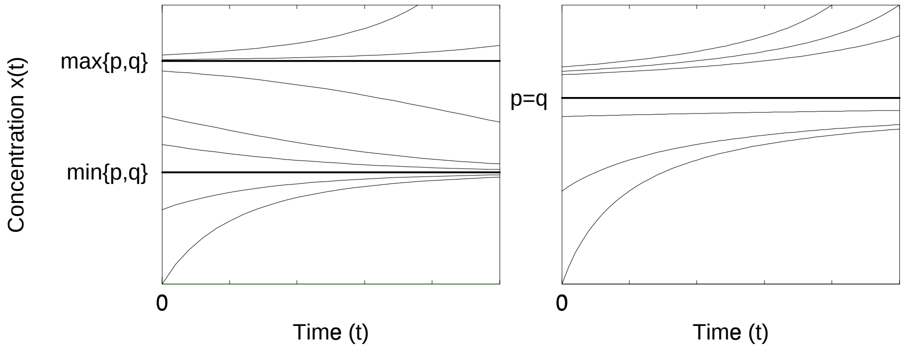

FIG. 26. *Qualitative depiction of the dynamics of the chemical reaction for (a) $p \neq q$ and (b) $p = q$. Note that in either case, the initial condition $x(0)$ reaches a limiting value.*

At this point, it turns out to be convenient to insert our initial condition $x(0) = 0$. This gives $C = q/p$. Inserting this into the above equation and rearranging then yields our solution

$$x(t) = pq\frac{\exp[\alpha(q-p)t] - 1}{q\exp[\alpha(q-p)t] - p}\,.$$

(b) Repeat part (a) but with $p = q$.

We begin by first investigating the limiting behavior as times gets large with $x(0) = 0$. The equation in this case is

$$\frac{dx}{dt} = \alpha(p-x)^2$$

which has the equilibrium solutions ($x' = 0$) when $x = p$. When then note the two cases:

$$\begin{aligned} &1.\ x < p: \quad \frac{dx}{dt} = \alpha(p-x)^2 > 0 \quad \text{growth} \\ &2.\ x > p: \quad \frac{dx}{dt} = \alpha(p-x)^2 > 0 \quad \text{growth}\,. \end{aligned}$$

The equilibrium solution $x = p$ is thus semi-stable (Fig. 1b) and

$$x(t) \to p \ \text{ as } \ x \to \infty\,.$$

To solve the problem, we use the method of separation. We thus find

$$\frac{dx}{dt} = \alpha(p-x)^2 \quad \to \quad \frac{dx}{(p-x)^2} = \alpha dt \quad \to \quad \int \frac{dx}{(p-x)^2} = \alpha t + c\,.$$

In contrast to part (a), the integral can be easily evaluated to give:

$$\frac{1}{p-x} = \alpha t + c \quad \to \quad p - x = \frac{1}{\alpha t + c}\,.$$

Inserting our initial condition $x(0) = 0$ gives $C = 1/p$. Inserting this into the above equation and rearranging then yields our solution

$$x(t) = \frac{p^2\alpha t}{\alpha p t + 1}\,.$$

Problem: (B&D 2.7 #7) A skydiver weighing 180 lbs falls vertically downward from an altitude of 5000 ft, and opens the parachute after 10 seconds of free fall. Assume that the force of air resistance is $0.75|v|$ when the chute is closed and $12|v|$ when the parachute is open.

Before proceeding to solve this problem, we first formulate our equations of motion. Both gravity and wind resistance are acting on the parachutter. Thus we have:

$$\sum F = ma \quad \rightarrow \quad mg - kv = ma = m\frac{dv}{dt} \quad \rightarrow \quad \frac{dv}{dt} + \frac{k}{m}v = g$$

where $g = 32.2$ ft/s^2, $mg = 180$ lbs ($m = 180/g = 5.6$ slugs), and k measures the wind resistance constant. This is a familiear first-order differential equation for v. Using the method of integrating factors, we note that $p(t) = k/m$ and $g(t) = g$. We can then calculate $\mu(t) = \exp(\int p(t)dt) = \exp(\int k/m dt) = \exp(kt/m)$. The solution $v(t)$ is then

$$v(t) = \frac{\int \mu(t)g(t)dt + C}{\mu(t)} = \exp(-kt/m)\left[\int g\exp(kt/m)dt + C\right] = \frac{gm}{k} + C\exp\left(-\frac{kt}{m}\right)$$

where C is an arbitrary constant determined by some initial condition. We can then integrate the velocity $v(t) = dx/dt$ to get the position $x(t)$:

$$x(t) = \frac{gm}{k}t - \frac{mC}{k}\exp\left(-\frac{kt}{m}\right) + C_1$$

where C_1 is a second constant of integration. We are now ready to proceed.

(a) Find the speed of the skydiver when the parachute opens

We have the solution for the velocity above provided we can calculate the constant C. We do this by noting that at time $t = 0$, $v(0) = 0$, which gives $C = -gm/k$. The velocity at $t = 10$ is then (with $k = 0.75$):

$$v(10) = \frac{gm}{k}\left(1 - \exp\left(-\frac{k}{m}10\right)\right) = 176 \text{ ft/s}.$$

(b) Find the distance fallen before the parachute opens.

We have also calculated the distance $x(t)$ above. We evaluate the constant C_1 by noting that at time $t = 0$, $x(0) = 0$, which gives $C_1 = mC/k = -gm^2/k^2$. At time $t = 10$ we then find:

$$x(10) = \frac{gm}{k}10 + \frac{gm^2}{k^2}\exp\left(-\frac{k}{m}10\right) - \frac{gm^2}{k^2} = 1075 \text{ ft}.$$

(c) What is the limiting velocity v_L after the parachute opens?

Assume that at time $t = 0$ the chute opens, i.e. we now have a new origin of time. Recall that the chute opens with an initial velocity $v(0) = 176$ ft/s. We can then calculate a new constant $C = 176 - gm/k$. The limiting velocity is determined by

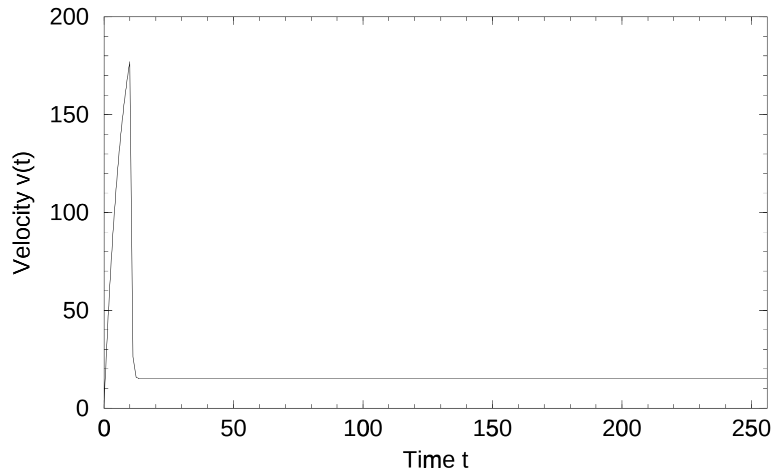

FIG. 27. *Plot of velocity $v(t)$ versus time t for the parachutter. Note that the chute is opened at time $t = 10$ and the skydiver quickly reaches the terminal velocity of $v_L = 15$ ft/s.*

finding the velocity as time goes to infinity. This is easy to do since $\exp(kt/m) \to 0$ as $t \to \infty$. This then gives (with $k = 12$ now):

$$v(t \gg 1) = v_L = \frac{mg}{k} = \frac{180}{12} = 15 \text{ ft/s}\,.$$

(d) Determine how long the skydiver is in the air after the parachute opens.

Following part (c), we also have that at time $t = 0$, $x(0) = 0$ so that we can determine the new constant $C_1 = mC/k = m(176 - gm/k)/k$. Unlike part (b) for which we are given the time and are only required to determine the position, here we are given the distance to the ground (5000-1075 ft=3925 ft) and are required to determine the time. Thus there is some time $t = t^*$ for which we hit the ground. This then gives

$$3925 = \frac{gm}{k}\,t^* + \left(176 - \frac{gm}{k}\right)\exp\left(-\frac{k}{m}t^*\right) - \frac{m}{k}\left(176 - \frac{gm}{k}\right)\,,$$

which is a transcendental equation for t^*. To solve it, we simply have to note that the parachutter is likely to fall with his parachute open for quite a bit longer than 10 seconds. Further $k/m \approx 2$ so that $\exp(-kt/m) \approx \exp(-20) = 10^{-9}$. So essentially it is zero and we can ignore the middle term above. This then leaves

$$3925 \approx \frac{gm}{k}\,t^* - \frac{m}{k}\left(176 - \frac{gm}{k}\right)\,,$$

which upon solving for t^* gives

$$t^* = \frac{k}{gm}\left[3925 + \frac{m}{k}\left(176 - \frac{gm}{k}\right)\right] = 256 \text{ sec}\,.$$

(e) Plot the graph of velocity versus time from the beginning of the fall until the skydiver reaches the ground.

This is plotted in Fig. 2 above.

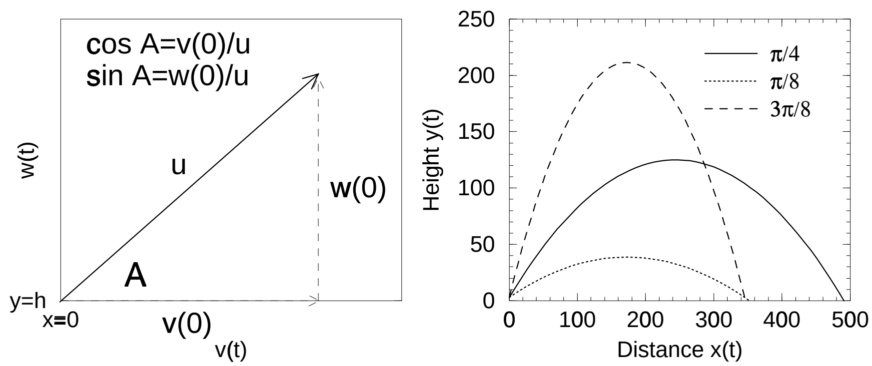

FIG. 28. *Plot of the initial velocity of the ball (a) and some sample trajectories of the flight of the ball (b) for u=125 ft/s, h=3 ft, and angles* $A = \pi/8, \pi/4, 3\pi/8$.

Problem: (B&D 2.7 #17) Let $v(t)$ and $w(t)$ be the horizontal and vertical component of a batted ball. Without air resistance, the equations satisfy $v' = 0$ and $w' = -g$.

(a) Show that $v = u\cos A$ and $w = -gt + u\sin A$ with and initial speed u and launch angle A.

We can integrate the governing equations to find the following

$$\frac{dv}{dt} = 0 \quad \rightarrow \quad v(t) = c_1 \quad \text{and} \quad \frac{dw}{dt} = -g \quad \rightarrow \quad w(t) = -gt + c_2\,,$$

where the constants c_1 and c_2 are determined from initial conditions. The initial conditions are easily found from the diagram of Fig. 3a where $\cos A = v(0)/u$ and $\sin A = w(0)/u$. Plugging these conditions in at time $t = 0$ yields $c_1 = u\cos A$ and $c_2 = u\sin A$ so that

$$v(t) = u\cos A \quad \text{and} \quad w(t) = -gt + u\sin A\,.$$

(b) Find $x(t)$ and $y(t)$, the horizontal and vertical distances respectively, given $x(0) = 0$ and $y(0) = h$.

Recall that $x' = v$ and $y' = w$ so that we can calculate $x(t)$ and $y(t)$ by integrating with respect to time:

$$x(t) = \int v(t)dt = ut\cos A \quad \text{and} \quad y(t) = \int w(t)dt = -\frac{gt^2}{2} + ut\sin A + h\,,$$

where we have used the fact that $x(0) = 0$ and $y(0) = h$ to determine the constants of integration.

(c) Plot the trajectories for several values of A.

See Fig. 3b. We have used $h = 3$ ft, $g = 32$ ft/s^2, and $u = 125$ ft/s.

(d) If the wall is L away and height H, determine a relationship between u and A so that the ball clears the wall.

For the ball to clear the wall, it must be equal to or higher than H and farther than or equal to L simultaneously. Thus the two equations must be satisfied:

$$ut\cos A \geq L \quad \text{and} \quad -\frac{gt^2}{2} + ut\sin A + h \geq H\,.$$

From the first equation we find that $t \geq L/u\cos A$ so that upon substituting into the second equation we get

$$-\frac{gL^2}{2u^2\cos^2 A} + L\tan A + h \geq H\,.$$

(e) If $L = 350$ ft and $H = 10$ ft, find the range of A values which clear the wall with $u = 110$ ft/s.

Two methods will be presented here for getting at the solution (or approximate solution). First, we can realize that $1/\cos^2 A = \sec^2 A = 1 + \tan^2 A$. Thus our previous relation becomes a quadratic equation for $\tan A$:

$$\tan^2 A - \frac{2u^2}{gL}\tan A + \left(1 + \frac{2u^2(H-h)}{gL^2}\right) = 0$$

The two roots of this equation give us what we are looking for

$$\tan A = \frac{u^2}{gL} \pm \sqrt{\frac{u^4}{g^2L^2} - 1 - \frac{2u^2(H-h)}{gL^2}} \quad \rightarrow \quad A = \tan^{-1}(1.080 \pm 0.3513) = 36^0, 55^0.$$

Alternatively, we note that dividing the relation in part (d) by L gives $H - h/L = 0.02 \approx 0.0$. By then multiplying through by $\cos^2 A$ we find

$$-\frac{gL}{2u^2} + \cos A\sin A = 0 \quad \rightarrow \quad \sin 2A = \frac{gL}{u^2} = 0.92 \quad \rightarrow \quad A = 34^0, 57^0\,.$$

(f) If $L = 350$ and $H = 10$, find the minimum velocity and angle for the ball to clear the wall.

For the quick method of approximating this, we note that the approximation above yields $u^2 = \frac{gl}{\sin 2A}$ which makes u^2 minimum when $\sin 2A = 1$ is maximum, i.e. one. Thus $2A = \pi/2$ and $A = \pi/4 = 45^0$ with $u = \sqrt{gL} = 106$ ft/s. To be more precise, we must recall that derivatives find minima and maxima of functions. And we would like to find the minimum u that satisfies the equality

$$-\frac{gL}{u^2} + \sin 2A - 2\frac{H-h}{L}\cos^2 A = 0$$

which is found by rearranging the result in part (d). Taking the derivative of the above and setting $du/dA = 0$ (recall that $u = u(A)$) gives:

$$\cos 2A + \frac{H-h}{L}\sin 2A = 0$$

where we have once again made use of $\sin 2A = \cos A\sin A$. This then gives

$$\tan 2A = -\frac{L}{H-h} = -50 \quad \rightarrow \quad 2A = -88.85^0 + 180^0 \times n$$

where n is an integer. Since we are looking for $0^0 < A < 90^0$, this then gives $n = 1$ and

$$2A = 91.15^0 \quad \rightarrow \quad A = 45.58^0 \quad \rightarrow \quad u = 106.9 \text{ ft/s}\,.$$

Example: (B&D 2.8 #15) Solve $(xy^2 + bx^2y)dx + (x+y)x^2dy = 0$.

This equation is exact as can be seen by calculating the following:

$$\begin{aligned} M(x,y) &= xy^2 + bx^2y & \qquad M_y &= 2xy + bx^2 \\ N(x,y) &= (x+y)x^2 & \qquad N_x &= 3x^2 + 2xy\,. \end{aligned}$$

So we find that $M_y = N_x$ provided that $b = 3$. We now proceed to find $\psi(x,y)$ from

$$\psi_x = M = xy^2 + 3x^2y \quad \text{and} \quad \psi_y = N = (x+y)x^2\,.$$

Integrating the first equation with respect to x gives

$$\psi(x,y) = x^2y^2/2 + x^3y + h(y)$$

which if we take the y-derivative gives:

$$\psi(x,y)_y = x^2y + x^3 + h'(y) = (x+y)x^2 + h'(y)\,.$$

Comparing to the above, we find that $h'(y) = 0$ so that $h(y) = C$. Thus we find that $\psi(x,y) = x^2y^2/2 + x^3y + c = \text{constant}$ so that our solution is

$$x^2y^2/2 + x^3y = C\,.$$

Problem: (B&D 2.8 #19) Solve $x^2y^3 + x(1+y^2)y' = 0$ with the integrating factor $\mu = 1/xy^3$.

This equation is not initially exact since

$$\begin{aligned} M(x,y) &= x^2y^3 & \qquad M_y &= 3x^2y^2 \\ N(x,y) &= x(1+y^2) & \qquad N_x &= (1+y^2)\,. \end{aligned}$$

So we find that $M_y \neq N_x$. Multiplying through by $\mu = 1/xy^3$ gives $x + (1+y^2)/y^3y'$ which is exact since

$$\begin{aligned} M(x,y) &= x & \qquad M_y &= 0 \\ N(x,y) &= (1+y^2)/y^3 = y^{-3} + y^{-1} & \qquad N_x &= 0\,. \end{aligned}$$

and $M_y = N_x$. We then have

$$\psi_x = M = x \quad \text{and} \quad \psi_y = N = y^{-3} + y^{-1}\,.$$

Integrating the first equation with respect to x gives

$$\psi(x,y) = x^2/2 + h(y)$$

which if we take the y-derivative gives $\psi(x,y)_y = h'(y)$. Comparing to the above, we find that $h'(y) = y^{-3} + y^{-1}$ so that $h(y) = -1/2y^2 + \ln y$ and $\psi(x,y) = c$ gives

$$x^2 - 1/y^2 + 2\ln y = C\,.$$

Example: (B&D 2.9 #5) Solve $\frac{dy}{dx} = \frac{4y-3}{2x-y}$.

This equation is homogenous since we can make the right side a function of $z = y/x$ alone. Dividing top and bottom by x we find:

$$\frac{dy}{dx} = \frac{4(y/x)-3}{2-(y/x)} = \frac{4z-3}{2-z} = f(z)\,.$$

We recall that the solution to the homogenous equation is $x = C\exp\left[\int \frac{dz}{f(z)-z}\right]$ so that we must evaluate

$$\int \frac{dz}{f(z)-z} = \int \frac{dz}{(4z-3)/(2-z)-z} = \int \frac{(2-z)dz}{z^2+2z-3} = \int \frac{(2-z)dz}{(z+3)(z-1)}\,.$$

This integral can be simplified by noting that $(2-z) = 1-(z-1)$ so that we get

$$= \int \frac{dz}{(z+3)(z-1)} - \int \frac{dz}{z+3} = \frac{1}{4}\left[\int \frac{dz}{z-1} - \int \frac{dz}{z+3}\right] - \int \frac{dz}{z+3}\,.$$

Integration then yields

$$\frac{1}{4}[\ln|z-1| - \ln|z+3|] - \ln|z+3| = \frac{1}{4}[\ln|z-1| - 5\ln|z+3|] = \ln\left[\frac{|z-1|}{|z+3|^5}\right]^{1/4}\,.$$

Plugging this into our solution above we find that

$$x = C\left[\frac{|z-1|}{|z+3|^5}\right]^{1/4} \quad\rightarrow\quad x^4|z+3|^5 = C|z-1|$$

which upon plugging in $z = y/x$ gives

$$|y+3x|^5 = C|y-x|\,.$$

Problem: (B&D 2.9 #13) Solve $\frac{dy}{dx} = \frac{3y^2-x^2}{2xy}$.

This equation is homogenous since we can make the right side a function of $z = y/x$ alone. Dividing the top through by xy gives:

$$\frac{dy}{dx} = \frac{3y}{2x} - \frac{x}{2y} = \frac{3}{2}z - \frac{1}{2z} = f(z)\,.$$

We recall that the solution to the homogenous equation is $x = C\exp\left[\int \frac{dz}{f(z)-z}\right]$ so that we must evaluate

$$\int \frac{dz}{f(z)-z} = \int \frac{dz}{3z/2-1/2z-z} = \int \frac{2zdz}{z^2-1} = \ln|z^2-1|\,.$$

Plugging this into our solution above we find that

$$|x| = C|z^2-1|$$

which upon plugging in $z = y/x$ gives

$$C|x|^3 = |y^2-x^2|\,.$$

Problem: (B&D 3.1 #1) Solve $y'' + 2y' - 3y = 0$.

We begin by letting $y = \exp(\lambda t)$ so that we find the characteristic equation

$$\lambda^2 + 2\lambda - 3 = 0 \quad \to \quad (\lambda+3)(\lambda-1) = 0 \quad \to \quad \lambda = -3, 1\,.$$

The solution is then given by

$$y(t) = c_1 \exp(-3t) + c_2 \exp(t)\,.$$

Problem: (B&D 3.1 #5) Solve $y'' + 5y' = 0$.

We begin by letting $y = \exp(\lambda t)$ so that we find the characteristic equation

$$\lambda^2 + 5\lambda = 0 \quad \to \quad \lambda(\lambda+5) = 0 \quad \to \quad \lambda = -5, 0\,.$$

The solution is then given by

$$y(t) = c_1 + c_2 \exp(-5t)\,.$$

Problem: (B&D 3.1 #12) Solve $y'' + 3y' = 0$ with $y(0) = -2$ and $y'(0) = 3$.

We begin by letting $y = \exp(\lambda t)$ so that we find the characteristic equation

$$\lambda^2 + 3\lambda = 0 \quad \to \quad \lambda(\lambda+3) = 0 \quad \to \quad \lambda = -3, 0\,.$$

The solution is then given by $y(t) = c_1 + c_2 \exp(-3t)$. The initial conditions give $y(0) = -2 = c_1 + c_2$ and $y'(0) = 3 = -3c_2$ so that $c_2 = -1$ and $c_1 = -1$ and

$$y(t) = -(1 + \exp(-3t))\,.$$

Problem: (B&D 3.1 #16) Solve $4y'' - y = 0$ with $y(-2) = 1$ and $y'(-2) = -1$.

We begin by letting $y = \exp(\lambda t)$ so that we find the characteristic equation

$$4\lambda^2 - 1 = 0 \quad \to \quad \lambda^2 = 1/4 \quad \to \quad \lambda = -1/2, 1/2\,.$$

The solution is then given by $y(t) = c_1 \exp(t/2) + c_2 \exp(-t/2)$. The initial conditions give $y(-2) = 1 = c_1/e + c_2 e$ and $2y'(-2) = -2 = c_1/e - c_2 e$ so that $c_2 = 3/(2e)$ and $c_1 = -e/2$ and

$$y(t) = (3/2)\exp[-(t/2+1)] - (1/2)\exp[(t/2+1)]\,.$$

Problem: (B&D 3.2 #5) Find the Wronskian of $f = \exp(t)\sin t$ and $g = \exp(t)\cos t$.

$$W = fg' - f'g = e^t \sin t(e^t \cos t - e^t \sin t) - e^t \cos t(e^t \sin t + e^t \cos t) = -e^{-2t}\,.$$

Problem: (B&D 3.2 #9) When does $t(t-4)y'' + 3ty' + 4y = 2$ with $y(3) = 0$ and $y'(3) = -1$ have a solution?

We first rewrite the equation in the proper form $(y'' + p(t)y' + q(t)y = g(t))$

$$y'' + 3/(t-4)y' + 4/(t(t-4))y = 2/(t(t-4))$$

and note that the coefficients blow up at $t = 0, 4$. Thus $0 < t < 4$ since we are given initial conditions at time $t = 3$.

Problem: (B&D 3.3 #6) Are $f = t$ and $g = 1/t$ linearly independent or dependent?

It is easy to evaluate the Wronskian for this case:

$$W(f, g) = fg' - f'g = t(-1/t^2) - 1/t = -2/t \neq 0\,.$$

Thus the functions are linearly independent.

Problem: (B&D 3.3 #16) Without solving it, what is the Wronskian of $(\cos t)y'' + (\sin t)y' - ty = 0$.

We first write the equation in the standard form

$$y'' + (\tan t)y' - (t \sec t)y = 0$$

Using Abel's theorem, we know that the Wronkian is given by

$$W(y_1, y_2) = C \exp\left[-\int p(t)dt\right] = C \exp\left[-\int \tan t dt\right] = C \exp\left[\ln(\cos t)\right] = C \cos t\,.$$

Problem: (B&D 3.4 #7) Solve $y'' - 2y' + 2y = 0$.

We begin by letting $y = \exp(\lambda t)$ so that we find the characteristic equation

$$\lambda^2 - 2\lambda + 2 = 0 \quad \rightarrow \quad \lambda_\pm = 1 \pm i \quad \rightarrow \quad \lambda = 1 + i, 1 - i\,.$$

The solution is then given by

$$y(t) = c_1 \exp(t) \cos t + c_2 \exp(t) \sin t\,.$$

Problem: (B&D 3.4 #17) Solve $y'' + 4y = 0$ with $y(0) = 0$ and $y'(0) = 1$.

We begin by letting $y = \exp(\lambda t)$ so that we find the characteristic equation

$$\lambda^2 + 4 = 0 \quad \rightarrow \quad \lambda_\pm = \pm 2i \quad \rightarrow \quad \lambda = 2i, -2i\,.$$

The solution is then given by $y(t) = c_1 \cos 2t + c_2 \sin 2t$. The initial conditions give $y(0) = 0 = c_1$ and $y'(0) = 1 = 2c_2$ so that $c_2 = 1/2$ and $c_1 = 0$ and

$$y(t) = \frac{\sin 2t}{2}\,.$$

Problem: (B&D 3.4 #22) Solve $y'' + 2y' + 2y = 0$ with $y(\pi/4) = 2$ and $y'(\pi/4) = -2$.

We begin by letting $y = \exp(\lambda t)$ so that we find the characteristic equation

$$\lambda^2 + 2\lambda + 2 = 0 \quad \rightarrow \quad \lambda_\pm = -1 \pm i \quad \rightarrow \quad \lambda = -1 + i, -1 - i\,.$$

The solution is then given by $y(t) = c_1 \exp(-t) \cos t + c_2 \exp(-t) \sin t$. The initial conditions give $y(\pi/4) = 2 = (2/\sqrt{2})(c_1 \exp(-\pi/4) + c_2 \exp(-\pi/4))$ and $y'(\pi/4) = -2 = (2/\sqrt{2})[(c_2 - c_1) \exp(-\pi/4) - (c_2 + c_1) \exp(-\pi/4)]$ so that $c_1 = \sqrt{2} \exp(\pi/4)$ and $c_2 = \sqrt{2} \exp(\pi/4)$ and

$$y(t) = \sqrt{2} \exp(-t + \pi/4) \cos t + \sqrt{2} \exp(-t + \pi/4) \sin t\,.$$

Problem: (B&D 3.5 #14) Solve $y'' + 4y' + 4y = 0$ with $y(-1) = 2$ and $y'(-1) = 1$.

We begin by letting $y = \exp(\lambda t)$ so that we find the characteristic equation

$$\lambda^2 + 4\lambda + 4 = (\lambda + 2)^2 = 0 \quad \rightarrow \quad \lambda_\pm = -2\,.$$

The solution is then given by $y(t) = c_1 \exp(-2t) + c_2 t \exp(-2t)$. The initial conditions give $y(-1) = 2 = c_1 \exp(2) - c_2 \exp(2)$ and $y'(-1) = 1 = -2c_1 \exp(2) + 2c_2 \exp(2) + c_2 \exp(2)$ so that $c_1 = 7/\exp(2)$ and $c_2 = 5/\exp(2)$ and

$$y(t) = 7\exp[-2(t+1)] + 5t\exp[-2(t+1)]\,.$$

Problem: (B&D 3.5 #25) Use reduction of order to find a second solution of $t^2y'' + 3ty' + y = 0$ given the solution $y_1 = 1/t$ for $t > 0$.

The reduction of order method assumes a second solution takes the form

$$y = v(t)y_1 = v(t)/t$$

Before plugging into our original equation, we note that

$$\begin{aligned} y' &= \frac{tv' - v}{t^2} \\ y'' &= \frac{(tv'')t^2 - 2t(tv' - v)}{t^4} = \frac{t^2v'' - 2tv' + 2v}{t^3}\,. \end{aligned}$$

Plugging this into our original equation yields

$$\begin{aligned} &\frac{1}{t}(t^2v'' - 2tv' + 2v) + \frac{1}{t}(3tv' - 3v)\frac{1}{t}v = 0 \\ &t^2v'' - 2tv' + 3tv' = 0 \\ &t^2v'' + tv' = 0 \\ &v'' = -\frac{1}{t}v'\,. \end{aligned}$$

We can easily solve this first order equation for v' by letting $u = v'$. Thus we have

$$u' = -\frac{1}{t}u \quad \rightarrow \quad \frac{du}{u} = -\frac{dt}{t}\,.$$

Integrating both sides gives

$$\ln u = -\ln t + c \quad \rightarrow \quad u = C/t\,.$$

Recalling that $u = v'$ then gives

$$v' = \frac{C}{t} \quad \rightarrow \quad v = C\ln t\,.$$

Finally we recall that our second solution was $y = v(t)y_1$ so that we have

$$y_2 = \frac{\ln t}{t}\,.$$

Problem: (B&D 3.6 #8) Solve $y'' + y = 3\sin 2t + t\cos 2t$.

We begin by finding the solution of the homogenous equation $y''+y = 0$ by letting $y = \exp(\lambda t)$. This gives the characteristic equation

$$\lambda^2 + 1 = 0 \quad \rightarrow \quad \lambda = \pm i$$

and the homogeneous solution $y_h = c_1 \cos t + c_2 \sin t$. The next step is to determine a particular solution y_p which gives rise to our nonhomogeneous term $g(t) = 3\sin 2t + t\cos 2t$. We thus guess the following:

$$y_p = A\sin 2t + B\cos 2t + Ct\sin 2t + Dt\cos 2t$$

where the constants A, B, C, and D are to be determined. We first calculate

$$\begin{aligned} y' &= (2A + D)\cos 2t + (C - 2B)\sin 2t + 2Ct\cos 2t - 2Dt\sin 2t \\ y'' &= -4(A + D)\sin 2t + 4(C - B)\cos 2t - 4Ct\sin 2t - 4Dt\cos 2t\,. \end{aligned}$$

Plugging into the original equation then gives

$$\begin{aligned} &(-4A - 4D + A)\sin 2t + (4C - 4B + B)\cos 2t \\ &\quad +(C - 4C)t\sin 2t + (D - 4D)t\cos 2t = 3\sin 2t + t\cos 2t \end{aligned}$$

which gives the four equations and four unknowns $-3A - 4D = 3$, $4C - 3B = 0$, $-3C = 0$, and $-3D = 1$. This gives $C = B = 0$, $D = -1/3$, $A = -5/9$, and the general solution

$$y = c_1 \cos t + c_2 \sin t - (5/9)\sin 2t - (t/3)\cos 2t\,.$$

Problem: (B&D 3.6 #12) Solve $y'' - y' - 2y = \cosh 2t$.

We begin by finding the solution of the homogenous equation $y'' - y' - 2y = 0$ by letting $y = \exp(\lambda t)$. This gives the characteristic equation

$$\lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1) = 0 \quad \rightarrow \quad \lambda = 2, -1$$

and the homogeneous solution $y_h = c_1 \exp(2t) + c_2 \exp(-t)$. The next step is to determine a particular solution y_p which gives rise to our nonhomogeneous term $g(t) = \cosh 2t = (\exp(2t) + \exp(-2t))/2$. We thus guess the following:

$$y_p = At\exp(2t) + B\exp(-2t)$$

where the $t\exp(2t)$ is necessary since $\exp(2t)$ is a solution to the homogeneous problem. We first calculate

$$\begin{aligned} y' &= A\exp(2t) + 2At\exp(2t) - 2B\exp(-2t) \\ y'' &= 4A\exp(2t) + 4At\exp(2t) + 4B\exp(-2t)\,. \end{aligned}$$

Plugging into the original equation then gives

$$(4A - A)e^{2t} + (4A - 2A - 2A)te^{2t} + (4B + 2B - 2B)e^{-2t} = (e^{2t} + e^{-2t})/2$$

which gives the three equations and two unknowns $3A = 1/2$, $4A - 4A = 0$, and $4B = 1/2$. This gives $B = 1/8$, $A = 1/6$, and the general solution

$$y = c_1 \exp(2t) + c_2 \exp(-t) + (t/6)\exp(2t) + (1/8)\exp(-2t)\,.$$

Problem: (B&D 3.6 #7) Solve $2y'' + 3y' + y = t^2 + 3\sin t$.

We begin by finding the solution of the homogenous equation $2y'' + 3y' + y = 0$ by letting $y = \exp(\lambda t)$. This gives the characteristic equation

$$2\lambda^2 + 3\lambda + 1 = (2\lambda + 1)(\lambda + 1) = 0 \quad \to \quad \lambda = -1/2,\ -1$$

and the homogeneous solution $y_h = c_1\exp(-t/2) + c_2\exp(-t)$. The next step is to determine a particular solution y_p which gives rise to our nonhomogeneous term $g(t) = t^2 + 3\sin t$. We thus guess the following:

$$y_p = A\sin t + B\cos t + Ct^2 + Dt + E$$

where the constants A, B, C, D, and E are to be determined. We first calculate

$$y' = A\cos t - B\sin t + 2Ct + D \quad \text{and} \quad y'' = -A\sin t - B\cos t + 2C\,.$$

Plugging into the original equation then gives

$$\begin{aligned}(-2A - 3B + A)\sin t + (-2B + 3A + B)\cos t \\ +(C)t^2 + (6C + D)t + (4C + 3D + E) = t^2 + 3\sin t\end{aligned}$$

which gives the equations: $-A - 3B = 3$, $3A - B = 0$, $C = 1$, $6C + D = 0$, and $4C + 3D + E = 0$. Thus we find $A = -3/10$, $B = -9/10$, $C = 1$, $D = -6$, $E = 14$, and the general solution

$$y = c_1\exp(-t/2) + c_2\exp(-t) - (3/10)\sin t - (9/10)\cos t + t^2 - 6t + 14\,.$$

Problem: (B&D 3.7 #3) Solve $y''+2y'+y = 3\exp(-t)$ using variation of parameters.

To find the solution of the homogenous equation $y'' + 2y' + y = 0$ we let $y = \exp(\lambda t)$. This gives the characteristic equation

$$\lambda^2 + 2\lambda + 1 = (\lambda + 1)^2 = 0 \quad \to \quad \lambda = -1 \ \text{(double root)}$$

and the homogeneous solution $y_h = c_1\exp(-t) + c_2 t\exp(-t)$. Thus $y_1 = \exp(-t)$ and $y_2 = t\exp(-t)$ and we calculate the Wronskian

$$W(y_1, y_2) = \exp(-2t) - t\exp(-2t) + t\exp(-2t) = \exp(-2t)\,.$$

It remains to evaluate the two integrals

$$\begin{aligned}\int \frac{y_2 g(t)}{W(t)}dt &= \int \frac{t\exp(-t)\cdot 3\exp(-t)}{\exp(-2t)}dt = 3\int t\,dt = 3t^2/2 \\ \int \frac{y_1 g(t)}{W(t)}dt &= \int \frac{\exp(-t)\cdot 3\exp(-t)}{\exp(-2t)}dt = 3\int dt = 3t\,.\end{aligned}$$

Our particular solution is then

$$y_p = y_2\int \frac{y_1 g(t)}{W(t)}dt - y_1\int \frac{y_2 g(t)}{W(t)}dt = t\exp(-t)\cdot 3t - \exp(-t)\cdot 3t^2/2 = (3/2)t^2\exp(-t)$$

which gives the general solution

$$y = c_1\exp(-t) + c_2 t\exp(-t) + (3/2)t^2\exp(-t)\,.$$

Problem: (B&D 3.7 #4) Solve $4y'' - 4y' + y = 16\exp(t/2)$ using variation of parameters and the method of undetermined coefficients.

We begin by using the method of undetermined coefficients. To find the solution of the homogenous equation $4y'' - 4y' + y = 0$ we let $y = \exp(\lambda t)$. This gives the characteristic equation

$$4\lambda^2 - 4\lambda + 1 = (2\lambda - 1)(2\lambda - 1) = 0 \quad \rightarrow \quad \lambda = 1/2 \ \text{(double root)}$$

and the homogeneous solution $y_h = c_1 \exp(t/2) + c_2 t \exp(t/2)$. The next step is to determine a particular solution y_p which gives rise to our nonhomogeneous term $g(t) = 16\exp(t/2)$. But since both $\exp(t/2)$ and $t\exp(t/2)$ are solutions of the homogeneous problem, we must guess the following

$$y_p = At^2 \exp(t/2)$$

where the constants A is to be determined. We first calculate

$$\begin{aligned} y' &= 2tA\exp(t/2) + (A/2)t^2\exp(t/2) \\ y'' &= 2A\exp(t/2) + 2At\exp(t/2) + (A/4)t^2\exp(t/2)\,. \end{aligned}$$

Plugging into the original equation then gives

$$\begin{aligned} &\exp(t/2)\left[2A + 2At + (A/4)t^2 - 2At - (A/2)t^2 + (A/4)t^2\right] = 4\exp(t/2) \\ &At^2(1/4 - 1/2 + 1/4) + At(2A - 2A) + 2A = 4 \end{aligned}$$

where we have divided through by a factor of four. Note that the terms proportional to both t^2 and t both cancel out automatically. This then gives $2A = 4$ so that $A = 2$. The general solution is then

$$y = c_1\exp(t/2) + c_2 t\exp(t/2) + 2t^2\exp(t/2)\,.$$

We can compare this to the variation of parameters method. We first must put the governing equation into the proper form:

$$y'' - y' + y/4 = 4\exp(t/2)\,.$$

Recall from the previous calculation of the homogeneous solution that $y_1 = \exp(t/2)$ and $y_2 = t\exp(t/2)$. This allows us to calculate the Wronskian

$$W(y_1, y_2) = \exp(t) + t\exp(t/2)/2 - t\exp(t/2)/2 = \exp(t)\,.$$

It remains to evaluate the two integrals

$$\begin{aligned} \int \frac{y_2 g(t)}{W(t)}dt &= \int \frac{t\exp(t/2)\cdot 4\exp(t/2)}{\exp(t)}dt = 4\int t dt = 2t^2 \\ \int \frac{y_1 g(t)}{W(t)}dt &= \int \frac{\exp(t/2)\cdot 4\exp(t/2)}{\exp(t)}dt = 4\int dt = 4t\,. \end{aligned}$$

Our particular solution is then

$$y_p = y_2\int\frac{y_1 g(t)}{W(t)}dt - y_1\int\frac{y_2 g(t)}{W(t)}dt = t\exp(t/2)\cdot 4t - \exp(t/2)\cdot 2t^2 = 2t^2\exp(t/2)$$

which gives the same general solution as found previously with undetermined coefficients.

Problem: (B&D 5.2 #5) Solve $(1-x)y'' + y = 0$ about $x_0 = 0$ using a power series solution.

We first note that the solution is not singular around $x = 0$. Thus a power series expansion can take the form $y(x) = \sum_{n=0}^{\infty} a_n x^n$ where the a_n are to be determined from plugging into the governing equation. We first note that taking the derivative twice gives $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$ and $y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$. Plugging in then gives

$$(1-x)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0\,.$$

And upon multiplying through by the $(1-x)$ factor we find

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0\,.$$

We simplify this by letting $m = n-2$ in the first sum and $m = n-1$ in the second sum. Simplifying the result gives

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=0}^{\infty} n(n+1)a_{n+1}x^n + \sum_{n=0}^{\infty} a_n x^n = 0\,.$$

where we have switched back to n as opposed to m and let the second sum start at $n = 0$. Combining these gives the single equation

$$\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2} - n(n+1)a_{n+1} + a_n\right]x^n = 0\,.$$

which results in the recursion relation

$$(n+2)(n+1)a_{n+2} - n(n+1)a_{n+1} + a_n = 0 \quad \rightarrow \quad a_{n+2} = \frac{n(n+1)a_{n+1} - a_n}{(n+2)(n+1)}\,.$$

So if we know a_0 and a_1, we can then calculate all other coefficients:

$$\begin{aligned}
a_2 &= -\frac{a_0}{2} \\
a_3 &= \frac{2a_2 - a_1}{2\cdot 3} = -\frac{a_0 + a_1}{2\cdot 3} \\
a_4 &= \frac{2\cdot 3a_3 - a_2}{3\cdot 4} = -\frac{a_0 + 2a_1}{2\cdot 3\cdot 4} \\
a_5 &= \frac{3\cdot 4a_4 - a_3}{4\cdot 5} = -\frac{2a_0 + 5a_1}{2\cdot 3\cdot 4\cdot 5}\,.
\end{aligned}$$

We can then write down the power series

$$y(x) = a_0 + a_1 x - \frac{a_0}{2}x^2 - \frac{a_0 + a_1}{2\cdot 3}x^3 - \frac{a_0 + 2a_1}{2\cdot 3\cdot 4}x^4 - \frac{2a_0 - 5a_1}{2\cdot 3\cdot 4\cdot 5}x^5 + \cdots\,.$$

Separating out the a_0 and a_1 terms gives us what we want

$$y(x) = a_0\left[1 - \frac{x^2}{2} - \frac{x^3}{2\cdot 3} - \frac{x^4}{2\cdot 3\cdot 4} + \cdots\right] + a_1\left[x - \frac{x^3}{2\cdot 3} - \frac{x^4}{3\cdot 4} - \frac{x^5}{2\cdot 3\cdot 4} + \cdots\right]\,.$$

Problem: (B&D 5.2 #7) Solve $y'' + xy' + 2y = 0$ about $x_0 = 0$ using a power series solution.

We first note that the solution is not singular around $x = 0$. Thus a power series expansion can take the form $y(x) = \sum_{n=0}^{\infty} a_n x^n$ where the a_n are to be determined from plugging into the governing equation. We first note that taking the derivative twice gives $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$ and $y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$. Plugging in then gives

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + x \sum_{n=1}^{\infty} n a_n x^{n-1} + 2 \sum_{n=0}^{\infty} a_n x^n = 0\,.$$

And upon multiplying through by the x factor we find

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} 2 a_n x^n = 0\,.$$

We simplify this by letting $m = n - 2$ in the first term which gives

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n + + \sum_{n=0}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} 2 a_n x^n = 0\,.$$

where we have switched back to n as opposed to m and let the second sum start at $n = 0$. Combining these gives the single equation

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} + n a_n + 2a_n\right] x^n = 0\,.$$

which results in the recursion relation

$$(n+2)(n+1)a_{n+2} + (n+2)a_n = 0 \quad \rightarrow \quad a_{n+2} = -\frac{a_n}{n+1}\,.$$

So if we know a_0 and a_1, we can then calculate all other coefficients:

$$\begin{aligned} a_2 &= -a_0 \\ a_3 &= -a_1/2 \\ a_4 &= -a_2/3 = a_0/3 \\ a_5 &= -a_3/4 = a_1/(2 \cdot 4) \\ a_6 &= -a_4/5 = -a_0/(3 \cdot 5) \\ a_7 &= -a_5/6 = -a_1/(2 \cdot 4 \cdot 6)\,. \end{aligned}$$

We can then write down the power series

$$y(x) = a_0 + a_1 x - \frac{a_0}{2} x^2 - \frac{a_0 + a_1}{2 \cdot 3} x^3 - \frac{a_0 + 2a_1}{2 \cdot 3 \cdot 4} x^4 - \frac{2a_0 - 5a_1}{2 \cdot 3 \cdot 4 \cdot 5} x^5 + \cdots\,.$$

Separating out the a_0 and a_1 terms gives us what we want

$$y(x) = a_0 \left[1 - x^2 + \frac{x^4}{3} - \frac{x^6}{3 \cdot 5} + \cdots\right] + a_1 \left[x - \frac{x^3}{2} + \frac{x^5}{2 \cdot 4} - \frac{x^7}{2 \cdot 4 \cdot 6} + \cdots\right]\,.$$

Problem: (B&D 5.5 #3) Solve $x^2y'' - 3xy' + 4y = 0$.

We begin by letting $y = x^r$ so that

$$r(r-1) - 3r + 4 = 0$$

and we find the characteristic equation

$$r^2 - 4r + 4 = 0 \quad \to \quad (r-2)^2 = 0 \quad \to \quad r = 2 \text{ (double root)}\,.$$

The solution is then given by

$$y(t) = c_1 x^2 + c_2 x^2 \ln|x|\,.$$

Problem: (B&D 5.5 #4) Solve $x^2y'' + 3xy' + 5y = 0$.

We begin by letting $y = x^r$ so that

$$r(r-1) + 3r + 5 = 0$$

and we find the characteristic equation

$$r^2 + 2r + 5 = 0 \quad \to \quad r = \frac{-2 \pm \sqrt{4-20}}{2} \quad \to \quad r = -1 \pm 2i\,.$$

The solution is then given by

$$y(t) = c_1 x^{-1}\cos(2\ln|x|) + c_2 x^{-1}\sin(2\ln|x|)\,.$$

Problem: (B&D 5.5 #12) Solve $x^2y'' - 4xy' + 4y = 0$.

We begin by letting $y = x^r$ so that

$$r(r-1) - 4r + 4 = 0$$

and we find the characteristic equation

$$r^2 - 5r + 4 = 0 \quad \to \quad (r-4)(r-1) = 0 \quad \to \quad r = 1, 4\,.$$

The solution is then given by

$$y(t) = c_1 x^4 + c_2 x\,.$$

Problem: (B&D 5.5 #13) Solve $2x^2y'' + xy' - 3y = 0$ with the initial conditions $y(1) = 1$ and $y'(1) = 4$.

We begin by letting $y = x^r$ so that

$$2r(r-1) + r - 3 = 0$$

and we find the characteristic equation

$$2r^2 - r - 3 = 0 \quad \to \quad (2r-3)(r+1) = 0 \quad \to \quad r = \frac{3}{2}, -1\,.$$

The solution is then given by $y(t) = c_1 x^{3/2} + c_2 x^{-1}$. With the initial conditions, we find that $y(1) = 1 = c_1 + c_2$ and $y'(1) = 4 = 3c_1/2 - c_2$ which gives $c_1 = 2$ and $c_2 = -1$. Our solution is then

$$y(t) = 2x^{3/2} - x^{-1}\,.$$

Problem: (B&D 5.6 #5) Solve $3x^2y'' + 2xy' + x^2y = 0$ about $x_0 = 0$ using a power series solution.

We first note that the solution is a regular singular point at $x = 0$ since $P(x) = 3x^2$, $Q(x) = 2x$ and $R(x) = x^2$, and

$$\lim_{x\to 0} x\frac{Q(x)}{P(x)} = \lim_{x\to 0} x\frac{2x}{3x^2} = \frac{2}{3}$$
$$\lim_{x\to 0} x^2\frac{R(x)}{P(x)} = \lim_{x\to 0} x^2\frac{x^2}{3x^2} = \lim_{x\to 0} 3x^2 = 0\,.$$

Thus a power series expansion can take the form $y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$ where the a_n and r are to be determined from plugging into the governing equation. We first note that taking the derivative twice gives $y' = \sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}$ and $y'' = \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}$. Plugging in then gives

$$3x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} + 2x\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} + x^2\sum_{n=0}^{\infty}a_n x^{n+r} = 0\,.$$

And upon multiplying through by the x factors we find

$$\sum_{n=0}^{\infty}3(n+r)(n+r-1)a_n x^{n+r} + \sum_{n=0}^{\infty}2(n+r)a_n x^{n+r} + \sum_{n=0}^{\infty}a_n x^{n+r+2} = 0\,.$$

We simplify this by letting $m = n+2$ in the last term and combining the first two terms which gives

$$\sum_{n=0}^{\infty}\left[3(n+r)(n+r-1) + 2(n+r)\right]a_n x^{n+r} + \sum_{n=2}^{\infty}a_{n-2}x^{n+r} = 0\,.$$

where we have switched back to n as opposed to m. In order to combine these terms, we must have the sums add over the same indices n. Therefore, we need to take out the first two terms, i.e. $n = 0$ and $n = 1$, of the first term so that the two sums are both in powers of x^{n+r} and summed from 2 to ∞. This then gives

$$a_0x^r\left[r(3r-1)\right] + a_1x^{r+1}\left[3r^2+5r+2\right] + \sum_{n=2}^{\infty}\left\{\left[3(n+r)(n+r-1)+2(n+r)\right]a_n + a_{n-2}\right\}x^{n+r} = 0\,.$$

We now equate each power of x^{n+r} to zero in order to satisfy the differential equation. Since a_0 is not zero, i.e. this term captures the singular behavior, we then have from the first term that

$$r(3r-1) = 0 \quad\to\quad r = 0, 1/3\,.$$

This then determines the behavior near the singular point $x = 0$. Equating the next coefficient to zero gives

$$a_1\left[3r^2+5r+2\right] = 0 \quad\to\quad a_1 = 0$$

which results in $a_1 = 0$ since with $r = 0$ or $r = 1/3$ the term in brackets does not go to zero.

We are now ready for our recursion relationship. This arises from the term which sums the remaining terms from 2 to ∞. Setting the coefficients of x^{n+r} to zero results in the recursion relation:

$$[3(n+r)(n+r-1) + 2(n+r)]\, a_n + a_{n-2} = 0 \quad \rightarrow \quad a_n = -\frac{a_{n-2}}{(n+r)(3(n+r)-1)}\,.$$

So if we know a_0, we can then determine a_2 which gives a_4 and so on. Similarly, since $a_1 = 0$ we find that all the odd terms must be zero, i.e. $a_3 = a_5 = a_7 = \cdots = 0$.

We can now construct the two linearly independent solutions. We first consider the $r = 1/3$ root. This gives the recursion relation

$$a_n = -\frac{a_{n-2}}{(n+1/3)(3(n+1/3)-1)} = -\frac{a_{n-2}}{n(3n+1)}$$

which gives the terms

$$\begin{aligned} a_2 &= -\frac{a_0}{2\cdot 7} \\ a_4 &= -\frac{a_2}{4\cdot 13} = \frac{a_0}{2\cdot 4\cdot 7\cdot 13} \\ a_6 &= -\frac{a_4}{6\cdot 19} = -\frac{a_0}{2\cdot 4\cdot 6\cdot 7\cdot 13\cdot 19}\,. \end{aligned}$$

We can then write down the power series associated with this root

$$y_1(x) = a_0 x^{1/3}\left[1 - \frac{x^2}{2\cdot 7} + \frac{x^4}{2\cdot 4\cdot 7\cdot 13} - \frac{x^6}{2\cdot 4\cdot 6\cdot 7\cdot 13\cdot 19} + \cdots\right]$$

where a_0 would be determined from initial conditions.

We can now construct the second linearly independent solution. Thus we consider the $r = 0$ root for which the recursion relation reduces to

$$a_n = -\frac{a_{n-2}}{n(3n-1)}\,.$$

This gives the first few terms

$$\begin{aligned} a_2 &= -\frac{a_0}{2\cdot 5} \\ a_4 &= -\frac{a_2}{4\cdot 11} = \frac{a_0}{2\cdot 4\cdot 5\cdot 11} \\ a_6 &= -\frac{a_4}{6\cdot 17} = -\frac{a_0}{2\cdot 4\cdot 6\cdot 5\cdot 11\cdot 17}\,. \end{aligned}$$

We can then write down the power series associated with this root

$$y_2(x) = a_0\left[1 - \frac{x^2}{2\cdot 5} + \frac{x^4}{2\cdot 4\cdot 5\cdot 11} - \frac{x^6}{2\cdot 4\cdot 6\cdot 5\cdot 11\cdot 17} + \cdots\right]$$

where $x^0 = 1$ and a_0 would be determined from initial conditions. The total solution is then

$$y(x) = c_1 x^{1/3}\left[1 - \frac{x^2}{2\cdot 7} + \frac{x^4}{2\cdot 4\cdot 7\cdot 13} + \cdots\right] + c_2\left[1 - \frac{x^2}{2\cdot 5} + \frac{x^4}{2\cdot 4\cdot 5\cdot 11} + \cdots\right]\,.$$

Problem: (B&D 5.6 #9) Solve $x^2y'' - x(x+3)y' + (x+3)y = 0$ about $x_0 = 0$ using a power series solution.

We first note that the solution is a regular singular point at $x = 0$ since $P(x) = x^2$, $Q(x) = -x(x+3)$ and $R(x) = x + 3$, and

$$\lim_{x\to 0} x\frac{Q(x)}{P(x)} = \lim_{x\to 0} x\frac{-x(x+3)}{x^2} = \lim_{x\to 0} -(x+3) = -3$$
$$\lim_{x\to 0} x^2\frac{R(x)}{P(x)} = \lim_{x\to 0} x^2\frac{(x+3)}{x^2} = \lim_{x\to 0} (x+3) = 3\,.$$

Thus a power series expansion can take the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$$

where the a_n and r are to be determined from plugging into the governing equation. We first note that taking the derivative twice gives

$$y' = \sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1} \quad \text{and} \quad y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-2}\,.$$

Plugging in then gives

$$x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} - x(x+3)\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} + (x+3)\sum_{n=0}^{\infty} a_n x^{n+r} = 0\,.$$

And upon multiplying through by the x^2 and x in the first two terms, we find

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} - (x+3)\sum_{n=0}^{\infty}(n+r)a_n x^{n+r} + (x+3)\sum_{n=0}^{\infty} a_n x^{n+r} = 0\,.$$

We simplify this by collecting the resulting terms of power x^{n+r} and x^{n+r+1}.

$$\sum_{n=0}^{\infty}\left[(n+r)(n+r-1) - 3(n+r) + 3\right] a_n x^{n+r} - \sum_{n=0}^{\infty}(n+r-1)a_n x^{n+r+1} = 0\,.$$

We now let $m = n + 1$ in the second term to arrive at

$$\sum_{n=0}^{\infty}\left[(n+r)(n+r-1) - 3(n+r) + 3\right] a_n x^{n+r} - \sum_{n=1}^{\infty}(n+r-2)a_{n-1} x^{n+r} = 0$$

where we have switched back to n as opposed to m. In order to combine these terms, we must have the sums add over the same indices n. Therefore, we need to take out the first term, i.e. $n = 0$, of the first term so that the two sums are both in powers of x^{n+r} and summed from 1 to ∞. This then gives

$$a_0 x^r\left[r(r-1) - 3r + 3\right] +$$
$$+ \sum_{n=1}^{\infty}\left\{\left[(n+r)(n+r-1) - 3(n+r) + 3\right] a_n - (n+r-2)a_{n-1}\right\} x^{n+r} = 0\,.$$

We now equate each power of x^{n+r} to zero in order to satisfy the differential equation. Since a_0 is not zero, i.e. this term captures the singular behavior, we then have from the first term that

$$r(r-1) - 3r + 3 = 0$$

which multiplied out gives

$$r^2 - 4r + 3 = (r-3)(r-1) = 0$$

whose roots are easily found to be

$$r = 3, 1\,.$$

This then determines the behavior near the singular point $x = 0$ to be given by x^3 and x^1. Unlike the previous example, this is the only coefficient that is equated to zero separately from the rest of the sum. Equating the next coefficient to zero gives the recursion relationship. This arises from the term which sums the remaining terms from 1 to ∞. Setting the coefficients of x^{n+r} to zero results in the recursion relation:

$$[(n+r)(n+r-1) - 3(n+r) + 3]\, a_n - (n+r-2)a_{n-1} = 0$$

So then

$$a_n = \frac{(n+r-2)a_{n-1}}{(n+r-1)(n+r-3)}\,.$$

So if we know a_0, we can then determine a_1 which gives a_2 and so on.

We can now construct the linearly independent solution associated with the highest root. Thus we take $r = 3$ which gives the recursion relation

$$a_n = \frac{(n+1)a_{n-1}}{n(n+2)}$$

which gives the terms

$$\begin{aligned} a_1 &= \frac{2a_0}{3} \\ a_2 &= \frac{3a_1}{2\cdot 4} = \frac{a_0}{4} \\ a_3 &= \frac{4a_2}{3\cdot 5} = \frac{a_0}{3\cdot 5} \\ a_4 &= \frac{5a_3}{4\cdot 6} = \frac{a_0}{3\cdot 4\cdot 6}\,. \end{aligned}$$

We can then write down the power series associated with this root

$$y_1(x) = a_0 x^3 \left[1 + \frac{2x}{3} + \frac{x^2}{4} + \frac{x^3}{3\cdot 5} + \frac{x^4}{3\cdot 4\cdot 6} + \cdots\right]$$

where a_0 would be determined from initial conditions.

Problem: (B&D 6.1 #5) Laplace transform (a) t, (b) t^2 and (c) t^n.

(a) The Laplace transform of t is given by (integrating by parts)

$$\mathcal{L}\{t\} = \int_0^\infty t\exp(-st)dt = \frac{-t}{s}\exp(-st)|_0^\infty + \frac{1}{s}\int_0^\infty \exp(-st)dt = -\frac{\exp(-st)|_0^\infty}{s^2} = \frac{1}{s^2}$$

(b) The Laplace transform of t^2 is given by (integrating by parts)

$$\mathcal{L}\{t^2\} = \int_0^\infty t^2\exp(-st)dt = \frac{-t^2}{s}\exp(-st)|_0^\infty + \frac{2}{s}\int_0^\infty t\exp(-st)dt = \frac{2}{s}\mathcal{L}\{t\} = \frac{2}{s^3}$$

(c) The Laplace transform of t^n is given by (integrating by parts)

$$\mathcal{L}\{t^n\} = \int_0^\infty t^n\exp(-st)dt = \frac{-t^n}{s}\exp(-st)|_0^\infty + \frac{n}{s}\int_0^\infty t^{n-1}\exp(-st)dt = \frac{n}{s}\mathcal{L}\{t^{n-1}\}\,.$$

We can then apply this recursively $(n-1)$ times to find

$$\mathcal{L}\{t^n\} = \frac{n(n-1)(n-2)\cdots 2\cdot 1}{s^{n-1}}\mathcal{L}\{t\} = \frac{n!}{s^{n-1}}\cdot\frac{1}{s^2} = \frac{n!}{s^{n+1}}$$

Problem: (B&D 6.1 #6) Laplace transform $f(t) = \cos(at)$.

We integrate by parts twice in this evaluation

$$\begin{aligned}\mathcal{L}\{\cos at\} &= \int_0^\infty e^{-st}\cos at dt = \frac{1}{a}e^{-st}\sin at|_0^\infty + \frac{s}{a}\int_0^\infty e^{-st}\sin at dt \\ &= -\frac{s}{a^2}e^{-st}\cos at|_0^\infty + \frac{s^2}{a^2}\int_0^\infty e^{-st}\cos at dt = \frac{s}{a^2} - \frac{s^2}{a^2}\mathcal{L}\{\cos at\}\end{aligned}$$

which can be rearranged to give

$$\left(1+\frac{s^2}{a^2}\right)\mathcal{L}\{\cos at\} = \frac{s}{a^2} \quad\rightarrow\quad \mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2} \quad (s>0)\,.$$

Problem: (B&D 6.1 #15) Laplace transform $t\exp(at)$.

The Laplace transform of $t\exp(at)$ is given by (integrating by parts)

$$\mathcal{L}\{te^{at}\} = \int_0^\infty te^{-(s-a)t}dt = \frac{-te^{-(s-a)t}|_0^\infty}{s-a} + \int_0^\infty \frac{e^{-(s-a)t}}{s-a}dt = -\frac{e^{-st}|_0^\infty}{(s-a)^2} = \frac{1}{(s-a)^2}$$

Problem: (B&D 6.2 #11) Solve $y'' - y' - 6y = 0$ with $y(0) = 1$ and $y'(0) = -1$.

We begin by Laplace transforming the equation

$$s^2Y - sy(0) - y'(0) - [sY - y(0)] - 6Y = Y(s^2 - s - 6) - s + 2 = 0\,.$$

Solving for $Y(s)$ then gives

$$Y(s) = \frac{s-2}{s^2-s-6} = \frac{s-2}{(s-3)(s+2)} = \frac{1}{5}\frac{1}{s-3} + \frac{4}{5}\frac{1}{s+2}$$

which upon inversion gives the solution

$$y(t) = (1/5)\exp(3t) + (4/5)\exp(-2t)\,.$$

Problem: (B&D 6.2 #13) Solve $y'' - 2y' + 2y = 0$ with $y(0) = 0$ and $y'(0) = 1$.

We begin by Laplace transforming the equation

$$s^2Y - sy(0) - y'(0) - 2\left[sY - y(0)\right] + 2Y = Y(s^2 - 2s + 2) - 1 = 0\,.$$

Solving for $Y(s)$ then gives

$$Y(s) = \frac{1}{s^2 - 2s + 2} = \frac{1}{(s-1)^2 + 1}$$

which upon inversion (#24 with $a = 1$ and $b = 1$) gives the solution

$$y(t) = \exp(t)\sin t\,.$$

Problem: (B&D 6.2 #15) Solve $y'' - 2y' - 2y = 0$ with $y(0) = 2$ and $y'(0) = 0$.

We begin by Laplace transforming the equation

$$s^2Y - sy(0) - y'(0) - 2\left[sY - y(0)\right] - 2Y = Y(s^2 - 2s - 2) - 2s + 4 = 0\,.$$

Solving for $Y(s)$ then gives

$$Y(s) = \frac{2s - 4}{s^2 - 2s - 2} = \frac{2(s-1) - 2}{(s-1)^2 - 3} = 2\frac{(s-1)}{(s-1)^2 - 3} - 2\frac{1}{(s-1)^2 - 3}$$

which upon inversion (#24 and #25 with $a = 1$ and $b = i\sqrt{3}$) gives the solution

$$y(t) = 2\exp(t)\cos(i\sqrt{3}t) - \frac{2}{i\sqrt{3}}\exp(t)\sin(i\sqrt{3}t)$$

which upon noting that $\cos(iAt) = \cosh(At)$ and $\sin(iAt) = i\sinh(At)$ gives

$$y(t) = 2\exp(t)\cosh(\sqrt{3}t) - \frac{2\sqrt{3}}{3}\exp(t)\sinh(\sqrt{3}t)$$

Problem: (B&D 6.3 #9) Laplace transform $f(t) = 0$ for $t < \pi$, $f(t) = t - \pi$ for $\pi < t < 2\pi$, and $f(t) = 0$ for $t > 2\pi$.

We note that the function $f(t)$ is given by

$$f(t) = \begin{cases} 0 & t < \pi \\ t - \pi & \pi < t < 2\pi \\ 0 & t > 2\pi \end{cases} \quad \rightarrow \quad f(t) = u_\pi(t)(t - \pi) - u_{2\pi}(t)(t - \pi)\,.$$

We simplify the second term by noting $t - \pi = t - 2\pi + \pi$ so that

$$f(t) = u_\pi(t)(t - \pi) - u_{2\pi}(t)(t - 2\pi) - u_{2\pi}(t)\pi\,.$$

Laplace transforming then gives

$$F(s) = \frac{\exp(-\pi s)}{s^2} - \frac{\exp(-2\pi s)}{s^2} - \pi\frac{\exp(-2\pi s)}{s} = \frac{\exp(-\pi s)}{s^2} - (1 + \pi s)\frac{\exp(-2\pi s)}{s^2}\,.$$

Problem: (B&D 6.4 #5) Solve $y'' + 3y' + 2y = f(t)$ with $y(0) = y'(0) = 0$ and $f(t) = 1$ for $0 < t < 10$ and $f(t) = 0$ for $t > 10$.

We begin by rewriting the forcing $f(t)$ in terms of the Heaviside function $f(t) = 1 - u_{10}(t)$ so that Laplace transforming the equation gives

$$s^2Y - sy(0) - y'(0) + 3\left[sY - y(0)\right] + 2Y = Y(s^2+3s+2) = \frac{1}{s} - \frac{\exp(-10s)}{s}\,.$$

Solving for $Y(s)$ then gives

$$Y(s) = \frac{1}{s(s^2+3s+2)}(1-\exp(-10s)) = \frac{1}{s(s+2)(s+1)}(1-\exp(-10s))$$

which upon inversion (#12 with $a = 0$, $b = -2$, and $c = -1$) gives the solution

$$y(t) = \frac{1}{2}[1+\exp(-2t) - 2\exp(-t)] - \frac{1}{2}u_{10}(t)\left[1+\exp(-2(t-10)) - 2\exp(-(t-10))\right].$$

Problem: (B&D 6.4 #9) Solve $y'' + y = f(t)$ with $y(0) = 0$ and $y'(0) = 1$ and $f(t) = t/2$ for $0 < t < 6$ and $f(t) = 3$ for $t \geq 6$.

We begin by rewriting the forcing $f(t)$ in terms of the Heaviside function $f(t) = \frac{t}{2} - u_6(t)\left(3 - \frac{t}{2}\right) = \frac{t}{2} - \frac{1}{2}u_6(t)\,(t-6)$ so that Laplace transforming the equation gives

$$s^2Y - sy(0) - y'(0) + Y = Y(s^2+1) - 1 = \frac{1}{2s^2} - \frac{\exp(-6s)}{2s^2}\,.$$

Solving for $Y(s)$ then gives

$$Y(s) = \frac{1}{s^2+1} + \frac{1}{2}\frac{1}{s^2(s^2+1)}[1-\exp(-6s)]$$

which upon inversion (#13 with $a = 1$ and #18 with $a = 1$) gives the solution $y(t) = \sin t + (1/2)(t - \sin t) - (1/2)u_6(t)\left[(t-6) - \sin(t-6)\right]$ which simplifies to

$$y(t) = \frac{1}{2}\sin t + \frac{t}{2} - \frac{1}{2}u_6(t)\left[(t-6) - \sin(t-6)\right]\,.$$

Problem: (B&D 6.4 #10) Solve $y'' + y' + (5/4)y = f(t)$ with $y(0) = y'(0) = 0$ and $f(t) = \sin t$ for $0 < t < \pi$ and $f(t) = 0$ for $t \geq \pi$.

We begin by rewriting the forcing $f(t)$ in terms of the Heaviside function $f(t) = \sin t - u_\pi(t)\sin t = \sin t + u_\pi(t)\sin(t-\pi)$ so that Laplace transforming the equation gives

$$s^2Y + sy(0) + y'(0) + sY - y(0) + \frac{5}{4}Y = Y\left(s^2 + s + \frac{5}{4}\right) = \frac{1}{s^2+1} - \frac{\exp(-\pi s)}{s^2+1}.$$

Solving for $Y(s)$ then gives (a messy partial fraction decomposition is required)

$$\begin{aligned} Y(s) &= \frac{1}{(s^2+1)(s^2+s+5/4)}[1-e^{-\pi s}] = \frac{4}{17}\left(\frac{-4s+1}{s^2+1} + \frac{4s+3}{(s+1/2)^2+1}\right)[1-e^{-\pi s}] \\ &= \frac{4}{17}\left(\frac{1}{s^2+1} - 4\frac{s}{s^2+1} + 4\frac{(s+1/2)}{(s+1/2)^2+1} + \frac{1}{(s+1/2)^2+1}\right)[1-e^{-\pi s}] \end{aligned}$$

which upon inversion (#13, #14, #24, and #25) gives the solution

$$y(t) = f(t) - u_\pi(t)f(t-\pi) \;\text{ where }\; f(t) = \frac{4}{17}\left[\sin t - 4\cos t + e^{-t/2}\sin t + 4e^{-t/2}\cos t\right].$$

Problem: (B&D 6.5 #5) Solve $y''+2y'+3y=\sin t+\delta(t-3\pi)$ with $y(0)=y'(0)=0$.

Laplace transforming the equation gives

$$s^2Y-sy(0)-y'(0)+2\left[sY-y(0)\right]+3Y=Y(s^2+2s+3)=Y((s+1)^2+2)=\frac{1}{s^2+1}+e^{-3\pi s}\,.$$

Solving for $Y(s)$ then gives (via a messy fractional decomposition)

$$Y(s)=\frac{1}{(s^2+1)((s+1)^2+2)}+\frac{e^{-3\pi s}}{(s+1)^2+2}=\frac{1}{4}\left(\frac{-s+1}{s^2+1}+\frac{s+1}{(s+1)^2+2}\right)+\frac{e^{-3\pi s}}{(s+1)^2+2}$$

which upon inversion (#13, #14, and #25) gives the solution

$$y(t)=\frac{\sin t}{4}-\frac{\cos t}{4}+\frac{e^{-t}\cos\sqrt{2}t}{4}+u_{3\pi}(t)\frac{e^{-(t-3\pi)}\sin\sqrt{2}(t-3\pi)}{\sqrt{2}}\,.$$

Problem: (B&D 6.5 #9) Solve $y''+y=u_{\pi/2}(t)+3\delta(t-3\pi/2)-u_{2\pi}(t)$ with $y(0)=y'(0)=0$.

Laplace transforming the equation gives

$$s^2Y-sy(0)-y'(0)+Y=Y(s^2+1)=\frac{e^{-\pi s/2}}{s}+3e^{-3\pi s/2}-\frac{e^{-2\pi s}}{s}\,.$$

Solving for $Y(s)$ then gives $Y(s)=\frac{1}{s(s^2+1)}\left(e^{-\pi s/2}-e^{-2\pi s}\right)+\frac{3e^{-3\pi s/2}}{s^2+1}$ which upon inversion (#17 with $a=1$ and #23 with $a=1$) gives the solution

$$y(t)=u_{\pi/2}(t)\left[1-\cos(t-\pi/2)\right]-u_{2\pi}(t)\left[1-\cos(t-2\pi)\right]+3u_{3\pi/2}(t)\left[\sin(t-3\pi/2)\right]\,.$$

Problem: (B&D 6.6 #13) Solve $y''+2y'+2y=\sin\alpha t$ with $y(0)=y'(0)=0$ using the convolution theorem.

Laplace transforming the equation gives

$$s^2Y-sy(0)-y'(0)+2(sY-y(0))+2Y=Y(s^2+2s+2)=Y((s+1)^2+1)=\frac{\alpha}{s^2+\alpha^2}\,.$$

Solving for $Y(s)$ gives $Y(s)=\frac{\alpha}{(s^2+\alpha^2)(s^2+2s+2)}=\frac{\alpha}{(s^2+\alpha^2)((s+1)^2+1)}$ which is a combination of #13 and #24. Using convolution we find

$$y(t)=\int_0^t\sin\alpha(t-\tau)e^{-\tau}\sin\tau d\tau\,.$$

Problem: (B&D 6.6 #17) Solve $y''+3y'+2y=\cos\alpha t$ with $y(0)=1$ and $y'(0)=0$ using the convolution theorem.

Laplace transforming the equation gives

$$s^2Y-sy(0)-y'(0)+3(sY-y(0))+2Y=Y(s^2+3s+2)-(s+3)=\frac{s}{s^2+\alpha^2}\,.$$

Solving for $Y(s)$ gives

$$Y(s)=\frac{s+3}{s^2+3s+2}+\frac{s}{(s^2+\alpha^2)(s^2+3s+2)}=\frac{(s+2)+1}{(s+2)(s+1)}+\frac{s}{(s^2+\alpha^2)}\cdot\frac{1}{(s+2)(s+1)}$$

whose first two terms are #7 and #10 and whose last term is a combination of #14 and #10. Using the convolution and simplifying we find

$$y(t)=2e^{-t}-e^{-2t}+\int_0^t\left[e^{-(t-\tau)}-e^{-2(t-\tau)}\right]\cos\alpha\tau d\tau\,.$$

Problem: (B&D 6.5 #7) Solve $y'' + y = \delta(t-2\pi)\cos t$ with $y(0) = 0$ and $y'(0) = 1$.

We first note that the delta function is zero everywhere except at $t = 2\pi$. Thus we find

$$\delta(t-2\pi)\cos t = \delta(t-2\pi)\cos 2\pi = \delta(t-2\pi)\,.$$

Laplace transforming the equation then gives

$$s^2Y - sy(0) - y'(0) + Y = Y(s^2+1) - 1 = e^{-2\pi s}\,.$$

Solving for $Y(s)$ then gives

$$Y(s) = \frac{1}{s^2+1} + \frac{e^{-2\pi s}}{s^2+1}$$

which upon inversion (#13 with $a = 1$) gives the solution

$$y(t) = \sin t + u_{2\pi}(t)\sin(t-2\pi)\,.$$

Problem: (B&D 6.5 #11) Solve $y''+2y'+2y = \cos t+\delta(t-\pi/2)$ with $y(0) = y'(0) = 0$.

Laplace transforming the equation gives

$$s^2Y - sy(0) - y'(0) + 2[sY - y(0)] + 2Y = Y(s^2+2s+2) = Y[(s+1)^2+1] = \frac{s}{s^2+1} + e^{-\pi s/2}\,.$$

Solving for $Y(s)$ then gives

$$Y(s) = \frac{s}{(s^2+1)[(s+1)^2+1]} + \frac{e^{-\pi s/2}}{(s+1)^2+1}\,.$$

To invert, we must first perform a partial fraction decomposition:

$$\frac{s}{(s^2+1)[(s+1)^2+1]} = \frac{As+B}{s^2+1} + \frac{Cs+D}{(s+1)^2+1}\,.$$

In order to make both sides equal, we collect powers of s and set them equal on both sides. This gives

$$\begin{aligned} s^3: &\quad A + C = 0 \\ s^2: &\quad 2A + B + D = 0 \\ s: &\quad 2A + 2B + C = 1 \\ \text{constant}: &\quad 2B + D = 0 \end{aligned}$$

which can be solved to give $A = 1/5$, $B = 2/5$, $C = -1/5$, and $D = -4/5$. Thus we have

$$Y(s) = \frac{1}{5}\frac{s}{(s^2+1)} + \frac{2}{5}\frac{1}{(s^2+1)} - \frac{1}{5}\frac{s+1}{(s+1)^2+1} - \frac{3}{5}\frac{1}{(s+1)^2+1} + \frac{e^{-\pi s/2}}{(s+1)^2+1}\,.$$

which upon inversion (#13, #14, #24, and #25) gives the solution

$$y(t) = \frac{1}{5}\cos t + \frac{2}{5}\sin t - \frac{1}{5}e^{-t}\cos t - \frac{3}{5}e^{-t}\sin t + u_{\pi/2}(t)e^{-(t-\pi/2)}\sin(t-\pi/2)\,.$$

Problem: (B&D 7.1 #1) Write $u''+u'/2+2u=0$ as a system of first-order equations.

We define $x=u$ and $y=u'$ so that $y'=u''$ and $x'=u'=y$ which leads to

$$\begin{aligned} x' &= y \\ y' &= -\frac{1}{2}y - 2x\,. \end{aligned}$$

Problem: (B&D 7.1 #3) Write $t^2u''+tu'+(t^2-1/4)u=0$ as a system of first-order equations.

We define $x=u$ and $y=u'$ so that $y'=u''$ and $x'=u'=y$ which leads to

$$\begin{aligned} x' &= y \\ y' &= -\frac{1}{t}y - \left(1-\frac{1}{4t^2}\right)x\,. \end{aligned}$$

Problem: (B&D 7.1 #9) Solve $x_1'=(5/4)x_1+(3/4)x_2$ and $x_2'=(3/4)x_1+(5/4)x_2$ with $x_1(0)=-2$ and $x_2(0)=1$.

From the first equation, we find that $x_2=(4/3)x_1'-(5/3)x_1$ which gives $x_2'=(4/3)x_1''-(5/3)x_1'$. Upon plugging both of these into the second equation we find

$$\frac{4}{3}x_1''-\frac{5}{3}x_1'=\frac{3}{4}x_1+\frac{3}{4}\left(\frac{4}{3}x_1'-\frac{5}{3}x_1\right) \quad\rightarrow\quad x_1''-\frac{5}{2}x_1'+x_1=0$$

We can solve this by letting $x_1=\exp(\lambda t)$ which gives

$$\lambda^2-\frac{5}{2}\lambda+1=0 \quad\rightarrow\quad \lambda=\frac{1}{2}\left(\frac{5}{2}\pm\sqrt{\frac{25}{4}-4}\right)=\frac{1}{2},\,2$$

Thus our solution is $x_1=c_1\exp(t/2)+c_2\exp(2t)$. Our initial conditions in this case are $x_1(0)=-2$ and $x_1'(0)=(5/4)x_1(0)+(3/4)x_2(0)=-7/4$. Plugging in and solving the two by two system gives $c_1=-3/2$ and $c_2=-1/2$. We can then calculate $x_2=(4/3)x_1'-(5/3)x_1$ so that

$$x_1(t)=-\frac{3}{2}\exp(t/2)-\frac{1}{2}\exp(2t) \quad\text{and}\quad x_2(t)=\frac{3}{2}\exp(t/2)-\frac{1}{2}\exp(2t)$$

Problem: (B&D 7.1 #9) Solve $x_1'{=}2x_2$ and $x_2'{=}{-}2x_1$ with $x_1(0){=}3$ and $x_2(0){=}4$.

From the first equation, we find that $x_2=x_1'/2$. Upon plugging into the second equation we find

$$\frac{1}{2}x_1''=-2x_1 \quad\rightarrow\quad x_1''+4x_1=0$$

We can solve this by letting $x_1=\exp(\lambda t)$ which gives

$$\lambda^2+4=0 \quad\rightarrow\quad \lambda=\pm 2i$$

Thus our solution is $x_1=c_1\cos(2t)+c_2\sin(2t)$. Our initial conditions in this case are $x_1(0)=3$ and $x_1'(0)=2x_2(0)=8$. Plugging in and solving the two by two system gives $c_1=3$ and $c_2=4$. We can then calculate $x_2=x_1'/2$ so that

$$x_1(t)=3\cos 2t+4\sin 2t \quad\text{and}\quad x_2(t)=-3\sin 2t+4\cos 2t$$

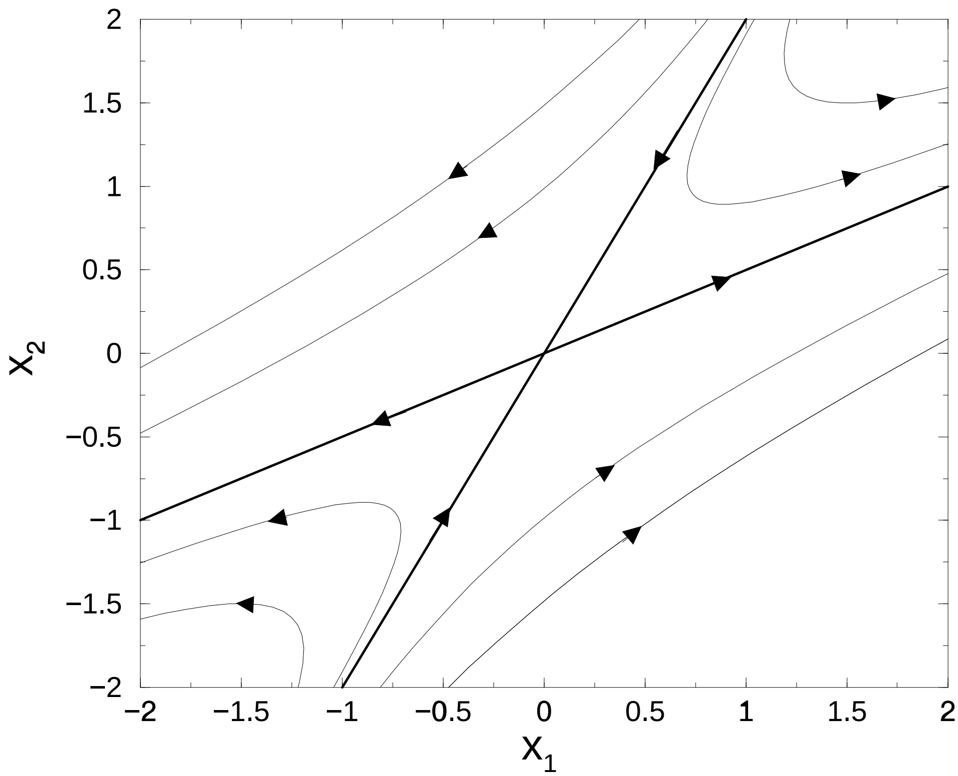

FIG. 29. *Phase–plane dynamics depicting the saddle-node dynamics. The bolded lines are the eigenvectors and the arrows indicate the direction of the flow of the solutions.*

Problem: (B&D 7.5 #1) Solve $\vec{x}' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \vec{x}$ and plot the solutions.

We turn this into an eigenvalue problem by letting $\vec{x} = \vec{v}\exp(\lambda t)$ so that

$$\begin{pmatrix} 3-\lambda & -2 \\ 2 & -2-\lambda \end{pmatrix} \vec{v} = 0$$

The eigenvalues are found by taking the determinant of the above matrix

$$\det\begin{vmatrix} 3-\lambda & -2 \\ 2 & -2-\lambda \end{vmatrix} = (3-\lambda)(-2-\lambda)+4 = \lambda^2-\lambda-2 = (\lambda+1)(\lambda-2) = 0 \ \rightarrow \ \lambda = -1, 2$$

which lead to the eigenvectors:

$$\lambda = -1: \ \begin{pmatrix} 4 & -2 \\ 2 & -1 \end{pmatrix} \vec{v} = 0 \ \rightarrow \ 2v_1 - v_2 = 0 \ \rightarrow \ \vec{v}^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\lambda = 2: \ \begin{pmatrix} 1 & -2 \\ 2 & -4 \end{pmatrix} \vec{v} = 0 \ \rightarrow \ v_1 - 2v_2 = 0 \ \rightarrow \ \vec{v}^{(2)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

and the solution

$$\vec{x}(t) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} \exp(-t) + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} \exp(2t)\,.$$

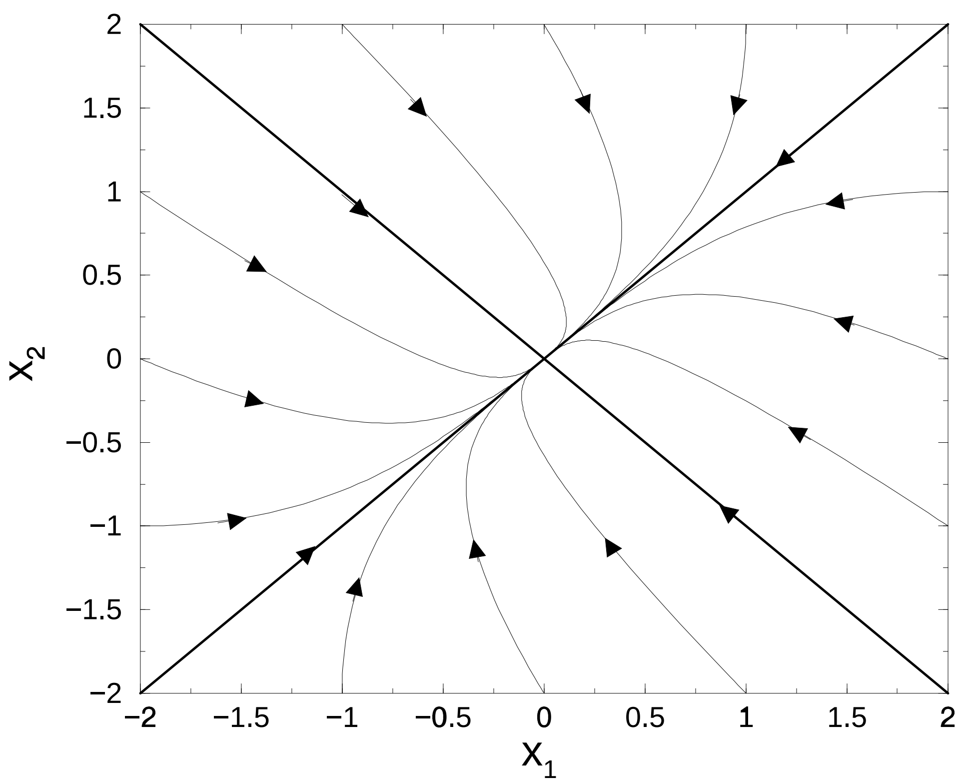

FIG. 30. *Phase–plane dynamics depicting a stable node dynamics. The bolded lines are the eigenvectors and the arrows indicate the direction of the flow of the solutions.*

Problem: (B&D 7.5 #5) Solve $\vec{x}' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \vec{x}$ and plot the solutions.

We turn this into an eigenvalue problem by letting $\vec{x} = \vec{v}\exp(\lambda t)$ so that

$$\begin{pmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{pmatrix} \vec{v} = 0$$

The eigenvalues are found by taking the determinant of the above matrix

$$\det\begin{vmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{vmatrix} = (2+\lambda)^2 - 1 = \lambda^2 + 4\lambda + 3 = (\lambda+3)(\lambda+1) = 0 \;\rightarrow\; \lambda = -1, -3$$

which lead to the eigenvectors:

$$\lambda = -1: \;\; \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \vec{v} = 0 \;\rightarrow\; v_1 - v_2 = 0 \;\rightarrow\; \vec{v}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\lambda = -3: \;\; \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \vec{v} = 0 \;\rightarrow\; v_1 + v_2 = 0 \;\rightarrow\; \vec{v}^{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

and the solution

$$\vec{x}(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \exp(-t) + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \exp(-3t)\,.$$

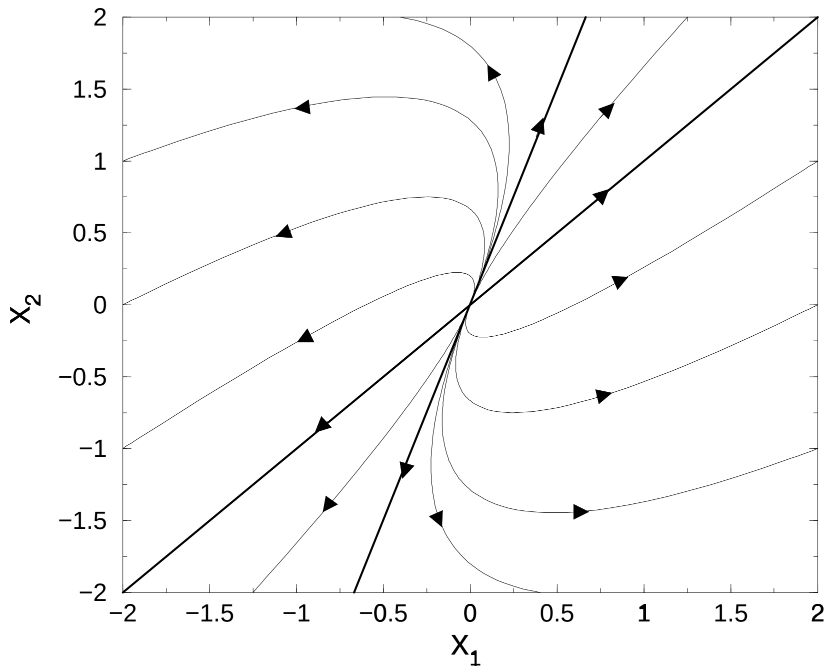

FIG. 31. *Phase–plane dynamics depicting an unstable node dynamics. The bolded lines are the eigenvectors and the arrows indicate the direction of the flow of the solutions.*

Problem: (B&D 7.5 #15) Solve $\vec{x}'=\begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix}\vec{x}$ and plot the solutions.

We turn this into an eigenvalue problem by letting $\vec{x}=\vec{v}\exp(\lambda t)$ so that

$$\begin{pmatrix} 5-\lambda & -1 \\ 3 & 1-\lambda \end{pmatrix}\vec{v}=0$$

The eigenvalues are found by taking the determinant of the above matrix

$$\det\begin{vmatrix} 5-\lambda & -1 \\ 3 & 1-\lambda \end{vmatrix} = (5-\lambda)(1-\lambda)+3=\lambda^2-6\lambda+8=(\lambda-2)(\lambda-4)=0 \ \rightarrow \ \lambda=2,\,4$$

which lead to the eigenvectors:

$$\lambda=2: \ \begin{pmatrix} 3 & -1 \\ 3 & -1 \end{pmatrix}\vec{v}=0 \ \rightarrow \ 3v_1-v_2=0 \ \rightarrow \ \vec{v}^{(1)}=\begin{pmatrix} 1 \\ 3 \end{pmatrix}$$
$$\lambda=4: \ \begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix}\vec{v}=0 \ \rightarrow \ v_1-v_2=0 \ \rightarrow \ \vec{v}^{(2)}=\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and the solution

$$\vec{x}(t)=c_1\begin{pmatrix} 1 \\ 3 \end{pmatrix}\exp(2t)+c_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}\exp(4t)\,.$$

We can easily put in our initial condition $\vec{x}(0)=(2\ \ -1)^T$ to find that $c_1+c_2=2$ and $3c_1+c_2=-1$. Thus $c_1=-3/2$ and $c_2=7/2$ giving

$$\vec{x}(t)=-\frac{3}{2}\begin{pmatrix} 1 \\ 3 \end{pmatrix}\exp(2t)+\frac{7}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\exp(4t)\,.$$

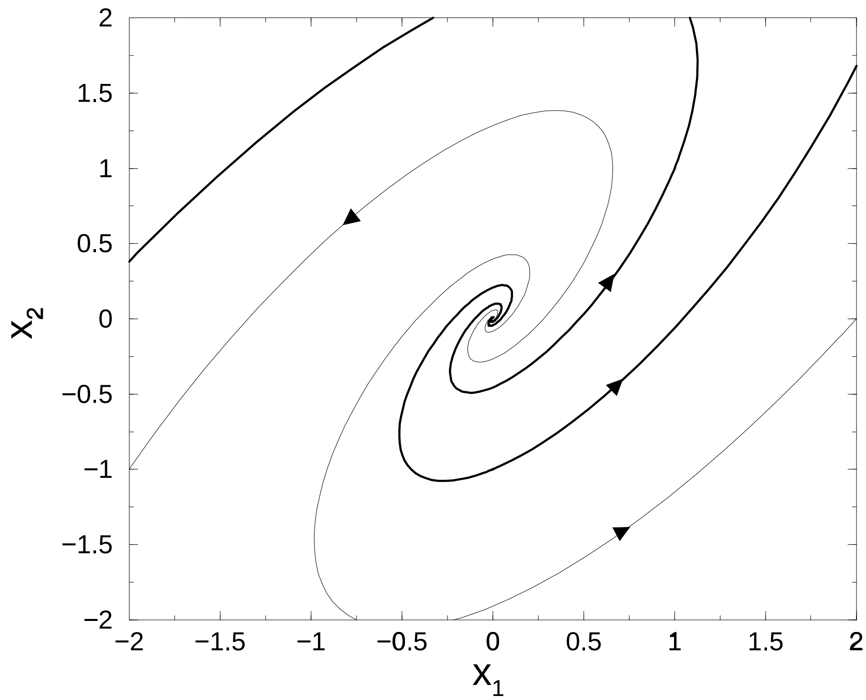

FIG. 32. *Phase–plane dynamics depicting an unstable spiral dynamics. The bolded lines are the eigenvectors and the arrows indicate the direction of the flow of the solutions.*

Problem: (B&D 7.6 #1) Solve $\vec{x}' = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix} \vec{x}$ and plot the solutions.

We turn this into an eigenvalue problem by letting $\vec{x} = \vec{v}\exp(\lambda t)$ so that

$$\begin{pmatrix} 3-\lambda & -2 \\ 4 & -1-\lambda \end{pmatrix} \vec{v} = 0$$

The eigenvalues are found by taking the determinant of the above matrix

$$\det\begin{vmatrix} 3-\lambda & -2 \\ 4 & -1-\lambda \end{vmatrix} = (3-\lambda)(-1-\lambda)+8 = \lambda^2 - 2\lambda + 5 = 0 \;\rightarrow\; \lambda = 1 \pm 2i$$

which leads to the eigenvector (recall the second eigenvector is the complex conjugate):

$$\lambda = 1+2i: \;\; \begin{pmatrix} 2-2i & -2 \\ 4 & -2-2i \end{pmatrix} \vec{v} = 0 \;\rightarrow\; (1-i)v_1 - v_2 = 0 \;\rightarrow\; \vec{v}^{(1)} = \begin{pmatrix} 1 \\ 1-i \end{pmatrix}$$

so then

$$\vec{x}^{(1)} = \begin{pmatrix} 1 \\ 1-i \end{pmatrix} e^{(1+2i)t} = \left[\begin{pmatrix} \cos 2t \\ \cos 2t + \sin 2t \end{pmatrix} + i \begin{pmatrix} \sin 2t \\ \sin 2t - \cos 2t \end{pmatrix} \right] e^t$$

and the solution

$$\vec{x}(t) = c_1 \begin{pmatrix} \cos 2t \\ \cos 2t + \sin 2t \end{pmatrix} e^t + c_2 \begin{pmatrix} \sin 2t \\ \sin 2t - \cos 2t \end{pmatrix} e^t .$$

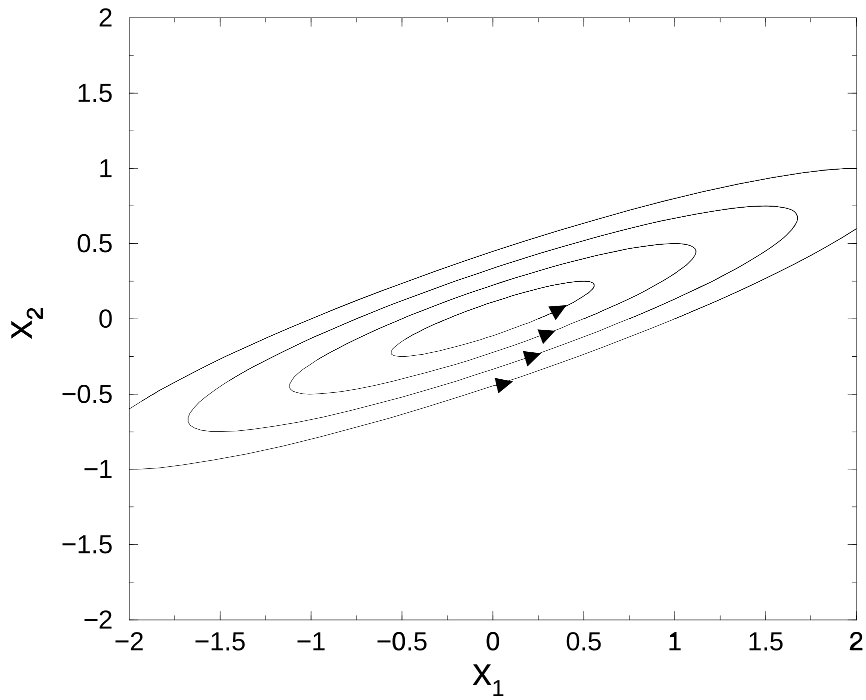

FIG. 33. *Phase–plane dynamics depicting a neutrally stable center dynamics.*

Problem: (B&D 7.6 #3) Solve $\vec{x}' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} \vec{x}$ and plot the solutions.

We turn this into an eigenvalue problem by letting $\vec{x} = \vec{v}\exp(\lambda t)$ so that

$$\begin{pmatrix} 2-\lambda & -5 \\ 1 & -2-\lambda \end{pmatrix} \vec{v} = 0$$

The eigenvalues are found by taking the determinant of the above matrix

$$\det \begin{vmatrix} 2-\lambda & -5 \\ 1 & -2-\lambda \end{vmatrix} = (2-\lambda)(-2-\lambda) + 5 = \lambda^2 + 1 = 0 \;\rightarrow\; \lambda = \pm i$$

which leads to the eigenvector (recall the second eigenvector is the complex conjugate):

$$\lambda = i: \;\; \begin{pmatrix} 2-i & -5 \\ 1 & -2-i \end{pmatrix} \vec{v} = 0 \;\rightarrow\; (2-i)v_1 - 5v_2 = 0 \;\rightarrow\; \vec{v}^{(1)} = \begin{pmatrix} 5 \\ 2-i \end{pmatrix}$$

so then

$$\vec{x}^{(1)} = \begin{pmatrix} 5 \\ 2-i \end{pmatrix} e^{it} = \left[\begin{pmatrix} 5\cos t \\ 2\cos t + \sin t \end{pmatrix} + i \begin{pmatrix} 5\sin t \\ 2\sin t - \cos t \end{pmatrix} \right] e^{t}$$

and the solution

$$\vec{x}(t) = c_1 \begin{pmatrix} 5\cos t \\ 2\cos t + \sin t \end{pmatrix} + c_2 \begin{pmatrix} 5\sin t \\ 2\sin t - \cos t \end{pmatrix} .$$

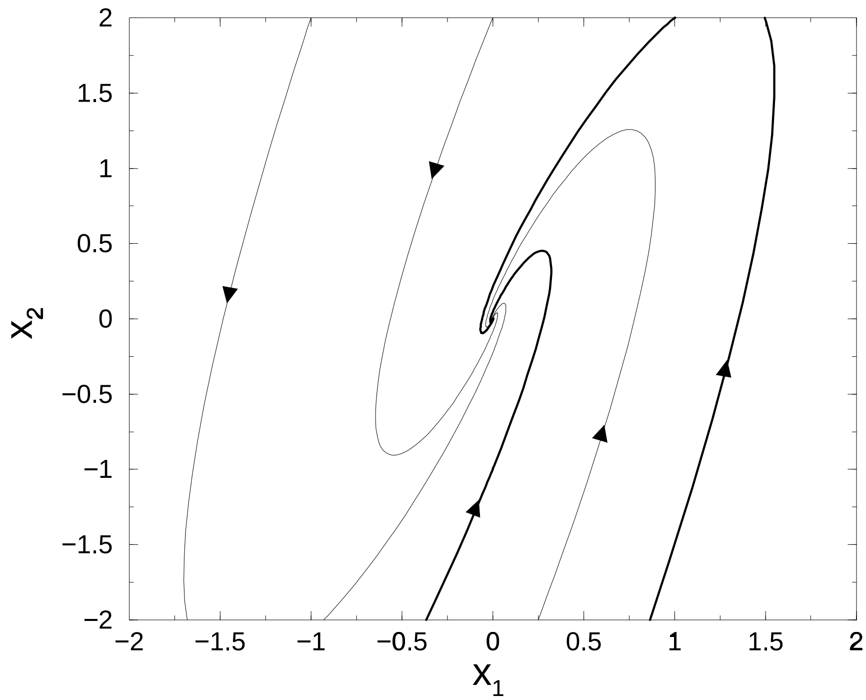

FIG. 34. *Phase–plane dynamics depicting a stable spiral dynamics. The bolded lines are the eigenvectors and the arrows indicate the direction of the flow of the solutions.*

Problem: (B&D 7.6 #5) Solve $\vec{x}' = \begin{pmatrix} 1 & -1 \\ 5 & -3 \end{pmatrix} \vec{x}$ and plot the solutions.

We turn this into an eigenvalue problem by letting $\vec{x} = \vec{v} \exp(\lambda t)$ so that

$$\begin{pmatrix} 1-\lambda & -1 \\ 5 & -3-\lambda \end{pmatrix} \vec{v} = 0$$

The eigenvalues are found by taking the determinant of the above matrix

$$\det \begin{vmatrix} 1-\lambda & -1 \\ 5 & -3-\lambda \end{vmatrix} = (1-\lambda)(-3-\lambda) + 5 = \lambda^2 + 2\lambda + 2 = 0 \;\rightarrow\; \lambda = -1 \pm i$$

which leads to the eigenvector (recall the second eigenvector is the complex conjugate):

$$\lambda = -1+i: \;\; \begin{pmatrix} 2-i & -1 \\ 5 & -2-i \end{pmatrix} \vec{v} = 0 \;\rightarrow\; (2-i)v_1 - v_2 = 0 \;\rightarrow\; \vec{v}^{(1)} = \begin{pmatrix} 1 \\ 2-i \end{pmatrix}$$

so then

$$\vec{x}^{(1)} = \begin{pmatrix} 1 \\ 2-i \end{pmatrix} e^{(-1+i)t} = \left[\begin{pmatrix} \cos t \\ 2\cos t + \sin t \end{pmatrix} + i \begin{pmatrix} \sin t \\ 2\sin t - \cos t \end{pmatrix} \right] e^{-t}$$

and the solution

$$\vec{x}(t) = c_1 \begin{pmatrix} \cos t \\ 2\cos t + \sin t \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} \sin t \\ 2\sin t - \cos t \end{pmatrix} e^{-t}.$$

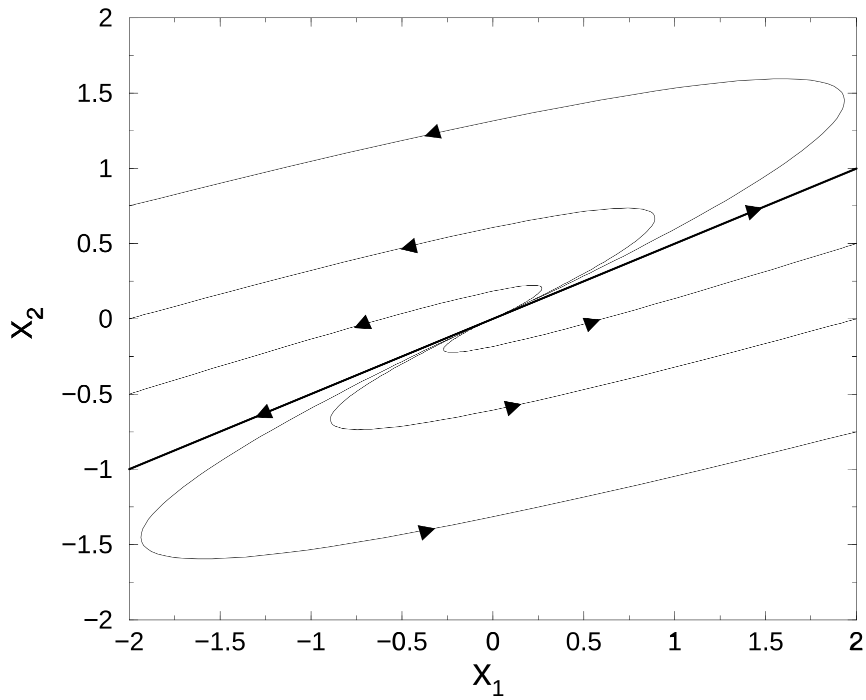

FIG. 35. *Phase–plane dynamics depicting the improper node dynamics. The bolded lines are the eigenvectors and the arrows indicate the direction of the flow of the solutions.*

Problem: (B&D 7.7 #1) Solve $\vec{x}' = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} \vec{x}$ and plot the solutions.

We turn this into an eigenvalue problem by letting $\vec{x} = \vec{v}\exp(\lambda t)$ so that

$$\begin{pmatrix} 3-\lambda & -4 \\ 1 & -1-\lambda \end{pmatrix} \vec{v} = 0$$

The eigenvalues are found by taking the determinant of the above matrix

$$\det\begin{vmatrix} 3-\lambda & -4 \\ 1 & -1-\lambda \end{vmatrix} = (3-\lambda)(-1-\lambda)+4 = \lambda^2 - 2\lambda + 1 = (\lambda-1)^2 = 0 \;\rightarrow\; \lambda = 1$$

which is a double root and leads to the eigenvector:

$$\lambda = 1: \;\; \begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix} \vec{v} = 0 \;\rightarrow\; v_1 - 2v_2 = 0 \;\rightarrow\; \vec{v}^{(1)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

The second eigenvector takes the form $\vec{v}^{(2)} = \vec{v}te^t + \vec{\eta}e^t$, which can be evaluated by plugging into the governing equation. Collecting te^t terms gives $\mathbf{A}\vec{v} = \vec{v}$ which has the eigenvalue $\lambda = 1$ and eigenvector $\vec{v} = \vec{v}^{(1)}$. The e^t terms give $(\mathbf{A} - \mathbf{I})\vec{\eta} = \vec{v}^{(1)}$. This leads to $\eta_1 - 2\eta_2 = 1$ for which we can choose $\vec{n} = (1\ 0)^T$. The solution is then

$$\vec{x}(t) = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} \exp(t) + c_2 \left[\begin{pmatrix} 2 \\ 1 \end{pmatrix} t\exp(t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \exp(t) \right].$$

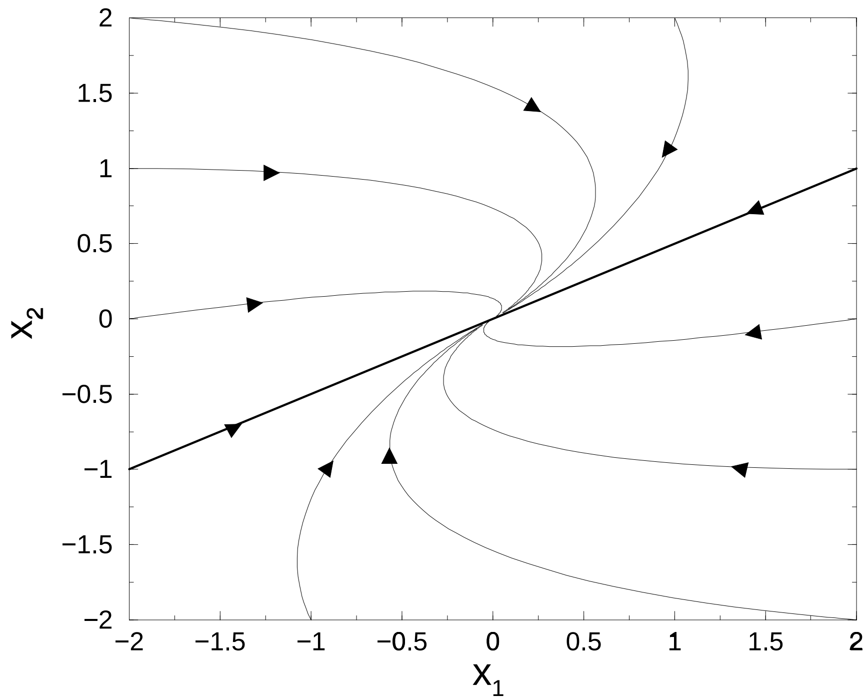

FIG. 36. *Phase–plane dynamics depicting the improper node dynamics. The bolded lines are the eigenvectors and the arrows indicate the direction of the flow of the solutions.*

Problem: (B&D 7.7 #3) Solve $\vec{x}' = \begin{pmatrix} -3/2 & 1 \\ -1/4 & -1/2 \end{pmatrix} \vec{x}$ and plot the solutions.

We turn this into an eigenvalue problem by letting $\vec{x} = \vec{v}\exp(\lambda t)$ so that

$$\begin{pmatrix} 3/2 - \lambda & 1 \\ -1/4 & -1/2 - \lambda \end{pmatrix} \vec{v} = 0$$

The eigenvalues are found by taking the determinant of the above matrix

$$\det \begin{vmatrix} 3/2 - \lambda & 1 \\ -1/4 & -1/2 - \lambda \end{vmatrix} = \lambda^2 + 2\lambda + 1 = (\lambda + 1)^2 = 0 \ \rightarrow \ \lambda = -1$$

which is a double root and leads to the eigenvector:

$$\lambda = 1 : \ \begin{pmatrix} -1/2 & 1 \\ -1/4 & 1/2 \end{pmatrix} \vec{v} = 0 \ \rightarrow \ v_1 - 2v_2 = 0 \ \rightarrow \ \vec{v}^{(1)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

The second eigenvector takes the form $\vec{v}^{(2)} = \vec{v}te^{-t} + \vec{\eta}e^{-t}$, which can be evaluated by plugging into the governing equation. Collecting te^{-t} terms gives $\mathbf{A}\vec{v} = -\vec{v}$ which has the eigenvalue $\lambda = -1$ and eigenvector $\vec{v} = \vec{v}^{(1)}$. The e^{-t} terms give $(\mathbf{A} + \mathbf{I})\vec{\eta} = \vec{v}^{(1)}$. This leads to $-\eta_1/2 + \eta_2 = 2$ for which we can choose $\vec{n} = (0\ 2)^T$. The solution is then

$$\vec{x}(t) = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} \exp(-t) + c_2 \left[\begin{pmatrix} 2 \\ 1 \end{pmatrix} t\exp(-t) + \begin{pmatrix} 0 \\ 2 \end{pmatrix} \exp(-t) \right].$$

Problem: (B&D 7.8 #1) Solve $\vec{x}' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \vec{x}$ and construct the fundamental solutions matrix $\mathbf{\Phi}(t)$.

We turn this into an eigenvalue problem by letting $\vec{x} = \vec{v}\exp(\lambda t)$ so that

$$\begin{pmatrix} 3-\lambda & -2 \\ 2 & -2-\lambda \end{pmatrix} \vec{v} = 0$$

The eigenvalues are found by taking the determinant of the above matrix

$$\det \begin{vmatrix} 3-\lambda & -2 \\ 2 & -2-\lambda \end{vmatrix} = (3-\lambda)(-2-\lambda)+4 = \lambda^2-\lambda-2 = (\lambda+1)(\lambda-2) = 0 \;\rightarrow\; \lambda = -1,\, 2$$

which lead to the eigenvectors:

$$\lambda = -1: \;\; \begin{pmatrix} 4 & -2 \\ 2 & -1 \end{pmatrix} \vec{v} = 0 \;\rightarrow\; 2v_1 - v_2 = 0 \;\rightarrow\; \vec{v}^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$
$$\lambda = 2: \;\; \begin{pmatrix} 1 & -2 \\ 2 & -4 \end{pmatrix} \vec{v} = 0 \;\rightarrow\; v_1 - 2v_2 = 0 \;\rightarrow\; \vec{v}^{(2)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

and the solution

$$\vec{x}(t) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} \exp(-t) + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} \exp(2t)\,.$$

In order to construct the fundamental matrix $\mathbf{\Phi(t)}$ which satisfies $\mathbf{\Phi(0)} = \mathbf{I}$, we need two new solutions. One $(X_1(t))$ which satisfies the initial condition $(1\ 0)^T$, and a second $(X_2(t))$ which satisfies $(0\ 1)^T$. Since $X_1(t)$ and $X_2(t)$ are just the general solution with some specific c_1 and c_2, we thus find:

$$X_1: \quad X_1(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$
$$X_2: \quad X_2(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

which yield the following two by two systems:

$$X_1: \quad c_1 + 2c_2 = 1 \;\text{ and }\; 2c_1 + c_2 = 0 \quad\rightarrow\quad c_1 = -\frac{1}{3},\; c_2 = \frac{2}{3}$$
$$X_2: \quad c_1 + 2c_2 = 0 \;\text{ and }\; 2c_1 + c_2 = 1 \quad\rightarrow\quad c_1 = \frac{2}{3},\; c_2 = -\frac{1}{3}\,.$$

Our two solutions are then

$$X_1(t) = -\frac{1}{3} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \exp(-t) + \frac{2}{3} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \exp(2t)$$
$$X_2(t) = \frac{2}{3} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \exp(-t) - \frac{1}{3} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \exp(2t)$$

and the fundamental matrix $\mathbf{\Phi}(t) = (X_1\ X_2)$ is

$$\mathbf{\Phi}(t) = \begin{pmatrix} \frac{4}{3}e^{2t} - \frac{1}{3}e^{-t} & -\frac{2}{3}e^{2t} + \frac{2}{3}e^{-t} \\ \frac{2}{3}e^{2t} - \frac{2}{3}e^{-t} & -\frac{1}{3}e^{2t} + \frac{4}{3}e^{-t} \end{pmatrix}$$

Problem: (B&D 7.8 #5) Solve $\vec{x}' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} \vec{x}$ and construct the fundamental solutions matrix $\mathbf{\Phi}(t)$.

We turn this into an eigenvalue problem by letting $\vec{x} = \vec{v}\exp(\lambda t)$ so that

$$\begin{pmatrix} 3-\lambda & -2 \\ 2 & -2-\lambda \end{pmatrix} \vec{v} = 0$$

The eigenvalues are found by taking the determinant of the above matrix

$$\det \begin{vmatrix} 2-\lambda & -5 \\ 1 & -2-\lambda \end{vmatrix} = (2-\lambda)(-2-\lambda) + 5 = \lambda^2 + 1 = 0 \;\rightarrow\; \lambda = \pm i$$

which leads to an eigenvector:

$$\lambda = i: \;\; \begin{pmatrix} 2-i & -5 \\ 1 & -2-i \end{pmatrix} \vec{v} = 0 \;\rightarrow\; (2-i)v_1 - 5v_2 = 0 \;\rightarrow\; \vec{v}^{(1)} = \begin{pmatrix} 5 \\ 2-i \end{pmatrix}.$$

The real and imaginary parts of the solution are then

$$x^{(1)} = \begin{pmatrix} 5 \\ 2-i \end{pmatrix} e^{it} = \begin{pmatrix} 5\cos t \\ 2\cos t + \sin t \end{pmatrix} + i \begin{pmatrix} 5\sin t \\ 2\sin t - \cos t \end{pmatrix}$$

which give the solution

$$\vec{x}(t) = c_1 \begin{pmatrix} 5\cos t \\ 2\cos t + \sin t \end{pmatrix} + c_2 \begin{pmatrix} 5\sin t \\ 2\sin t - \cos t \end{pmatrix}.$$

In order to construct the fundamental matrix $\mathbf{\Phi(t)}$ which satisfies $\mathbf{\Phi(0)} = \mathbf{I}$, we need two new solutions. One $(X_1(t))$ which satisfies the initial condition $(1\ 0)^T$, and a second $(X_2(t))$ which satisfies $(0\ 1)^T$. Since $X_1(t)$ and $X_2(t)$ are just the general solution with some specific c_1 and c_2, we thus find:

$$X_1: \quad X_1(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = c_1 \begin{pmatrix} 5 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$
$$X_2: \quad X_2(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = c_1 \begin{pmatrix} 5 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

which yield the following two by two systems:

$$X_1: \quad 5c_1 = 1 \;\text{ and }\; 2c_1 - c_2 = 0 \quad \rightarrow \quad c_1 = \frac{1}{5},\; c_2 = \frac{2}{5}$$
$$X_2: \quad 5c_1 = 0 \;\text{ and }\; 2c_1 - c_2 = 1 \quad \rightarrow \quad c_1 = 0,\; c_2 = -1\,.$$

Our two solutions are then

$$X_1(t) = \frac{1}{5} \begin{pmatrix} 5\cos t \\ 2\cos t + \sin t \end{pmatrix} + \frac{2}{5} \begin{pmatrix} 5\sin t \\ 2\sin t - \cos t \end{pmatrix}$$
$$X_2(t) = -\begin{pmatrix} 5\sin t \\ 2\sin t - \cos t \end{pmatrix}$$

and the fundamental matrix $\mathbf{\Phi}(t) = (X_1\ X_2)$ is

$$\mathbf{\Phi}(t) = \begin{pmatrix} \cos t + 2\sin t & -5\sin t \\ \sin t & \cos t - 2\sin t \end{pmatrix}$$

Problem: (B&D 7.9 #1) Solve $\vec{x}'=\begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}\vec{x}+\begin{pmatrix} e^t \\ t \end{pmatrix}$ and plot the solutions.

We solve the homogeneous part first by turning it into an eigenvalue problem. Letting $\vec{x}=\vec{v}\exp(\lambda t)$ gives

$$\begin{pmatrix} 2-\lambda & -1 \\ 3 & -2-\lambda \end{pmatrix}\vec{v}=0$$

whose eigenvalues are found by taking the determinant of the above matrix

$$\det\begin{vmatrix} 2-\lambda & -1 \\ 3 & -2-\lambda \end{vmatrix}=(2-\lambda)(-2-\lambda)+3=\lambda^2-1=0 \ \rightarrow\ \lambda=\pm 1$$

which lead to the eigenvectors:

$$\lambda=1: \ \begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix}\vec{v}=0 \ \rightarrow\ v_1-v_2=0 \ \rightarrow\ \vec{v}^{(1)}=\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$\lambda=-1: \ \begin{pmatrix} 3 & -1 \\ 3 & -1 \end{pmatrix}\vec{v}=0 \ \rightarrow\ 3v_1-v_2=0 \ \rightarrow\ \vec{v}^{(1)}=\begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

The homogeneous solution is then

$$\vec{x}_h(t)=c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^t+c_2\begin{pmatrix} 1 \\ 3 \end{pmatrix}e^{-t}.$$

To find the particular solution and the resulting general solution given $\vec{g}(t)=(e^t\ t)^T$, we can use either the method of undetermined coefficients or variation of parameters. For the method of undetermined coefficients, we guess the following particular solution

$$\vec{x}_p=\vec{a}te^t+\vec{b}e^t+\vec{c}t+\vec{d}$$

where the te^t is necessary since e^t is already a homogeneous solution. Plugging this into the governing equation and collecting like terms results in the following sets of equations:

$$\begin{aligned} te^t: &\quad \mathbf{A}\vec{a}=\vec{a} \\ e^t: &\quad \vec{a}+\vec{b}=\mathbf{A}\vec{b}+\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ t: &\quad \mathbf{A}\vec{c}=-\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \text{constant}: &\quad \vec{c}=\mathbf{A}\vec{d} \end{aligned}$$

We can solve each equation in turn. We have in fact already solved the first equation $\mathbf{A}\vec{a}=\vec{a}$. This problem corresponds to the eigenvalue $\lambda=1$ case for which we found the eigenvector $v^{(1)}$ above. Thus we have

$$\vec{a}=k\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

where k is some arbitrary constant at this point. The next equation is then given by

$$(\mathbf{A}-\mathbf{I})\vec{b}=\vec{a}-\begin{pmatrix} 1 \\ 0 \end{pmatrix}=\begin{pmatrix} k-1 \\ k \end{pmatrix} \ \rightarrow\ b_1-b_2=k-1 \ \text{ and } \ b_1-b_2=\frac{k}{3}.$$

For consistency, we must then have $k-1=k/3$ which gives $k=3/2$. Once this is established, we have $b_1-b_2=1/2$ so that we can choose $b_1=0$ and $b_2=-1/2$. Thus

$$\vec{b}=-\frac{1}{2}\begin{pmatrix}0\\1\end{pmatrix}.$$

Solving for $\vec{c}$ results in the equations $2c_1-c_2=0$ and $3c_1-2c_2=-1$ so that $c_1=1$ and $c_2=2$. Plugging this into the last equation gives $2d_1-d_2=1$ and $3d_1-2d_2=2$ so that $d_1=0$ and $d_2=-1$. Thus

$$\vec{c}=\begin{pmatrix}1\\2\end{pmatrix} \quad \text{and} \quad \vec{d}=-\begin{pmatrix}0\\1\end{pmatrix}.$$

The solution is then given by ($\vec{x}=\vec{x}_h+\vec{x}_p$)

$$\vec{x}(t)=c_1\begin{pmatrix}1\\1\end{pmatrix}e^t+c_2\begin{pmatrix}1\\3\end{pmatrix}e^{-t}+\frac{3}{2}\begin{pmatrix}1\\1\end{pmatrix}te^t-\frac{1}{2}\begin{pmatrix}0\\1\end{pmatrix}e^t+\begin{pmatrix}1\\2\end{pmatrix}t-\begin{pmatrix}0\\1\end{pmatrix}.$$

Variation of parameters requires the construction of the fundamental matrix

$$\psi(t)=\begin{pmatrix}e^t & e^{-t}\\ e^t & 3e^{-t}\end{pmatrix}.$$

We then have that $\psi\vec{u}'=\vec{g}(t)$ so that

$$\begin{pmatrix}e^t & e^{-t}\\ e^t & 3e^{-t}\end{pmatrix}\begin{pmatrix}u_1'\\u_2'\end{pmatrix}=\begin{pmatrix}e^t\\t\end{pmatrix}.$$

This yields two equations and two unknowns (u_1' and u_2') so that

$$\begin{aligned}u_1'&=-\frac{1}{2}te^{-t}+\frac{3}{2} \quad : \quad u_1=\frac{1}{2}te^{-t}+\frac{1}{2}e^{-t}+\frac{3}{2}t+c_1\\ u_2'&=\frac{1}{2}te^t-\frac{1}{2}e^2t \quad : \quad u_2=\frac{1}{2}te^t-\frac{1}{2}e^t-\frac{1}{4}e^{2t}+c_2\end{aligned}$$

where we integrated to get u_1 and u_2. The vector $\vec{u}$ is then:

$$\vec{u}=\begin{pmatrix}\frac{1}{2}te^{-t}+\frac{1}{2}e^{-t}+\frac{3}{2}t+c_1\\ \frac{1}{2}te^t-\frac{1}{2}e^t-\frac{1}{4}e^{2t}+c_2\end{pmatrix}.$$

The solution is then

$$\begin{aligned}\vec{x}=\psi\vec{u}&=\begin{pmatrix}e^t & e^{-t}\\ e^t & 3e^{-t}\end{pmatrix}\begin{pmatrix}\frac{1}{2}te^{-t}+\frac{1}{2}e^{-t}+\frac{3}{2}t+c_1\\ \frac{1}{2}te^t-\frac{1}{2}e^t-\frac{1}{4}e^{2t}+c_2\end{pmatrix}\\ &=\begin{pmatrix}\frac{1}{2}+\frac{t}{2}+\frac{3}{2}te^t+c_1e^t+\frac{t}{2}-\frac{1}{2}-\frac{1}{4}e^t+c_2e^{-t}\\ \frac{1}{2}+\frac{t}{2}+\frac{3}{2}te^t+c_1e^t+\frac{3t}{2}-\frac{3}{2}-\frac{3}{4}e^t+3c_2e^{-t}\end{pmatrix}\end{aligned}$$

which upon separating out the constituative parts greatly simplifies to the following form

$$\vec{x}(t)=c_1\begin{pmatrix}1\\1\end{pmatrix}e^t+c_2\begin{pmatrix}1\\3\end{pmatrix}e^{-t}+\frac{3}{2}\begin{pmatrix}1\\1\end{pmatrix}te^t-\frac{1}{4}\begin{pmatrix}1\\3\end{pmatrix}e^t+\begin{pmatrix}1\\2\end{pmatrix}t-\begin{pmatrix}0\\1\end{pmatrix}.$$

We note that this solution is slightly different than that generated by the method of undetermined coefficients. That is okay since the particular solution is not unique.

Problem: (B&D 7.9 #3) Solve $\vec{x}' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} \vec{x} + \begin{pmatrix} -\cos t \\ \sin t \end{pmatrix}$.

We solve the homogeneous part first by turning it into an eigenvalue problem. Letting $\vec{x} = \vec{v}\exp(\lambda t)$ gives

$$\begin{pmatrix} 2-\lambda & -5 \\ 1 & -2-\lambda \end{pmatrix} \vec{v} = 0 .$$

We have solved this problem previously (7.6 #3 and 7.8 #5). The homogeneous solution is then

$$\vec{x}_h(t) = c_1 \begin{pmatrix} 5\cos t \\ 2\cos t + \sin t \end{pmatrix} + c_2 \begin{pmatrix} 5\sin t \\ 2\sin t - \cos t \end{pmatrix} .$$

To find the particular solution and general solution given $\vec{g}(t) = (-\cos t \;\; \sin t)^T$, we use variation of parameters. We first construct the fundamental matrix

$$\psi(t) = \begin{pmatrix} 5\cos t & 5\sin t \\ 2\cos t + \sin t & 2\sin t - \cos t \end{pmatrix} .$$

We then have that $\psi \vec{u}' = \vec{g}(t)$ so that

$$\begin{pmatrix} 5\cos t & 5\sin t \\ 2\cos t + \sin t & 2\sin t - \cos t \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} -\cos t \\ \sin t \end{pmatrix} .$$

From the first equation, we find $u_1' = -1/5 - u_2' \tan t$. When inserted into the second equation, this yields an equation for u_2' alone. Thus we find

$$\begin{aligned} u_1' &= \frac{1}{5}\sin 2t + \frac{6}{5}\sin^2 t - \frac{1}{5} \quad : \quad u_1 = \frac{1}{5}\left(2t - \frac{3}{2}\sin 2t - \frac{1}{2}\cos 2t + c_1\right) \\ u_2' &= -\frac{3}{5}\sin 2t - \frac{2}{5}\cos^2 t \quad : \quad u_2 = \frac{1}{5}\left(-t - \frac{1}{2}\sin 2t + \frac{3}{2}\cos 2t + c_2\right) \end{aligned}$$

where we integrated to get u_1 and u_2. The vector $\vec{u}$ is then:

$$\vec{u} = \frac{1}{5} \begin{pmatrix} 2t - \frac{3}{2}\sin 2t - \frac{1}{2}\cos 2t + c_1 \\ -t - \frac{1}{2}\sin 2t + \frac{3}{2}\cos 2t + c_2 \end{pmatrix} .$$

The solution is then

$$\vec{x} = \psi \vec{u} = \frac{1}{5} \begin{pmatrix} 5\cos t & 5\sin t \\ 2\cos t + \sin t & 2\sin t - \cos t \end{pmatrix} \begin{pmatrix} 2t - \frac{3}{2}\sin 2t - \frac{1}{2}\cos 2t + c_1 \\ -t - \frac{1}{2}\sin 2t + \frac{3}{2}\cos 2t + c_2 \end{pmatrix}$$

which upon separating out the constituative parts into the two different eigenvectors simplifies to the following form

$$\begin{aligned} \vec{x}(t) = \frac{1}{5}\left(2t - \frac{3}{2}\sin 2t - \frac{1}{2}\cos 2t + c_1\right) \begin{pmatrix} 5\cos t \\ 2\cos t + \sin t \end{pmatrix} \\ + \frac{1}{5}\left(-t - \frac{1}{2}\sin 2t + \frac{3}{2}\cos 2t + c_2\right) \begin{pmatrix} 5\sin t \\ 2\sin t - \cos t \end{pmatrix} . \end{aligned}$$

We note that generating this solution via the the method of undetermined coefficients is extremely difficult. However, it can be done.

Problem: (B&D 7.9 #5) Solve $\vec{x}' = \begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix} \vec{x} + \begin{pmatrix} t^{-3} \\ -t^{-2} \end{pmatrix}$ and plot the solutions.

We solve the homogeneous part first by turning it into an eigenvalue problem. Letting $\vec{x} = \vec{v}\exp(\lambda t)$ gives

$$\begin{pmatrix} 4-\lambda & -2 \\ 8 & -4-\lambda \end{pmatrix} \vec{v} = 0$$

whose eigenvalues are found by taking the determinant of the above matrix

$$\det\begin{vmatrix} 4-\lambda & -2 \\ 8 & -4-\lambda \end{vmatrix} = (4-\lambda)(-4-\lambda) + 16 = \lambda^2 = 0 \;\rightarrow\; \lambda = 0$$

which is a double root and has the eigenvector:

$$\lambda = 0: \;\; \begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix} \vec{v} = 0 \;\rightarrow\; 2v_1 - v_2 = 0 \;\rightarrow\; \vec{v}^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

and the generalized eigenvector $\vec{v}^{(2)} = \vec{v}^{(1)}t + \vec{\eta}$ where

$$\mathbf{A}\vec{\eta} = \vec{v}^{(1)} \quad\rightarrow\quad 4\eta_1 - 2\eta_2 = 1 \quad\rightarrow\quad \vec{\eta} = -\frac{1}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The homogeneous solution is then

$$\vec{x}_h(t) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \left[\begin{pmatrix} 1 \\ 2 \end{pmatrix} t - \frac{1}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \right].$$

To find the particular solution given $\vec{g}(t) = (t^{-3} \;\; - t^{-2})^T$, we can use either the method of undetermined coefficients or variation of parameters. For the method of undetermined coefficients, we guess the following particular solution

$$\vec{x}_p = \vec{a}t^{-2} + \vec{b}t^{-1} + \vec{c}\ln t + \vec{d}$$

where the $\ln t$ is necessary since it can generate a $1/t$ contribution. Plugging this into the governing equation and collecting like terms results in the following sets of equations:

$$\begin{aligned} t^{-3}: &\;\; -2\vec{a} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ t^{-2}: &\;\; -\vec{b} = \mathbf{A}\vec{a} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ t^{-1}: &\;\; \vec{c} = \mathbf{A}\vec{b} \\ \ln t: &\;\; \mathbf{A}\vec{c} = 0 \\ \text{constant}: &\;\; \mathbf{A}\vec{d} = 0 \end{aligned}$$

We can solve each equation in turn. The first equation is trivial. We simply divide through by -2 to get

$$\vec{a} = -\frac{1}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

The next equation is then given by

$$\vec{b} = -\mathbf{A}\vec{a} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

The last two are also very easily solved since they simply correspond to the case of a zero eigenvalue so that

$$\vec{c} = k\begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad \text{and} \qquad \vec{d} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

where we keep the constant k in $\vec{c}$ since we still need to solve the last equation. This gives us a certain degree of freedom that becomes necessary in the solution. Note further that $\vec{d}$ is just part of the homogeneous solution and can be folded into the constant c_1. The last equation (for the t^{-1} terms) gives

$$\vec{c} = \mathbf{A}\vec{b} \quad \rightarrow \quad \vec{c} = \begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix}\begin{pmatrix} 2 \\ 5 \end{pmatrix} = -2\begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

For consistency then, the k found above must be -2. The solution is then given by ($\vec{x} = \vec{x}_h + \vec{x}_p$)

$$\vec{x}(t) = c_1\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2\left[\begin{pmatrix} 1 \\ 2 \end{pmatrix}t - \frac{1}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] - \frac{1}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix}t^{-2} + \begin{pmatrix} 2 \\ 5 \end{pmatrix}t^{-1} - 2\begin{pmatrix} 1 \\ 2 \end{pmatrix}\ln t\,.$$

Variation of parameters requires the construction of the fundamental matrix

$$\psi(t) = \begin{pmatrix} 1 & t \\ 2 & 2t - 1/2 \end{pmatrix}.$$

We then have that $\psi\vec{u}' = \vec{g}(t)$ so that

$$\begin{pmatrix} 1 & t \\ 2 & 2t - 1/2 \end{pmatrix}\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} t^{-3} \\ -t^{-2} \end{pmatrix}.$$

This yields two equations and two unknowns (u_1' and u_2') so that

$$\begin{aligned} u_1' = t^{-3} - 4t^{-2} - 2t^{-1} \quad &: \quad u_1 = -\frac{1}{2}t^{-2} + 4t^{-1} - 2\ln t + c_1 \\ u_2' = 4t^{-3} + 2t^{-2} \quad &: \quad u_2 = -2t^{-2} - 2t^{-1} + c_2 \end{aligned}$$

where we integrated to get u_1 and u_2. The solution is then

$$\begin{aligned} \vec{x} = \psi\vec{u} &= \begin{pmatrix} 1 & t \\ 2 & 2t - 1/2 \end{pmatrix}\begin{pmatrix} -\frac{1}{2}t^{-2} + 4t^{-1} - 2\ln t + c_1 \\ -2t^{-2} - 2t^{-1} + c_2 \end{pmatrix} \\ &= \begin{pmatrix} -\frac{1}{2}t^{-2} + 4t^{-1} - 2\ln t + c_1 - 2t^{-1} - 2 + c_2 t \\ -t^{-2} + 8t^{-1} - 4\ln t + 2c_1 - 4t^{-1} - 4 + 2c_2 t + t^{-2} + t^{-1} - \frac{1}{2}c_2 \end{pmatrix} \end{aligned}$$

which upon separating out the constituative parts greatly simplifies to the following form

$$\vec{x}(t) = c_1\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2\left[\begin{pmatrix} 1 \\ 2 \end{pmatrix}t - \frac{1}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] - \frac{1}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix}t^{-2} + \begin{pmatrix} 2 \\ 5 \end{pmatrix}t^{-1} - 2\begin{pmatrix} 1 \\ 2 \end{pmatrix}\ln t\,.$$

We note that this solution is exactly the same as that generated by the method of undetermined coefficients. Note further that we have folded the constant vector in as part of the homogeneous solution with constant c_1.

Problem: (B&D 7.9 #7) Solve $\vec{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \vec{x} + \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^t$ using undetermined coefficients and variation of parameters.

We solve the homogeneous part first by turning it into an eigenvalue problem. Letting $\vec{x} = \vec{v}\exp(\lambda t)$ gives

$$\begin{pmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{pmatrix} \vec{v} = 0$$

whose eigenvalues are found by taking the determinant of the above matrix

$$\det\begin{vmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{vmatrix} = (1-\lambda)(1-\lambda) - 4 = \lambda^2 - 2\lambda - 3 = (\lambda-3)(\lambda+1) = 0 \rightarrow \lambda = 3, -1$$

which has the eigenvectors:

$$\lambda = 3 : \begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix} \vec{v} = 0 \rightarrow 2v_1 - v_2 = 0 \rightarrow \vec{v}^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$
$$\lambda = -1 : \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix} \vec{v} = 0 \rightarrow 2v_1 + v_2 = 0 \rightarrow \vec{v}^{(1)} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

The homogeneous solution is then

$$\vec{x}_h(t) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}.$$

To find the particular solution given $\vec{g}(t) = (2\ -1)^T e^t$, we can use either the method of undetermined coefficients or variation of parameters. For undetermined coefficients, we guess the solution

$$\vec{x}_p = \vec{a}e^t \quad \rightarrow \quad \vec{a} = \mathbf{A}\vec{a} + \begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad \rightarrow \quad \vec{a} = \frac{1}{4}\begin{pmatrix} 1 \\ -8 \end{pmatrix} \quad \rightarrow \quad \vec{x}_p = \frac{1}{4}\begin{pmatrix} 1 \\ -8 \end{pmatrix} e^t.$$

Variation of parameters requires the construction of the fundamental matrix $\psi(t)$. We then have that $\psi\vec{u}' = \vec{g}(t)$ so that

$$\begin{pmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 2e^t \\ -e^t \end{pmatrix}.$$

This yields two equations and two unknowns (u_1' and u_2') so that

$$u_1' = \frac{3}{4}e^{-2t} \quad : \quad u_1 = -\frac{3}{8}e^{-2t} + c_1$$
$$u_2' = \frac{5}{4}e^{2t} \quad : \quad u_2 = \frac{5}{8}e^{2t} + c_2$$

where we integrated to get u_1 and u_2. The solution is then

$$\vec{x} = \psi\vec{u} = \begin{pmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{pmatrix} \begin{pmatrix} -\frac{3}{8}e^{-2t} + c_1 \\ \frac{5}{8}e^{2t} + c_2 \end{pmatrix}$$

which upon separating out the constituative parts greatly simplifies to

$$\vec{x}(t) = c_1\begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t} + \frac{1}{4}\begin{pmatrix} 1 \\ -8 \end{pmatrix} e^t.$$

We note that this solution is exactly the same as that generated by the method of undetermined coefficients.

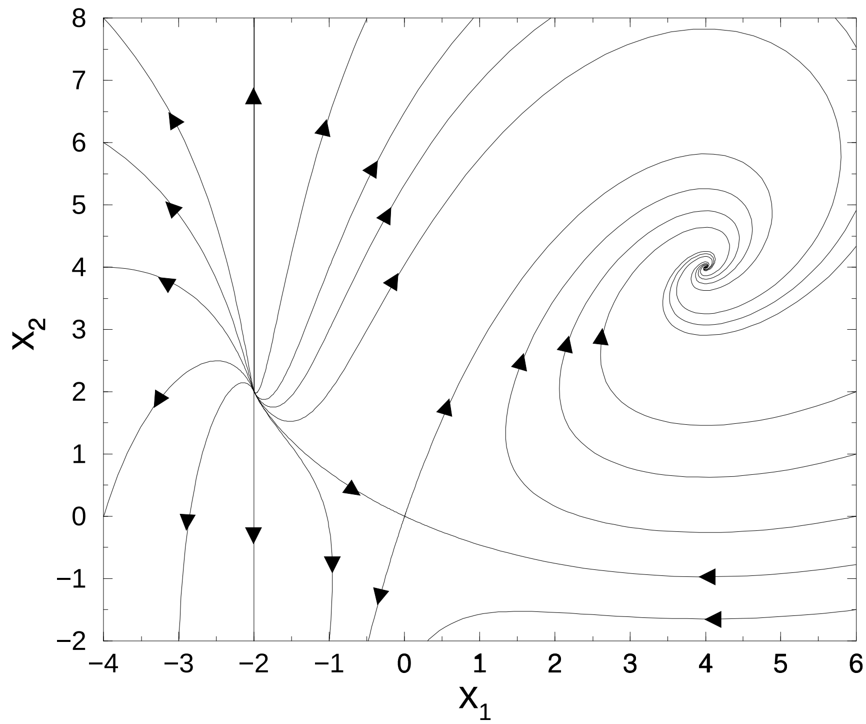

FIG. 37. *Phase–plane dynamics depicting the nonlinear dynamics. The arrows indicate the direction of the flow of the solutions.*

Problem: (B&D 9.3 #5) Consider $x' = (2+x)(y-x)$ and $y' = (4-x)(y+x)$ and plot the solutions.

We begin by considering the equilibrium solutions for which $F = G = 0$:

$$\begin{aligned} F &= (2+x)(y-x) = 0 \\ G &= (4-x)(y+x) = 0 \end{aligned} \quad \rightarrow \quad \text{Equilibrium: } (0,0),\ (-2,2),\ (4,4)\,.$$

To proceed to stability, we first note that $F_x = -2+y-2x$, $F_y = 2+x$, $G_x = 4-y-2x$, and $G_y = 4-x$. For each equilibrium we then find:

$$(0,0): \ \vec{x}' = \begin{pmatrix} -2 & 2 \\ 4 & 4 \end{pmatrix} \vec{x} \ \rightarrow \ \begin{pmatrix} -2-\lambda & 2 \\ 4 & 4-\lambda \end{pmatrix} \vec{v} = 0 \ \rightarrow \ \lambda^2 - 2\lambda - 16 = 0 \ \rightarrow \ \lambda_\pm = 1 \pm \sqrt{17}$$

$$(-2,2): \ \vec{x}' = \begin{pmatrix} 4 & 0 \\ 6 & 6 \end{pmatrix} \vec{x} \ \rightarrow \ \begin{pmatrix} 4-\lambda & 0 \\ 6 & 6-\lambda \end{pmatrix} \vec{v} = 0 \ \rightarrow \ (\lambda-4)(\lambda-6) = 0 \ \rightarrow \ \lambda = 4, 6$$

$$(4,4): \ \vec{x}' = \begin{pmatrix} -6 & 6 \\ -8 & 0 \end{pmatrix} \vec{x} \ \rightarrow \ \begin{pmatrix} -6-\lambda & 6 \\ -8 & -\lambda \end{pmatrix} \vec{v} = 0 \ \rightarrow \ \lambda^2 + 6\lambda + 48 = 0 \ \rightarrow \ \lambda_\pm = -3 \pm i\sqrt{39}$$

The combination of critical points contains an unstable saddle at (0,0), and unstable node at (4,6), and a stable spiral (counter–clockwise) at (4,4).

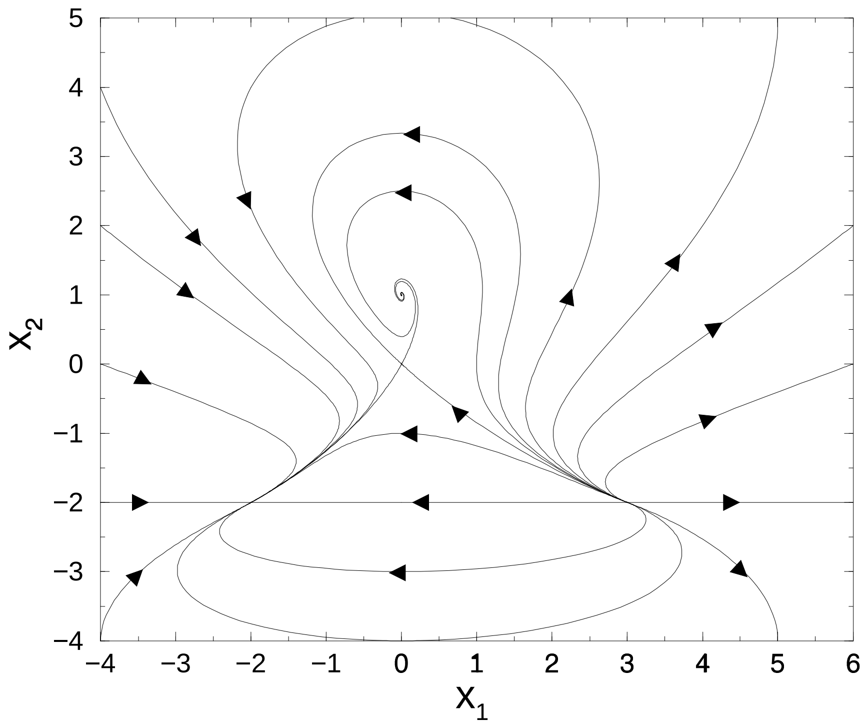

FIG. 38. *Phase–plane dynamics depicting the nonlinear dynamics. The arrows indicate the direction of the flow of the solutions.*

Problem: (B&D 9.3 #9) Consider $x' = -(x-y)(1-x-y)$ and $y' = x(2+y)$ and plot the solutions.

We begin by considering the equilibrium solutions for which $F = G = 0$:

$$\begin{array}{l} F = (x-y)(1-x-y) = 0 \\ G = x(2+y) = 0 \end{array} \quad \rightarrow \quad \text{Equilibrium: } (0,0),\ (0,1),\ (-2,-2),\ (3,-2)\,.$$

To proceed to stability, we first note that $F_x = -1+2x$, $F_y = 1-2y$, $G_x = 2+y$, and $G_y = x$. For each equilibrium we then find:

$$(0,0):\ \vec{x}' = \begin{pmatrix} -1 & 1 \\ 2 & 0 \end{pmatrix}\vec{x} \rightarrow \begin{pmatrix} -1-\lambda & 1 \\ 2 & -\lambda \end{pmatrix}\vec{v} = 0 \rightarrow (\lambda-1)(\lambda+2) = 0 \rightarrow \lambda_\pm = 1, -2$$

$$(0,1):\ \vec{x}' = \begin{pmatrix} -1 & -1 \\ 3 & 0 \end{pmatrix}\vec{x} \rightarrow \begin{pmatrix} -1-\lambda & -1 \\ 3 & -\lambda \end{pmatrix}\vec{v} = 0 \rightarrow \lambda^2+\lambda+3 = 0 \rightarrow \lambda = (-1 \pm i\sqrt{11})/2$$

$$(-2,-2):\ \vec{x}' = \begin{pmatrix} -5 & 5 \\ 0 & -2 \end{pmatrix}\vec{x} \rightarrow \begin{pmatrix} -5-\lambda & 5 \\ 0 & -2-\lambda \end{pmatrix}\vec{v} = 0 \rightarrow (\lambda+5)(\lambda+2) = 0 \rightarrow \lambda_\pm = -2, -5$$

$$(3,-2):\ \vec{x}' = \begin{pmatrix} 5 & 5 \\ 0 & 3 \end{pmatrix}\vec{x} \rightarrow \begin{pmatrix} 5-\lambda & 5 \\ 0 & 3-\lambda \end{pmatrix}\vec{v} = 0 \rightarrow (\lambda-5)(\lambda-3) = 0 \rightarrow \lambda_\pm = 3, 5$$

The critical points contain an unstable saddle at (0,0), an unstable node at (3,5), a stable node at (-2,-2), and a stable spiral (counter–clockwise) at (0,1).

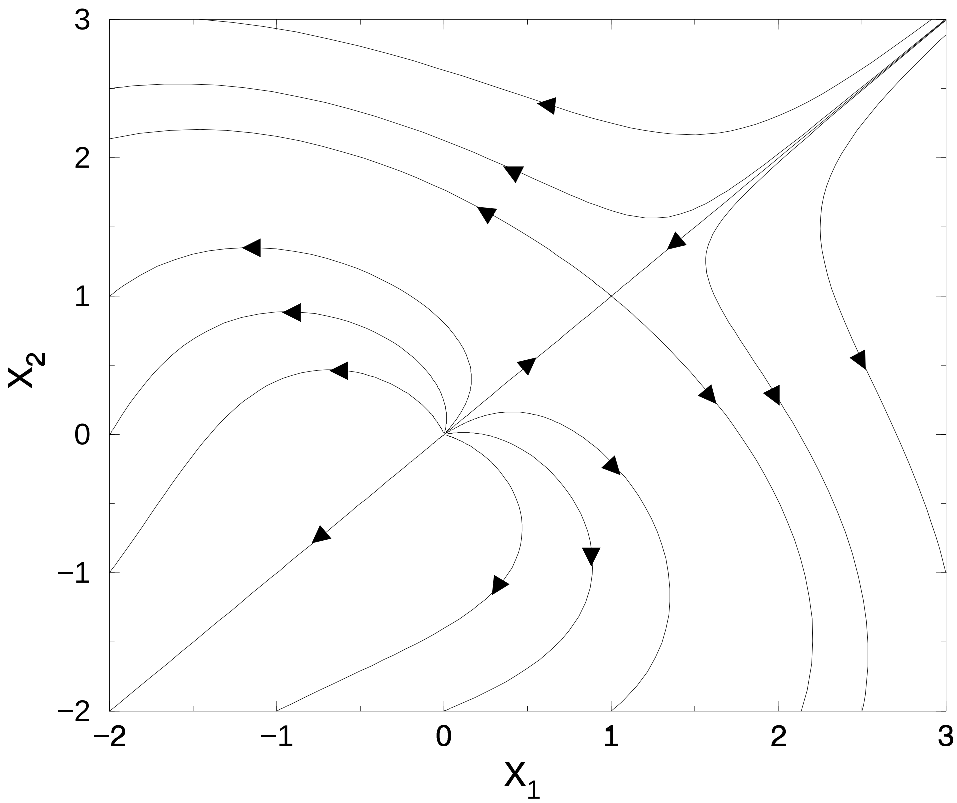

FIG. 39. *Phase–plane dynamics depicting the nonlinear dynamics. The arrows indicate the direction of the flow of the solutions.*

Problem: (B&D 9.3 #13) Consider $x' = x - y^2$ and $y' = y - x^2$ and plot the solutions.

We begin by considering the equilibrium solutions for which $F = G = 0$:

$$\begin{array}{l} F = x - y^2 = 0 \\ G = y - x^2 = 0 \end{array} \quad \rightarrow \quad \text{Equilibrium: } (0,0),\ (1,1)\,.$$

To proceed to stability, we first note that $F_x = 1$, $F_y = -2y$, $G_x = -2x$, and $G_y = 1$. For each equilibrium we then find:

$$(0,0): \ \vec{x}' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \vec{x} \ \rightarrow \ \begin{pmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{pmatrix} \vec{v} = 0 \ \rightarrow \ (\lambda - 1)^2 = 0 \ \rightarrow \ \lambda = 1, 1$$

$$(1,1): \ \vec{x}' = \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix} \vec{x} \ \rightarrow \ \begin{pmatrix} 1-\lambda & -2 \\ -2 & 1-\lambda \end{pmatrix} \vec{v} = 0 \ \rightarrow \ (\lambda - 3)(\lambda + 1) = 0 \ \rightarrow \ \lambda = 3, -1$$

The critical points contain a double root at the origin which produces an unstable improper node at (0,0). In addition, there is a unstable saddle at (1,1). Note that all solutions eventually end up at infinity as time goes to infinity.

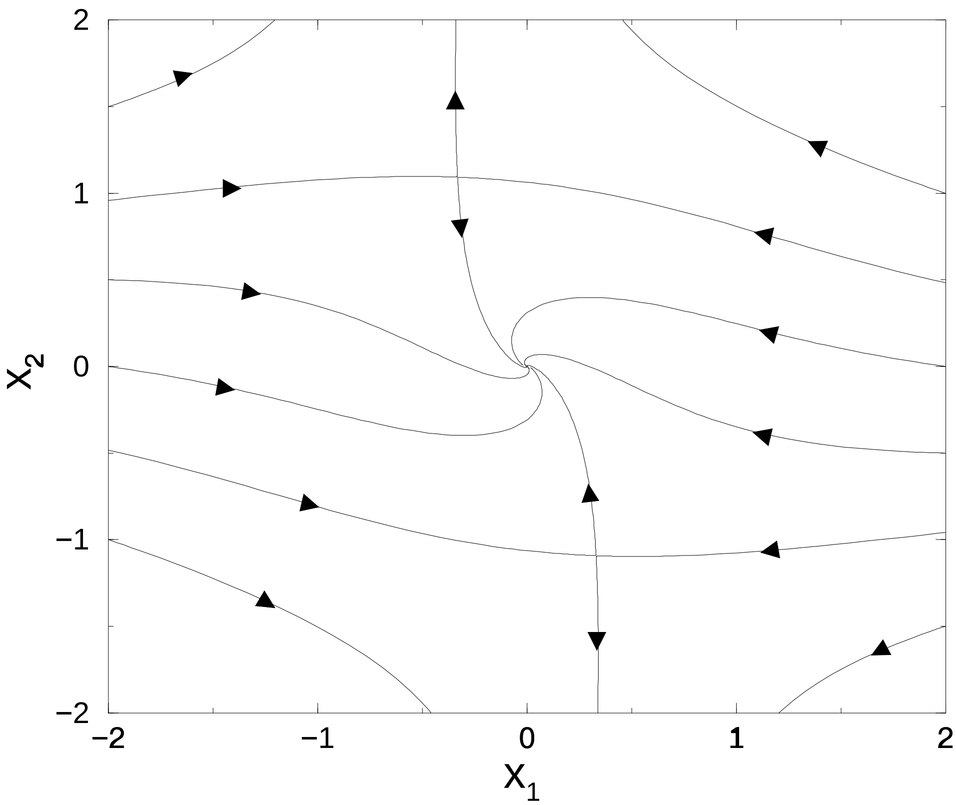

FIG. 40. *Phase–plane dynamics depicting the nonlinear dynamics. The arrows indicate the direction of the flow of the solutions.*

Problem: (B&D 9.3 #15) Consider $x' = -2x - y - x(x^2+y^2)$ and $y' = x - y + y(x^2+y^2)$ and plot the solutions.

We begin by considering the equilibrium solutions for which $F = G = 0$. Besides (0,0), there are two remaining equilibrium points ($x \neq 0$ and $y \neq 0$).

$$\begin{array}{ccc} F = -2x - y - x(x^2+y^2) = 0 & & -2 - y/x - (x^2+y^2) = 0 \\ & \rightarrow & \\ G = x - y + y(x^2+y^2) = 0 & & x/y - 1 + (x^2+y^2) = 0 \end{array},$$

which upon adding gives $x/y - 3 - y/x = 0$. Multiplying by x/y then gives

$$\left(\frac{x}{y}\right)^2 - 3\left(\frac{x}{y}\right) - 1 = 0 \;\rightarrow\; \frac{x}{y} = \frac{3 \pm \sqrt{13}}{2} \;\rightarrow\; x = \left(\frac{3 \pm \sqrt{13}}{2}\right) y\,.$$

We can then plug this back into the second reduced equation above to find

$$\frac{3 \pm \sqrt{13}}{2} - 1 + \left[\left(\frac{3 \pm \sqrt{13}}{2}\right)^2 y^2 + y^2\right] = 0 \;\rightarrow\; y = \pm\sqrt{\frac{-(1-\sqrt{13})}{2 + (3-\sqrt{13})^2/2}}$$

where we take the negative root of $\sqrt{13}$ so that we have a real equilibrium point. To proceed to stability, we first note that $F_x = -2 - 3x^2 - y^2$, $F_y = -1 - 2xy$, $G_x = 1 + 2xy$, and $G_y = -1 + x^2 + 3y^2$. For the equilibrium at the origin we then find that the eigenvalues are:

$$(0,0): \ \vec{x}' = \begin{pmatrix} -2 & -1 \\ 1 & -1 \end{pmatrix} \vec{x} \ \rightarrow \ \begin{pmatrix} -2-\lambda & -1 \\ 1 & -1-\lambda \end{pmatrix} \vec{v} = 0 \ \rightarrow \ \lambda^2 + 3\lambda + 3 = 0 \ \rightarrow \ \lambda_\pm = \frac{-3 \pm i\sqrt{3}}{2}$$

which gives a stable spiral at the origin (0,0). The stability of the remaining two points can now be calculated. First we note that

$$y = \pm \sqrt{\frac{-(1-\sqrt{13})}{2 + (3-\sqrt{13})^2/2}} = \pm 1.0924\,,$$

which gives the the x–location of

$$x = \left(\frac{3-\sqrt{13}}{2}\right) y = \mp 0.33076\,.$$

The two remaining critical points are then

$$(0.33076, -1.0924) \qquad \text{and} \qquad (-0.33076, 1.0924)\,.$$

To determine stability, we must now calculate the eigenvalues of the linearized problem. In both cases we find:

$$F_x = -3.52150$$

$$F_y = -0.27736$$

$$G_x = 0.27736$$

$$G_y = 2.6893\,.$$

Thus we calculate the linearized system for both points to be:

$$\vec{x}' = \begin{pmatrix} -3.52150 & -0.27736 \\ 0.27736 & 2.6893 \end{pmatrix} \vec{x}$$

which yields the eigenvalue problem:

$$\begin{pmatrix} -3.52150-\lambda & -0.27736 \\ 0.27736 & 2.6893-\lambda \end{pmatrix} \vec{v} = 0 \ \rightarrow \ \lambda^2 + 0.8322\lambda - 9.3935 = 0 \ \rightarrow \ \lambda_\pm = -3.509, 2.677\,.$$

Thus the two remaining points are both saddles since the eigenvalues are purely real and of opposite sign. The dynamics is then determined by the interaction between the stable spiral at the origin and the two unstable saddles away from the origin.

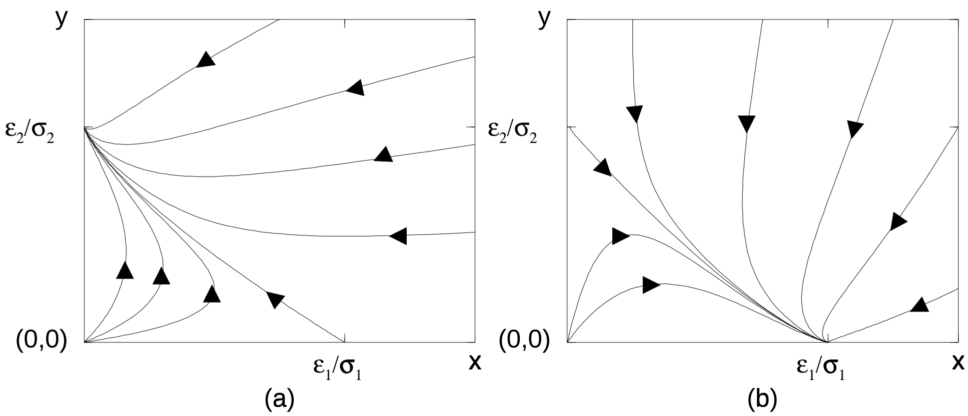

FIG. 41. *Phase–plane dynamics depicting the nonlinear dynamics. The arrows indicate the direction of the flow of the solutions. We consider the two cases: (a)* $\epsilon_2/\alpha_2 > \epsilon_1/\sigma_1$ *and* $\epsilon_2/\sigma_2 > \epsilon_1/\alpha_1$ *and (b)* $\epsilon_2/\alpha_2 < \epsilon_1/\sigma_1$ *and* $\epsilon_2/\sigma_2 < \epsilon_1/\alpha_1$.

Problem: (B&D 9.4 #8) Consider $x' = x(\epsilon_1 - \sigma_1 x - \alpha_1 y)$ and $y' = y(\epsilon_2 - \sigma_2 y - \alpha_2 x)$. Plot the solutions where all the constants are positive and consider the two cases: (a) $\epsilon_2/\alpha_2 > \epsilon_1/\sigma_1$ and $\epsilon_2/\sigma_2 > \epsilon_1/\alpha_1$ and (b) $\epsilon_2/\alpha_2 < \epsilon_1/\sigma_1$ and $\epsilon_2/\sigma_2 < \epsilon_1/\alpha_1$.

We begin by considering the equilibrium solutions for which $F = G = 0$:

$$\begin{array}{l} F = x(\epsilon_1 - \sigma_1 x - \alpha_1 y) = 0 \\ G = y(\epsilon_2 - \sigma_2 y - \alpha_2 x) = 0 \end{array} \quad \rightarrow \quad \text{Equilibrium: } (0,0),\ (0,\epsilon_2/\sigma_2),\ (\epsilon_1/\sigma_1, 0)\,.$$

To proceed to stability, we first note that $F_x = \epsilon_1 - 2\sigma_1 x - \alpha_1 y$, $F_y = -\alpha_1 x$, $G_x = -\alpha_2 y$, and $G_y = \epsilon_2 - 2\sigma_2 y - \alpha_2 x$. For each equilibrium we then find:

$$(0,0):\ \vec{x}' = \begin{pmatrix} \epsilon_1 & 0 \\ \epsilon_2 & 0 \end{pmatrix} \vec{x} \rightarrow \begin{pmatrix} \epsilon_1 - \lambda & 0 \\ 0 & \epsilon_2 - \lambda \end{pmatrix} \vec{v} = 0 \rightarrow (\lambda - \epsilon_1)(\lambda - \epsilon_2) = 0 \rightarrow \lambda_\pm = \epsilon_1, \epsilon_2$$

$$(0, \epsilon_2/\sigma_2):\ \vec{x}' = \begin{pmatrix} \epsilon_1 - \alpha_1\epsilon_2/\sigma_2 & 0 \\ -\alpha_2\epsilon_2/\sigma_2 & -\epsilon_2 \end{pmatrix} \vec{x} \rightarrow \lambda = -\epsilon_2 \text{ and } \lambda = \alpha_1 \left(\frac{\epsilon_1}{\alpha_1} - \frac{\epsilon_2}{\sigma_2} \right)$$

$$(\epsilon_1/\sigma_1, 0):\ \vec{x}' = \begin{pmatrix} -\epsilon_1 & -\alpha_1\epsilon_1/\sigma_1 \\ 0 & \epsilon_2 - \alpha_2\epsilon_1/\sigma_1 \end{pmatrix} \vec{x} \rightarrow \lambda = -\epsilon_1 \text{ and } \lambda = \alpha_2 \left(\frac{\epsilon_2}{\alpha_2} - \frac{\epsilon_1}{\sigma_1} \right)$$

There are two parts two this problem:

$$(a)\ \frac{\epsilon_2}{\alpha_2} > \frac{\epsilon_1}{\sigma_1} \text{ and } \frac{\epsilon_2}{\sigma_2} > \frac{\epsilon_1}{\alpha_1}:\ (0, \epsilon_2/\sigma_2) \rightarrow \text{stable node } (\epsilon_1/\sigma_1, 0) \rightarrow \text{saddle}$$

$$(b)\ \frac{\epsilon_2}{\alpha_2} < \frac{\epsilon_1}{\sigma_1} \text{ and } \frac{\epsilon_2}{\sigma_2} < \frac{\epsilon_1}{\alpha_1}:\ (0, \epsilon_2/\sigma_2) \rightarrow \text{saddle } (\epsilon_1/\sigma_1, 0) \rightarrow \text{stable node}$$

Thus for the two cases, the dynamics are depicted in the figure with (a) giving rise to an attractor at $(0, \epsilon_2/\sigma_2)$ and (b) giving an attractor at $(\epsilon_1/\sigma_1, 0)$. In case (a), only the y species survives: causing the extinction of bluegill. Alternatively, in (b) on the x species survives: causing the extinction of redear.