

7.A Vectors

A scalar is a real number.
 The set of all n -tuples $X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$
 of scalars is denoted by \mathbb{R}^n ,
 and X is called a vector in \mathbb{R}^n .

If c is a scalar and X, Y two vectors
 in \mathbb{R}^n , we define

$$cX = \begin{pmatrix} cX_1 \\ \vdots \\ cX_n \end{pmatrix}, \quad X \pm Y = \begin{pmatrix} X_1 \pm Y_1 \\ \vdots \\ X_n \pm Y_n \end{pmatrix}$$

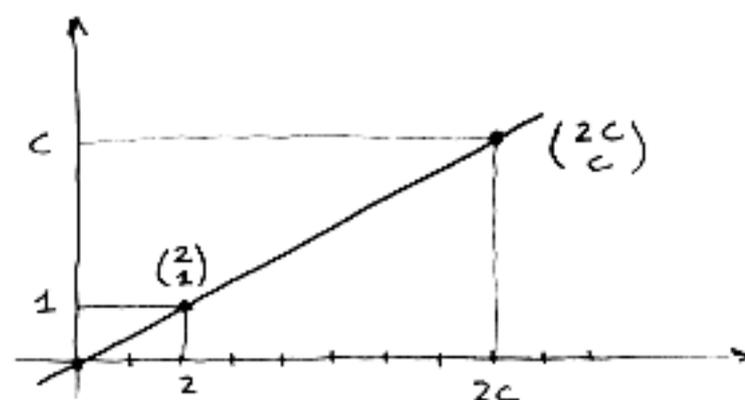
If X^1, X^2, \dots, X^m are given vectors in \mathbb{R}^n ,
 then the set of all linear combinations

$$c_1 X^1 + c_2 X^2 + \dots + c_m X^m$$

is called a linear subspace of \mathbb{R}^n ,
 spanned by X^1, \dots, X^m .

Example: Some linear subspaces of \mathbb{R}^2

- (a) the point $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$
- (b) all points of the form $c \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix}$
- (c) \mathbb{R}^2



\mathbb{R}^2 can be spanned by any two (or more)
 vectors which are not multiples of each
 other

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_2 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (2x_2 - x_1) \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} + (x_1 - x_2) \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix} : \text{do not span } \mathbb{R}^2$$

Problem 1: Show that the vectors

$$x^1 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \quad x^2 = \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}, \quad x^3 = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$$

span \mathbb{R}^3

A set of vectors x^1, x^2, \dots, x^m is called linearly independent, if

$$c_1 x^1 + c_2 x^2 + \dots + c_m x^m \neq \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

except if $c_1 = c_2 = \dots = c_m = 0$.

Example:

(a) $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ are lin. independent

(b) $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}$ are lin. dependent

Problem 2: Show that the vectors in problem 1 are lin. independent

If a linear subspace S of \mathbb{R}^n is spanned by a set of lin. independent vectors x^1, \dots, x^m , then x^1, \dots, x^m are called a basis for S .

Example: The vectors

$$\delta^1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \delta^2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \dots, \delta^n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

form a basis for \mathbb{R}^n , since for all x

$$x = x_1 \delta^1 + x_2 \delta^2 + \dots + x_n \delta^n$$

with $x=0$ only if $x_1 = \dots = x_n = 0$.

Example: The three vectors

$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ span the x - y -plane S in \mathbb{R}^3 . But they are not a basis for S since they are lin. dependent.

Theorem 1: If x^1, \dots, x^m and y^1, \dots, y^k are both a basis for S , then $m=k$.

The number $m=k$ is called the dimension of S . Note that \mathbb{R}^n has dimension n .

Example: The lin. subspaces of \mathbb{R}^3 are

dimension 0 : (by definition) the point 0
 dimension 1 : straight lines through 0
 dimension 2 : planes through 0
 dimension 3 : \mathbb{R}^3

The same can be done with complex scalars. The set of complex n -tuples is denoted by \mathbb{C}^n . Given two vectors x, y in \mathbb{C}^n we define their scalar product

$$(x, y) = \bar{x}_1 y_1 + \bar{x}_2 y_2 + \dots + \bar{x}_n y_n$$

The length of x is defined by

$$\|x\| = \sqrt{(x, x)}$$

Problem: Show that

$$(u + \alpha x, v + \beta y) = (u, v) + \bar{\alpha}(x, v) + \beta(u, y) + \bar{\alpha}\beta(x, y)$$

and show that (x, x) is always real and positive

Two vectors x, y are said to be orthogonal if $(x, y) = 0$.

Problem:

7. B Matrices

Here we consider maps A from \mathbb{R}^m to \mathbb{R}^n which are linear, i.e.

$$A(c_1 x + c_2 y) = c_1 A(x) + c_2 A(y)$$

The vectors

$$a^i = A(\delta^i), \quad \delta^i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow \text{component } i$$

are called column vectors of A .
For every x in \mathbb{R}^m we get

$$\begin{aligned} A(x) &= A(x_1 \delta^1 + \dots + x_m \delta^m) \\ &= x_1 A(\delta^1) + \dots + x_m A(\delta^m) \\ &= x_1 a^1 + x_2 a^2 + \dots + x_m a^m \end{aligned}$$

Note that $A(x)$ is in the linear subspace of \mathbb{R}^n spanned by a^1, a^2, \dots, a^m . This subspace is called the range of A , and its dimension is called the rank of A .

We will identify the map A with the $n \times m$ matrix

$$A = (a_j^i) = \begin{pmatrix} a_1^1 & \cdots & a_1^m \\ \vdots & & \vdots \\ a_n^1 & \cdots & a_n^m \end{pmatrix}$$

and write $A \cdot x$ or Ax for $A(x)$.

Example:

$$\begin{pmatrix} 2 & 0 & 4 \\ 1 & 0 & 2 \\ 0 & 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2x_1 + 0 + 4x_3 \\ x_1 + 0 + 2x_3 \\ 0 + x_2 - x_3 \end{pmatrix}$$

The three column vectors of A are dependent,

$$-2 \cdot \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The range of A is not the whole space \mathbb{R}^3 , but only a plane.

$$Ax = (x_1 + 2x_3) \cdot \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + x_2 \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

so the rank of A is 2.

Example: The $n \times n$ matrix

$$I = (\delta_j^i) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & 1 & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}$$

is called the identity matrix. We have $Ix = x$ for all x . Thus the range of I is \mathbb{R}^n and the rank of I is n .

Problem 1: Determine the rank of A , $A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & 2 & -1 \end{pmatrix}$

Matrices can be multiplied with scalars

$$cA = (ca_j^i)$$

and we can define the sum (difference) of two $n \times n$ matrices

$$A \pm B = (a_j^i \pm b_j^i)$$

Example:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + 2 \cdot \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 11 & 14 \\ 17 & 20 \end{pmatrix}$$

B4

Problem 2: Show that $A(cX) = cAX$,
and that $(A+B)X = AX + BX$.

Suppose that A is $n \times k$ and that B is
 $k \times m$. Then

$$\begin{aligned} BAX &= B(x_1 a^1 + \dots + x_m a^m) \\ &= x_1 B a^1 + \dots + x_m B a^m \end{aligned}$$

Therefore $BAX = TX$ where T is the $n \times m$
matrix with column vectors

$$t^i = B a^i$$

We write $T = BA$ and call T the
product of A and B .

$$\begin{array}{|c|} \hline n & m \\ \hline T \\ \hline \end{array} = \begin{array}{|c|} \hline n & k \\ \hline B \\ \hline \end{array} \cdot \begin{array}{|c|} \hline k & m \\ \hline A \\ \hline \end{array}$$

B5

Example:

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$$

$$AB = \begin{pmatrix} 5 \cdot 1 + 7 \cdot 2 & 6 \cdot 1 + 8 \cdot 2 \\ 5 \cdot 3 + 7 \cdot 4 & 6 \cdot 3 + 8 \cdot 4 \end{pmatrix} = \begin{pmatrix} 19 & 22 \\ 43 & 50 \end{pmatrix}$$

Problem 3: With the same A, B as above,
show that $BA \neq AB$

If A is a square matrix then we can compute

$$A^n = A \cdot A \cdot \dots \cdot A \quad (n \text{ times})$$

More general, for every analytic function

$$f(t) = \sum_{n=0}^{\infty} c_n t^n$$

we can define the matrix

$$f(A) = \sum_{n=0}^{\infty} c_n A^n$$

Example: Compute e^{nA} for $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Solution

$$A^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

$$n = 2k \text{ even: } A^n = (A^2)^k = I^k = I$$

$$n = 2k+1 \text{ odd: } A^n = (A^2)^k A = I \cdot A = A$$

$$\begin{aligned} e^{rA} &= \sum_{n=0}^{\infty} \frac{1}{n!} (rA)^n = \sum_{n \text{ even}} \frac{r^n}{n!} \cdot I + \sum_{n \text{ odd}} \frac{r^n}{n!} A \\ &= (\cosh r)I + (\sinh r)A = \begin{pmatrix} \cosh r & \sinh r \\ \sinh r & \cosh r \end{pmatrix} \end{aligned}$$

Problem 4: Compute e^A for

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

7.C Systems of 1st order DE's, part I

If a matrix depends on a parameter t ,

$$B(t) = (\zeta_j^i(t))$$

then we can compute its derivative

$$\frac{d}{dt} B(t) = \lim_{h \rightarrow 0} \frac{1}{h} (B(t+h) - B(t)) = \left(\frac{d}{dt} \zeta_j^i(t) \right)$$

Theorem 2: Let A be a square matrix. Then

$$(a) \frac{d}{dt} e^{tA} = A e^{tA}$$

$$(b) e^{-tA} \cdot e^{tA} = I$$

Example:

$$\text{If } A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ then } e^{\pm tA} = \begin{pmatrix} c & \pm s \\ \pm s & c \end{pmatrix},$$

where $c = \cosh t$ and $s = \sinh t$.

$$e^{-tA} \cdot e^{tA} = \begin{pmatrix} c & -s \\ -s & c \end{pmatrix} \cdot \begin{pmatrix} c & s \\ s & c \end{pmatrix} = \begin{pmatrix} c^2 - s^2 & cs - sc \\ -cs + cs & -s^2 + c^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

C2

$$Ae^{tA} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c & s \\ s & c \end{pmatrix} = \begin{pmatrix} s & c \\ c & s \end{pmatrix}$$

$$\frac{d}{dt} e^{tA} = \begin{pmatrix} \dot{c} & \dot{s} \\ \dot{s} & \dot{c} \end{pmatrix} = \begin{pmatrix} s & c \\ c & s \end{pmatrix} = Ae^{tA}$$

Note that we just solved the initial value problem

$$\frac{d}{dt} X(t) = AX(t), \quad X(0) = y$$

The solution is

$$X(t) = e^{tA} y$$

If y^1, \dots, y^n are lin. independent, then

$$x_1(t) = e^{tA} y^1, \dots, x_n(t) = e^{tA} y^n$$

are also lin. independent. This follows since

$$0 = c_1 x^1(t) + \dots + c_n x^n(t)$$

implies that

$$\begin{aligned} 0 &= e^{-tA} (c_1 x^1(t) + \dots + c_n x^n(t)) \\ &= c_1 y^1 + \dots + c_n y^n \end{aligned}$$

C3

Example: Find the general solution of

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -x_1 + 2x_2$$

Solution: $\dot{x} = Ax$, where $A = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}$.

The solutions are $x = e^{tA} y = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n y$

$$A^2 = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} -2 & 3 \\ -3 & 4 \end{pmatrix}$$

$$\vdots$$

$$A^n = \begin{pmatrix} -n+1 & n \\ -n & n+1 \end{pmatrix} = n \cdot \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$e^{tA} = \sum_{n=0}^{\infty} \frac{nt^n}{n!} \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} + \sum_{n=0}^{\infty} \frac{t^n}{n!} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -te^t + e^t & te^t \\ -te^t & te^t + e^t \end{pmatrix}$$

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = e^{tA} y = \begin{pmatrix} y_1 \cdot (1-t)e^t + y_2 \cdot te^t \\ -y_1 e^t + y_2 \cdot (1+t)e^t \end{pmatrix}$$

Problem: Compute $x = e^{tA} y$ and check that $\dot{x} = Ax$.

$$(a) A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (z) A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

As in the 1-dimensional case it is also possible to find an explicit solution for the inhomogeneous problem

$$\dot{x}(t) = Ax(t) + z(t), \quad x(0) = y$$

Namely

$$\dot{x} - Ax = z$$

$$e^{-tA} \dot{x} - Ae^{-tA} x = e^{-tA} z$$

$$\frac{d}{dt} (e^{-tA} x) = e^{-tA} z$$

$$e^{-tA} x = \int_0^t e^{-sA} z ds + y$$

$$x(t) = \int_0^t e^{(t-s)A} z(s) ds + e^{tA} y$$

7.D The inverse of a matrix

A $n \times n$ matrix A is called invertible if for every vector y the equation

$$Ax = y$$

has exactly one solution x . By using Theorem 1 it is possible to prove

Theorem 3: The following five statements are equivalent

- (a) The rank of A is n
- (b) a^1, a^2, \dots, a^n are lin. independent
- (c) $Ax = 0$ only if $x = 0$
- (d) A is invertible
- (e) There is a $n \times n$ matrix A^{-1} such that

$$A^{-1}A = AA^{-1} = I$$

A matrix which violates (a), ..., (e), or which is not square, is called singular.

D2

Example: The inverse of e^A is e^{-A}

Problem 1: Show that $(AB)^{-1} = B^{-1}A^{-1}$

Note that if A is invertible, then $Ax = y$ has exactly one solution, namely $x = A^{-1}y$. If A is singular, then there are vectors z which satisfy $Az = 0$. As a consequence $A(x + cz) = Ax$. Therefore $Ax = y$ has either no solution or infinitely many.

Problem 2: Find all solutions of

$$\begin{pmatrix} 1 & b \\ c & 2 \end{pmatrix} x = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ for all choices of } b, c.$$

Example: Solve

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= 7 \\ -x_1 + x_2 - 2x_3 &= -5 \\ 2x_1 - x_2 - x_3 &= 4 \end{aligned}$$

Solution: Add 1^{st} to 2^{nd} and add (-2) times 1^{st} to 3^{rd} :

D3

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= 7 \\ -x_2 + x_3 &= 2 \\ 3x_2 - 7x_3 &= -10 \end{aligned}$$

Add 3 times 2^{nd} to 3^{rd} and then multiply 2^{nd} by (-1)

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= 7 \\ x_2 - x_3 &= -2 \\ -4x_3 &= -4 \end{aligned}$$

Divide 3^{rd} by (-4) . We get a triangular system

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= 7 \\ x_2 - x_3 &= -2 \\ x_3 &= 1 \end{aligned}$$

Add 3^{rd} to 2^{nd} and add (-3) times 3^{rd} to 1^{st}

$$\begin{aligned} x_1 - 2x_2 &= 4 \\ x_2 &= -1 \\ x_3 &= 1 \end{aligned}$$

D4

Add 2 times 2nd to 1st. We get a diagonal system (solution)

$$\begin{aligned} x_1 &= 2 \\ x_2 &= -1 \\ x_3 &= 1 \end{aligned}$$

Solving $Ax = y$ for arbitrary y is the same as to invert A . This can be done by solving

$$Ax^i = \delta^i$$

simultaneously for all basis vectors δ^i . Then the solutions

$$x^i = A^{-1} \delta^i$$

are just the column vectors of A^{-1} .

Example:

$$A \quad \delta^1 \delta^2 \delta^3$$

$$\begin{pmatrix} 1 & -2 & 3 \\ -1 & 1 & -2 \\ 2 & -1 & -1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

D5

Add 1st to 2nd and add (-2) times 1st to 3rd

$$\begin{pmatrix} 1 & -2 & 3 \\ 0 & -1 & 1 \\ 0 & 3 & -7 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$$

Add 3 times 2nd to 3rd and then multiply 2nd by (-1)

$$\begin{pmatrix} 1 & -2 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & -4 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ -1 & -1 & 0 \\ 1 & 3 & 1 \end{pmatrix}$$

Divide 3rd by (-4)

$$\begin{pmatrix} 1 & -2 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \quad -\frac{1}{4} \begin{pmatrix} -4 & 0 & 0 \\ 4 & 4 & 0 \\ 1 & 3 & 1 \end{pmatrix}$$

Add 3rd to 2nd and add (-3) times 3rd to 1st

$$\begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad -\frac{1}{4} \begin{pmatrix} -7 & -9 & -3 \\ 5 & 7 & 1 \\ 1 & 3 & 1 \end{pmatrix}$$

D6

Add 2 times 2nd to 1st

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \underbrace{-\frac{1}{4} \begin{pmatrix} 3 & 5 & -1 \\ 5 & 7 & 1 \\ 1 & 3 & 1 \end{pmatrix}}_{A^{-1}}$$

The inverse of A can also be obtained by using determinants. Consider the submatrices $A[\begin{smallmatrix} i \\ j \end{smallmatrix}]$ of A , obtained from A by deleting column i and row j .

The determinant of A is defined by

$$(*) \quad \det A = \sum_{i=1}^n (-1)^{i+j} a_j^i \det A[\begin{smallmatrix} i \\ j \end{smallmatrix}]$$

with $j=1$. The determinant of a $n \times 1$ or $1 \times n$ matrix is zero, except for $n=1$. If a is the only element of A , then $\det A = a$.

Example:

$$\begin{aligned} \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= (-1)^{1+1} a \det(d) + (-1)^{2+1} b \det(c) \\ &= ad - bc \end{aligned}$$

D7

Example: If we denote $\det(a_j^i)$ by $|a_j^i|$, then

$$\begin{aligned} \begin{vmatrix} 1 & -2 & 3 \\ -1 & 1 & -2 \\ 2 & -1 & -1 \end{vmatrix} &= (-1)^{1+1} \cdot 1 \cdot \begin{vmatrix} 1 & -2 \\ -1 & -1 \end{vmatrix} \\ &+ (-1)^{2+1} \cdot (-2) \cdot \begin{vmatrix} -1 & -2 \\ 2 & -1 \end{vmatrix} + (-1)^{3+1} \cdot 3 \cdot \begin{vmatrix} -1 & 1 \\ 2 & -1 \end{vmatrix} = \\ &= 1 \cdot (-3) + 2 \cdot 5 + 3 \cdot (-1) = 4 \end{aligned}$$

Theorem 4:

- (a) The expression (*) does not depend on the choice of j .
- (b) A is invertible if and only if $\det A \neq 0$
- (c) The matrix elements of A^{-1} are

$$b_j^i = (-1)^{i+j} \frac{\det A[\begin{smallmatrix} i \\ j \end{smallmatrix}]}{\det A}$$

Problem 3: Find the inverse of

$$A = \begin{pmatrix} 1 & -1 & 2 \\ -2 & 1 & -1 \\ 3 & -2 & -1 \end{pmatrix}$$

7. E Eigenvalues and Eigenvectors

If there is a scalar λ and a vector $x \neq 0$ such that

$$Ax = \lambda x$$

then λ is called an eigenvalue of A , and x is called an eigenvector of A (corresponding to λ).

Example: The diagonal matrix

$$D = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \\ & & & \lambda_n \end{pmatrix}$$

has eigenvalues $\lambda_1, \dots, \lambda_n$, and $x = \begin{pmatrix} 0 \\ \vdots \\ x_j \\ \vdots \\ 0 \end{pmatrix}$ is an eigenvector of D corresponding to the eigenvalue λ_j .

Note that if $Ax - \lambda x = 0$, then

$$(A - \lambda I)x = 0.$$

Therefore either $x = 0$ or $A - \lambda I$ is singular. This shows

Theorem 5: λ is an eigenvalue of A if and only if $\det(A - \lambda I) = 0$.

Example: Find the eigenvalues and eigenvectors of $A = \begin{pmatrix} 5 & 1 \\ -2 & 2 \end{pmatrix}$

Solution:

$$A - \lambda I = \begin{pmatrix} 5 & 1 \\ -2 & 2 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 5-\lambda & 1 \\ -2 & 2-\lambda \end{pmatrix}$$

$$\det(A - \lambda I) = (5-\lambda)(2-\lambda) + 2 = \lambda^2 - 7\lambda + 12$$

The eigenvalues of A are $\lambda = 3, \lambda = 4$. To get the eigenvectors we solve

$$\begin{pmatrix} 5-\lambda & 1 \\ -2 & 2-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

For $\lambda = 3$:

For $\lambda = 4$:

Note that for fixed λ the solutions x of $(A - \lambda I)x = 0$ form a linear subspace

E3

Example: For $A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}$ and

$\lambda = 2$: the plane spanned by $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$
 $\lambda = 3$: the line spanned by $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

Theorem 6: If y is an eigenvector of A with eigenvalue λ , then

$$x(t) = e^{\lambda t} y \text{ solves } \dot{x} = Ax$$

Proof 1: Use the solution $x = e^{\lambda t} y$ and the fact that

$$A^n y = A^{n-1} A y = A^{n-1} \lambda y = \lambda A^{n-1} y = \dots = \lambda^n y$$

Then we get

$$x = e^{\lambda t} y = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n y = \sum_{n=0}^{\infty} \frac{t^n}{n!} \lambda^n y = e^{\lambda t} y$$

Proof 2: Compute $Ax - \dot{x}$ for $x = e^{\lambda t} y$

$$Ax - \dot{x} = Ae^{\lambda t} y - \lambda e^{\lambda t} y = e^{\lambda t} (A - \lambda I)y = 0$$

E4

As for complex eigenvalues λ , eigenvectors y , and solutions $x(t)$, it is easy to see that everything comes in complex conjugate pairs,

$$x^{\pm} = X^1 \pm iX^2 \text{ etc.}$$

provided that A is a real matrix. Real solutions are obtained as usual

$$c_1 X^1 + c_2 X^2 = \frac{c_1 + ic_2}{2} X^+ + \frac{c_1 - ic_2}{2} X^-$$

Example: $\begin{cases} \dot{x}_1 = 3x_1 + 5x_2 \\ \dot{x}_2 = -2x_1 + x_2 \end{cases}$

Solution: Analyze the matrix $A = \begin{pmatrix} 3 & 5 \\ -2 & 1 \end{pmatrix}$. The eigenvalues λ of A satisfy

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 5 \\ -2 & 1-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 13 = 0$$

For each eigenvalue $\lambda = 2 \pm 3i$ we determine one eigenvector y .

$$(A - \lambda I)y = \begin{pmatrix} 3-\lambda & 5 \\ -2 & 1-\lambda \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$(3-\lambda)y_1 + 5y_2 = 0$$

$$y = \begin{pmatrix} 5 \\ \lambda - 3 \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \pm 3i \end{pmatrix}$$

E5

We find two independent solutions

$$\begin{aligned} x &= e^{\lambda t} & y &= e^{(2 \pm 3i)t} \begin{pmatrix} 5 \\ -1 \pm 3i \end{pmatrix} \\ & & &= e^{2t} \begin{pmatrix} 5 \cos 3t \\ -\cos 3t - 3 \sin 3t \end{pmatrix} \\ & & &\pm i e^{2t} \begin{pmatrix} 5 \sin 3t \\ 3 \cos 3t - \sin 3t \end{pmatrix} \end{aligned}$$

The general solution is

$$\begin{aligned} x_1 &= c_1 e^{2t} \cdot 5 \cos 3t + c_2 e^{2t} \cdot 5 \sin 3t \\ x_2 &= c_1 e^{2t} (-\cos 3t - 3 \sin 3t) \\ &\quad + c_2 e^{2t} (3 \cos 3t - \sin 3t) \end{aligned}$$

Problem 1: Solve $\begin{cases} \dot{x}_1 = 5x_1 + 8x_2 \\ \dot{x}_2 = cx_1 + 3x_2 \end{cases}$

for $c=1$ and $c=3$.

Theorem 7: Eigenvectors y^1, \dots, y^k with different eigenvalues $\lambda_1, \dots, \lambda_k$ are lin. independent

E6

Proof: Suppose that the theorem holds for $k=m-1$. To prove it for $k=m$, we show that the assumption

$$(*) \quad c_1 y^1 + \dots + c_m y^m = 0$$

implies that $c_1 = \dots = c_m = 0$. Starting now with $(*)$, we get

$$\begin{aligned} 0 &= (A - \lambda_m I)(c_1 y^1 + \dots + c_m y^m) \\ &= c_1 \underbrace{(\lambda_1 - \lambda_m)}_{\neq 0} y^1 + \dots + c_{m-1} \underbrace{(\lambda_{m-1} - \lambda_m)}_{\neq 0} y^{m-1} \end{aligned}$$

Since y^1, \dots, y^{m-1} are lin. independent, it follows that $c_1 = \dots = c_{m-1} = 0$. But $y^m \neq 0$, and therefore $(*)$ can only be satisfied if also $c_m = 0$.

Corollary: A $n \times n$ matrix with n different eigenvalues has n linearly independent eigenvectors, namely (up to a constant) exactly one for each eigenvalue.

Whenever a $n \times n$ matrix A has n independent eigenvectors y^1, \dots, y^n with (not necessarily different) eigenvalues $\lambda^1, \dots, \lambda^n$, then the general solution of $\dot{x} = Ax$ is simply

$$x(t) = a_1 e^{\lambda_1 t} y^1 + \dots + a_n e^{\lambda_n t} y^n.$$

This can also be written as

$$x(t) = Y a(t), \quad a_i(t) = a_i e^{\lambda_i t}$$

where Y is the (nonsingular) matrix with column vectors y^1, \dots, y^n . In an initial value problem, the coefficients a_i are determined by

$$a(0) = Y^{-1} x(0).$$

Problem 2: Find the general solution of

$$\begin{aligned} \dot{x}_1 &= x_1 + x_2 + x_3 \\ \dot{x}_2 &= 2x_1 + x_2 - x_3 \\ \dot{x}_3 &= -8x_1 - 5x_2 - 3x_3 \end{aligned}$$

A matrix A is called symmetric, if the matrix elements satisfy

$$a_{ij} = \bar{a}_{ji} \quad \text{for all } i, j$$

Theorem 8: If A is a symmetric $n \times n$ matrix, then

- A has n lin. independent eigenvectors
- The eigenvalues of A are real
- The eigenvectors y^1, \dots, y^n are (or can be chosen) mutually orthogonal.

Problem 3: Check (a), (b), (c) for $A = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$.

Find real solutions of $\dot{x} = zAx$ by

- using $e^{i\lambda_1 t} y^1$ and $e^{i\lambda_2 t} y^2$
- computing $e^{z t A}$
- using the method of elimination.

F1

7.7 Systems of 1st order DE's, part II

We already know the general solution of

$$\dot{x} = Ax,$$

namely

$$x(t) = e^{tA} y,$$

where y is an arbitrary vector. The only problem is to find n linearly indep. vectors y^1, y^2, \dots, y^n for which $e^{tA} y$ is easy to compute!

Candidates #1: For each eigenvalue λ of A find the eigenvectors y by solving

$$(A - \lambda I)y = 0.$$

Then we have

$$x(t) = e^{tA} y = e^{\lambda t} y.$$

F2

"Unfortunately" not every $n \times n$ matrix has n linearly independent eigenvectors.

Example:

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix}$$

Here $\lambda = 2$ is a triple eigenvalue, since

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & 1 & 2-\lambda \end{vmatrix} \\ &= (2-\lambda) \begin{vmatrix} 2-\lambda & 0 \\ 1 & 2-\lambda \end{vmatrix} = (2-\lambda)^3 \end{aligned}$$

But A has only two lin. independent eigenvectors y , since the equation

$$(A - 2I)y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

requires that $y_2 = 0$.

F3

Candidates # 2, 3, 4, ...

Consider the case where A has a multiple eigenvalue λ . Assume that we can find a sequence of vectors

$$(*) \quad 0 \rightsquigarrow v^1 \rightsquigarrow v^2 \rightsquigarrow v^3 \rightsquigarrow \dots$$

by successively solving the equations

$$(A - \lambda I)v^1 = 0 \quad (\text{eigenvector})$$

$$(A - \lambda I)v^2 = v^1$$

$$(A - \lambda I)v^3 = v^2$$

$$\vdots$$

$$(A - \lambda I)v^k = v^{k-1}$$

$$\vdots$$

Then $e^{tA}v^k$ is easy to compute,

$$e^{tA}v^k = e^{t\lambda I} e^{t(A-\lambda I)} v^k$$

$$= e^{t\lambda} \left[v^k + t \underbrace{(A-\lambda I)v^k}_{v^{k-1}} + \frac{t^2}{2} \underbrace{(A-\lambda I)^2 v^k}_{v^{k-2}} + \dots \right].$$

Thus for each v_k in the sequence we obtain an explicit solution of $\dot{x} = Ax$,

F4

$$x(t) = e^{tA} v^k$$

$$= e^{t\lambda} \left[v^k + t v^{k-1} + \frac{t^2}{2} v^{k-2} + \dots + \frac{t^{k-1}}{(k-1)!} v^1 \right]$$

Example:

$$A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix}$$

Step 1: Find all eigenvalues of A :

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 0 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & 1 & 2-\lambda \end{vmatrix} = (3-\lambda)(2-\lambda)^2 = 0$$

$\lambda = 3$ is a simple eigenvalue and $\lambda = 2$ is a double eigenvalue.

Step 2: Find all eigenvectors

$\lambda = 3$:

$$(A - 3I)u = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -u_2 \\ u_2 - u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

We get $u = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

F5

 $\lambda = 2$:

$$(A - 2I)v = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} v_1 \\ 0 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Therefore $v = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

A has only two eigenvectors. We need

Step 3: There is a third vector associated with $\lambda = 2$. We solve $(A - 2I)w = v$,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} w_1 \\ 0 \\ w_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

and obtain $w = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

The general solution of $\dot{x} = Ax$ is therefore

$$x(t) = a \cdot e^{3t} u + b \cdot e^{2t} v + c \cdot e^{2t} [w + t \cdot v],$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} a e^{3t} \\ (b + ct) e^{2t} \\ c e^{2t} \end{pmatrix}$$

F6

Problem: Find three independent solutions of

$$(a) \quad \dot{x} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & -4 \\ 0 & 1 & -1 \end{pmatrix} \cdot x$$

$$(b) \quad \dot{x} = \begin{pmatrix} 5 & -3 & -2 \\ 8 & -5 & -4 \\ -4 & 3 & 3 \end{pmatrix} \cdot x$$

Theorem 9: Suppose that A is a $n \times n$ matrix. Then the multiplicities of its eigenvalues add up to n , i.e.

$$\det(A - \lambda I) = \text{const} \cdot (\lambda - \lambda_1)^{m_1} (\lambda - \lambda_2)^{m_2} \dots$$

with $m_1 + m_2 + \dots = n$. If λ is an eigenvalue of multiplicity m , then the sequence (*) leads to m linearly independent vectors v^k . All the vectors v^k together (from all eigenvalues) span \mathbb{R}^n .