The Distribution of Prime Numbers

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Preface

Notation

For functions f and g with $g \geq 0$, we write $f(x) = O(g(x))$ or $f(x) \ll g(x)$ when there is a constant c such that $|f(x)| \leq cg(x)$; when f and g are both non-negative, we may also write $f \gg g$ instead of $g \ll f$. We write $f(x) \sim g(x)$ when $\lim f(x)/g(x) = 1$ as x tends to some limit to be specified at each occurrence. We use $c, c_0, c_1, c_2, \dots$ to denote implicit constants; these and the constants implied by O- and $\ll$-symbols are presumed absolute, unless stated otherwise. Throughout these notes (and much of number theory outside them) the letter p, with or without subscripts or superscripts, is reserved for prime numbers. We also use $s = \sigma + \mathrm{i}t$ to denote a complex variable.

For the most part, we use the standard notations for common number-theoretic functions. These are usually defined at their first appearance, but for convenience we also list them here:

$\|\theta\|$	the distance from the real number θ to the nearest integer;
$[\theta]$	the integral part of the real number θ;
$\{\theta\}$	the fractional part of the real number θ;
$e(z)$	$e^{2\pi \mathrm{i} z}$;
$\operatorname{Log} z$	the principal branch of the complex logarithm ($\operatorname{Log} x = \ln x$ when $x > 0$);
$d(n)$	the number of positive divisors of n;
$\phi(n)$	Euler's totient function: the number of reduced residue classes modulo n;
$\mu(n)$	the Möbius function (see (1.1));
$\Lambda(n)$	von Mangoldt's function (see (1.3));
$\pi(x)$	the number of primes $p \leq x$;
$\pi(x; q, a)$	the number of primes $p \leq x$, with $p \equiv a \pmod q$;
$\theta(x)$	Chebyshev's function (see (1.4));
$\psi(x)$	the sum of the values of $\Lambda(n)$ over $n \leq x$;
$\psi(x; q, a)$	the sum of the values of $\Lambda(n)$ over $n \leq x$, with $n \equiv a \pmod q$;
$\psi(x, \chi)$	the sum of the values of $\Lambda(n)\chi(n)$ over $n \leq x$ (see (3.50));
$\tau(\chi, a)$	the Gaussian sum (see (3.7));

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Chapter 0

Historical background

0.1 Early history

The first result on the distribution of primes is Euclid's theorem (*circa* 300 B.C.) on the infinitude of the primes. In 1737 Euler went a step further and proved that, in fact, the series of the reciprocals of the primes diverges. In the opposite direction, Euler observed that the rate of divergence of this series is much slower than the rate of divergence of the harmonic series:

> "*The sum of the series of the reciprocals of the prime numbers,*
>
> $$\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \frac{1}{13} + \cdots,$$
>
> *is infinitely large, but it is infinitely many times less than the sum of the harmonic series,*
>
> $$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots.$$
>
> *Furthemore, the sum of the former series is like the logarithm of the sum of the latter series.*"

This statement appears to be the earliest attempt to quantify the frequency of the primes among the positive integers.

Consider the prime counting function

$$\pi(x) = \sum_{p \le x} 1.$$

In 1798 Legendre conjectured that $\pi(x)$ satisfies the asymptotic relation

$$\lim_{x \to \infty} \frac{\pi(x)}{x/(\log x)} = 1; \tag{0.1}$$

this is the *prime number theorem* (PNT). Years later, Gauss wrote that back in his adolescent years he had observed that the logarithmic integral

$$\operatorname{Li} x = \int_2^x \frac{dt}{\log t}$$

seemed to provide a very good approximation to $\pi(x)$. This, of course, is consistent with (0.1), as can be seen from the formula

$$\mathrm{Li}\, x = \frac{x}{\log x} + \frac{1!x}{(\log x)^2} + \cdots + \frac{k!x}{(\log x)^{k+1}} + O\left(\frac{x}{(\log x)^{k+2}}\right). \tag{0.2}$$

The first theoretical evidence in support of the PNT was given by Chebyshev in the 1850s. He proved that:

- (0.1) predicts correctly the order of magnitude of $\pi(x)$, that is, there exist absolute constants $c_2 > c_1 > 0$ such that
$$\frac{c_1 x}{\log x} \le \pi(x) \le \frac{c_2 x}{\log x}. \tag{0.3}$$
Chebyshev showed that for sufficiently large x one may take $c_1 = 0.9212$ and $c_2 = 1.1056$. In his honor, bounds for $\pi(x)$ of this type are now known as *Chebyshev's estimates*.

- If the limit on the left side of (0.1) exists, then it must be equal to 1.

Chebyshev used the methods that he developed for the proof of (0.3) to establish *Bertrand's postulate:* the interval $(n, 2n]$ contains a prime number for all integers $n \ge 1$. Furthermore, in 1874 Mertens used Chebyshev's estimates (0.3) to show that

$$\sum_{p \le x} \frac{1}{p} = \log\log x + B + O\big((\log x)^{-1}\big), \tag{0.4}$$

B being an absolute constant. This provided the first rigorous proof of Euler's observation that "the sum of the [series of the reciprocals of the primes] is like the logarithm of the sum of the [harmonic series]." We sketch the proofs of (0.3), (0.4), and some related results in §1.2.

0.2 The Riemann ζ-function and the prime number theorem

The *Riemann zeta-function* is defined in the half-plane $\mathrm{Re}(s) > 1$ as

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \prod_p \left(1 - p^{-s}\right)^{-1}. \tag{0.5}$$

The identity between the infinite series and the infinite product on the right (which runs over all primes) is an analytic expression of the fundamental theorem of arithmetic and was discovered by Euler in 1737 (in the same paper as his proof of the infinitude of the primes), at least in the case when s is real. The first to consider $\zeta(s)$ as a function of a complex variable was Riemann. In 1859 he published his seminal paper [39] (his only paper on number theory), in which he observed that $\zeta(s)$ is holomorphic in the half-plane $\mathrm{Re}(s) > 1$ and that it can be continued analytically to a meromorphic function, whose only singularity is a simple pole at $s = 1$. Riemann was interested in $\zeta(s)$, because Euler's identity (0.5) provides a connection between the analytic properties of $\zeta(s)$

and the PNT. It is not difficult to deduce from (0.5) that $\zeta(s)$ does not vanish in the half-plane $\mathrm{Re}(s) > 1$. Riemann proved that $\zeta(s)$ satisfies the *functional equation*

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{(s-1)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s),$$

from which it is easy to deduce that the only zeros of $\zeta(s)$ in the half-plane $\mathrm{Re}(s) < 0$ are the negative even integers; these are the *trivial zeros* of $\zeta(s)$. Besides the trivial zeros, $\zeta(s)$ has infinitely many zeros in the strip $0 \leq \mathrm{Re}(s) \leq 1$: the *non-trivial zeros* of $\zeta(s)$. Riemann proposed several conjectures about the non-trivial zeros of $\zeta(s)$:

C1. If

$$N(T) = \#\big\{\rho \in \mathbb{C} : \zeta(\rho) = 0,\ 0 \leq \mathrm{Re}(\rho) \leq 1,\ 0 < \mathrm{Im}(\rho) \leq T\big\},$$

then

$$N(T) = \frac{T}{2\pi}\log\left(\frac{T}{2\pi e}\right) + O(\log T).$$

C2. The entire function

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

has a product representation

$$\xi(s) = e^{A+Bs}\prod_{\rho}\left(1-\frac{s}{\rho}\right)e^{-s/\rho},$$

the product being over all non-trivial zeros of $\zeta(s)$.

C3. If $x > 1$, there is an explicit formula that represents $\pi(x)$ as a series over the non-trivial zeros of $\zeta(s)$.

C4. **Riemann Hypothesis (RH).** All zeros of $\zeta(s)$ with $0 \leq \mathrm{Re}(s) \leq 1$ lie on the line $\mathrm{Re}(s) = \frac{1}{2}$.

By the end of the 19th century, conjectures C1–C3 were proved: C1 and C3 were established by von Mangoldt, and C2 is a consequence of the general theory of entire functions of finite order developed by Hadamard. In particular, the Riemann–Mangoldt explicit formula for $\pi(x)$ demonstrated that the PNT follows from the nonvanishing of $\zeta(s)$ on the line $\mathrm{Re}(s) = 1$. Thus, when in 1896 Hadamard and de la Vallée Poussin proved (independently) that $\zeta(1 + it) \neq 0$ for all real t, the PNT was finally proved. In contrast, the Riemann Hypothesis is still an open problem that has been selected by the Clay Mathematics Institute as one of the seven Millennium Problems. We remark that under RH, the Riemann–Mangoldt formula implies the asymptotic formula

$$\pi(x) = \mathrm{Li}\,x + O\big(x^{1/2}\log x\big), \tag{0.6}$$

which is essentially best possible.

The last observation has motivated the investigations of the error term in the PNT. In 1899 de la Vallée Poussin refined the original proof that $\zeta(1 + it) \neq 0$ and showed that, in fact, $\zeta(\sigma + it)$ does not vanish in the region

$$\sigma \geq 1 - \frac{c}{\log(|t| + 10)}, \tag{0.7}$$

for some absolute constant $c > 0$. This suffices to establish the following quantitative version of the PNT, which will be the main subject of Chapter 2 of these notes.

Theorem 1. *There exists an absolute constant* $c > 0$ *such that*

$$\pi(x) = \operatorname{Li} x + O\big(x \exp\big(-c\sqrt{\log x}\big)\big).$$

Further improvements on the error term in the PNT have been quite limited. In 1922 Littlewood proved that

$$\pi(x) - \operatorname{Li} x \ll x \exp\big(-c\sqrt{\log x \log\log x}\big), \tag{0.8}$$

while the best result to date was obtained by Korobov [31] and I. M. Vinogradov [50] in 1958:

$$\pi(x) - \operatorname{Li} x \ll x \exp\big(-c(\log x)^{3/5}(\log\log x)^{-1/5}\big). \tag{0.9}$$

Both (0.8) and (0.9) are consequences of repsective improvements on the estimate of the zerofree region (0.7). Unfortunately, it is known that the approach employed in these works can never yield a bound of the form $\pi(x) - \operatorname{Li} x \ll x^{\theta}$, with a fixed $\theta < 1$.

0.3 Primes in arithmetic progressions

In a couple of memoirs published in 1837 and 1840, Dirichlet proved that if a and q are natural numbers with $(a, q) = 1$, then the arithmetic progression $a, a + q, a + 2q, \dots$ contains infinitely many primes. By refining Dirichlet's argument, Mertens established the asymptotic formula

$$\sum_{\substack{p \leq x \\ p \equiv a \ (\mathrm{mod}\ q)}} \frac{1}{p} \sim \frac{1}{\phi(q)} \sum_{p \leq x} \frac{1}{p} \qquad \text{as } x \to \infty, \tag{0.10}$$

where $\phi(q)$ is Euler's totient function. Fix q and consider the various reduced residue classes modulo q. Since all but finitely many primes lie in residue classes a mod q with $(a, q) = 1$, (0.10) suggests that the primes are uniformly distributed among the reduced residue classes to a given modulus q. Thus, one may expect that if $(a, q) = 1$, then

$$\pi(x; q, a) = \sum_{\substack{p \leq x \\ p \equiv a \ (\mathrm{mod}\ q)}} 1 \sim \frac{\operatorname{Li} x}{\phi(q)} \qquad \text{as } x \to \infty. \tag{0.11}$$

This is the *prime number theorem for arithmetic progressions*. We may approach this statement in two different ways. First, we may fix a and q and ask whether (0.11) holds (allowing the convergence to depend on q and a). Posed in this form, the problem is a minor generalization of the PNT. In fact, shortly after proving Theorem 1, de la Vallée Poussin showed that

$$\pi(x;q,a) = \frac{\mathrm{Li}\, x}{\phi(q)} + O\big(x \exp\big(-c\sqrt{\log x}\big)\big),$$

where $c = c(q,a) > 0$ and the O-implied constant depends on q and a. The problem becomes much more difficult if we want an estimate that is explicit in q and uniform in a. The first result of this kind was obtained by Page [35], who proved that there exists a (small) positive number δ such that

$$\pi(x;q,a) = \frac{\mathrm{Li}\, x}{\phi(q)} + O\big(x \exp\big(-(\log x)^{\delta}\big)\big),$$

whenever $1 \le q \le (\log x)^{2-\delta}$ and $(a,q) = 1$. In 1935 Siegel [41] proved the following result, which we will establish in Chapter 3.

Theorem 2. *For any fixed $A > 0$, there exists a constant $c = c(A) > 0$ such that*

$$\pi(x;q,a) = \frac{\mathrm{Li}\, x}{\phi(q)} + O\big(x \exp\big(-c\sqrt{\log x}\big)\big),$$

whenever $1 \le q \le (\log x)^A$ and $(a,q) = 1$.

Remark. While this result is clearly sharper than Page's, it does have one significant drawback: it is ineffective, that is, given a particular value of A, the proof does not allow the constant $c(A)$ or the O-implied constant to be computed.

The proofs of the above results rely on the analytic properties of a class of generalizations of the Riemann zeta-function known as *Dirichlet L-functions*. For each positive integer q there are $\phi(q)$ functions $\chi : \mathbb{Z} \to \mathbb{C}$ called *Dirichlet characters modulo q* (we will define these in Chapter 3). Given a character χ modulo q, we define the respective Dirichlet L-function by

$$L(s,\chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s} \qquad (\mathrm{Re}(s) > 1).$$

Similarly to $\zeta(s)$, $L(s,\chi)$ is holomorphic in the half-plane $\mathrm{Re}(s) > 1$ and can be continued analytically to a meromorphic function that has at most one pole, which (if present) must be a simple pole at $s = 1$. Furthermore, just as $\zeta(s)$, the continued $L(s,\chi)$ has infinitely many zeros in the strip $0 \le \mathrm{Re}(s) \le 1$, and the horizontal distribution of those zeros has important implications on the distribution of primes in arithmetic progressions. For example, the results of de la Vallée Poussin, Page, and Siegel mentioned above were proved by showing that no L-function has zeros "close" to the line $\mathrm{Re}(s) = 1$. We also have the following generalization of the Riemann Hypothesis:

> **Generalized Riemann Hypothesis (GRH).** Let $L(s,\chi)$ be a Dirichlet L-function. Then all zeros of $L(s,\chi)$ with $0 \le \mathrm{Re}(s) \le 1$ lie on the line $\mathrm{Re}(s) = \frac{1}{2}$.

Assuming GRH, we can deduce easily that

$$\pi(x; q, a) = \frac{\operatorname{Li} x}{\phi(q)} + O\big(x^{1/2} \log x\big), \tag{0.12}$$

whenever $(a, q) = 1$. This estimate establishes (0.11) for $1 \le q \le x^{\theta}$, $\theta < \frac{1}{2}$.

In many applications one only needs approximations like (0.12) in some average sense over the moduli q. During the 1950s and 1960s several authors obtained such results. In particular, the following quantity was studied extensively:

$$E(x, Q) = \sum_{q \le Q} \max_{(a,q)=1} \max_{y \le x} \left| \pi(y; q, a) - \frac{\operatorname{Li} y}{\phi(q)} \right|.$$

The trivial bound for this sum is $E(x, Q) \ll x$, whereas (0.12) implies

$$E(x, Q) \ll Q x^{1/2} \log x. \tag{0.13}$$

In 1965 Bombieri [5] and A. I. Vinogradov [47] proved (independently) the following result.

Theorem 3. *For any fixed $A > 0$, there exists a constant $B = B(A) > 0$ such that*

$$E(x, Q) \ll x(\log x)^{-A},$$

whenever $Q \le x^{1/2}(\log x)^{-B}$.

We observe that this result provides a nontrivial estimate for $E(x, Q)$ under essentially the same restrictions on Q as GRH. Because of this fact, the Bombieri–Vinogradov theorem has seen numerous applications in which it has been used as a *de facto* replacement for GRH. In Chapter 5 we will give a proof of Theorem 3 with $B = A + 4$.

It should be noted that unlike the error term in (0.6), the error term in (0.12) is not necessarily best possible. In fact, there is some evidence in support of the bold conjecture that

$$\pi(x; q, a) = \frac{\operatorname{Li} x}{\phi(q)} + O_{\epsilon}\big((x/q)^{1/2+\epsilon}\big)$$

for any fixed $\epsilon > 0$. In Chapter 5 we will establish the so-called *Barban–Davenport–Halberstam theorem*, which asserts that this bound holds in the mean-square over all arithmetic progressions with differences $q \le x^{1-\epsilon}$. We should also mention that during the mid 1980s Bombieri, Friedlander, and Iwaniec [6, 7, 8] obtained several variants of the Bombieri–Vinogradov theorem, in which the value of Q can exceed $x^{1/2}$. However, since their methods go beyond the reach of these notes, we will only state one of their results (see [7]).

Theorem 4. *Let $a \ne 0$ and $x \ge y \ge 3$. Then*

$$\sum_{\substack{q \le \sqrt{xy} \\ (q,a)=1}} \left| \pi(x; q, a) - \frac{\operatorname{Li} x}{\phi(q)} \right| \ll (\operatorname{Li} x) \left(\frac{\log y}{\log x} \right)^2 (\log\log x)^c.$$

Here c is an absolute constant and the $\ll$-implied constant depends only on a.

0.4 Primes in short intervals

It is an old problem in the theory of prime numbers to prove that for any integer $n \geq 1$, the interval $(n^2, (n+1)^2]$ contains a prime number. This problem leads quickly to the more general question of estimating the differences between consecutive primes. Cramér was the first to study this question systematically. Let p_n denote the nth prime number. In 1920 Cramér [10] proved that under RH

$$p_{n+1} - p_n \ll p_n^{1/2} \log p_n.$$

Cramér also proposed a probabilistic model of the prime numbers that leads to very precise (and very bold) predictions of the asymptotic properties of the primes. In particular, he conjectured [11] that

$$\limsup_{n\to\infty} \frac{p_{n+1} - p_n}{(\log p_n)^2} = 1. \tag{0.14}$$

Nontrivial upper bounds for $p_{n+1} - p_n$ can be derived from the quantitative versions of the PNT stated above, but the ensuing results are rather poor, because the known bounds for the error term in the PNT are barely smaller than the main term. However, in 1930 Hoheisel [22] obtained a much sharper result. He proved (unconditionaly) the asymptotic formula

$$\pi(x + x^\theta) - \pi(x) \sim x^\theta (\log x)^{-1} \qquad \text{as } x \to \infty, \tag{0.15}$$

with $\theta = 1 - (33000)^{-1}$. Subsequently several authors made further contributions that produced the following improvements on Hoheisel's result:

Heilbronn [21] (1933)	$\theta = 0.996$
Chudakov [9] (1936)	$\theta > 3/4 \ = 0.750$
Ingham [26] (1937)	$\theta > 5/8 \ = 0.625$
Montgomery [33] (1971)	$\theta > 3/5 \ = 0.600$
Huxley [23] (1972)	$\theta > 7/12 = 0.583\ldots$
Heath-Brown [19] (1988)	$\theta = 7/12 = 0.583\ldots$

We will see the proof of Huxley's result in Chapter 5 of these notes. Furthermore, since the late 1970s, several mathematicians have shown that even shorter intervals must contain primes (without establishing an asymptotic formula for the number of primes in such intervals). Such results usually take the form

$$\pi(x + x^\theta) - \pi(x) \gg x^\theta (\log x)^{-1} \qquad \text{for } x \geq x_0(\theta). \tag{0.16}$$

The following list displays the progress in that direction over the last 30 years:

Iwaniec and Jutila [27] (1979)	$\theta = 13/23 = 0.565\ldots$
Heath-Brown and Iwaniec [20] (1979)	$\theta > 11/20 = 0.550$
Iwaniec and Pintz [28] (1984)	$\theta = 23/42 = 0.547\ldots$
Lou and Yao [32] (1992)	$\theta = 6/11 \ = 0.545\ldots$
Baker and Harman [1] (1996)	$\theta = 0.535$
Baker, Harman, and Pintz [2] (2001)	$\theta = 0.525$

Selberg [40] considered the distribution of primes in “almost all short intervals.” Let $h(x)$ be an increasing function of x. We say that *almost all* intervals $(x, x+h(x)]$ contain primes if the measure of the set of $x \in (1, X]$ for which the interval $(x, x+h(x)]$ contains no prime is $o(X)$. Selberg proved that if $h(x)$ grows faster than $(\log x)^2$ as $x \to \infty$, the Riemann Hypothesis implies that almost all intervals $(x, x+h(x)]$ contain a prime number (and also that the asymptotic formula (0.15) holds for each inexceptional interval). Further, Selberg showed unconditionally that if $\theta > 1/4$, then almost all intervals $(x, x+x^{\theta}]$ contain a prime number. The latter result has been the subject of a long series of successive improvements, similar to the improvements on Hoheisel’s result described above. In particular, the best result to date obtained in 1996 by Jia [29] extends the range for θ in Selberg’s result to $\theta > 1/20$.

In the opposite direction, Erdös [14] showed in 1935 that

$$p_{n+1} - p_n \geq c \log p_n \log\log p_n (\log\log\log p_n)^{-2} \tag{0.17}$$

infinitely often. In 1938 Rankin [38] showed that one can replace the right side of (0.17) by

$$(1/3 + o(1)) \log p_n \log\log p_n \log\log\log\log p_n (\log\log\log p_n)^{-2},$$

but subsequent attempts at further improvements have not been very successful: the best result to date (see Pintz [36]) replaces the constant 1/3 in Rankin’s bound by $2e^{\gamma}$, where γ is Euler’s constant. In fact, the problem appears to be so notoriously difficult that Erdös—who was known for offerring monetary prizes for solutions of problems he was intrigued by—announced that he would pay $10,000 to anyone who proved that the constant 1/3 in Rankin’s result can be taken arbitrarily large!

Chapter 1

Introduction: basic estimates

The purpose of this chapter is to introduce some of the basic techniques and functions appearing in the later chapters. The results are mostly elementary and the reader may be familiar with some (and possibly all) of them.

1.1 Multiplicative functions

A function $f : \mathbb{N} \to \mathbb{C}$ is said to be *multiplicative* if it is not identically zero and

$$f(mn) = f(m)f(n) \qquad \text{whenever } \gcd(m,n) = 1.$$

If f satisfies the stronger condition that $f(mn) = f(m)f(n)$ for all pairs m, n, it is said to be *completely (or totally) multiplicative*.

Some functions, such as $f(n) = n^s$ $(s \in \mathbb{C})$, are obviously multiplicative. Others are defined so that they are. One such function is the *Möbius function*

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ (-1)^r & \text{if } n \text{ is the product of } r \text{ distinct primes}, \\ 0 & \text{if } n \text{ is divisible by the square of a prime}. \end{cases} \tag{1.1}$$

The following lemma provides an easy way to deduce the multiplicativity of a large class of arithmetic functions. We leave its proof as an exercise.

Lemma 1.1. *Suppose that f and g are multiplicative functions. Then the arithmetic function $f * g$, defined by*

$$(f * g)(n) = \sum_{d|n} f(d)g(n/d),$$

is also multiplicative.

The next lemma contains the most important property of the Möbius function.

Lemma 1.2. $\sum_{d|n}\mu(d) = \begin{cases} 1 & \textit{if } n = 1, \\ 0 & \textit{otherwise}. \end{cases}$

Proof. It suffices to consider the case when n is squarefree. Suppose that $n = p_1 p_2 \cdots p_r$, where $p_1, p_2, \dots, p_r$ are distinct primes, and write $m = p_1 p_2 \cdots p_{r-1}$. Then

$$\sum_{d|n}\mu(d) = \sum_{d|m}\mu(d) + \sum_{d|m}\mu(dp_r) = \sum_{d|m}\mu(d) + \sum_{d|m}(-\mu(d)) = 0,$$

where the second to last step uses the multiplicativity of μ. ■

Corollary 1.3. *(Möbius inversion formula) Suppose that $f : \mathbb{N} \to \mathbb{C}$ is an arithmetic function and define*

$$F(n) = \sum_{d|n} f(d).$$

Then $f = F * \mu$, that is,

$$f(n) = \sum_{d|n} F(d)\mu(n/d).$$

In particular, if F is multiplicative, so is f.

Proof. We have

$$\sum_{d|n} F(d)\mu(n/d) = \sum_{d|n}\sum_{k|d} f(k)\mu(n/d) = \sum_{k|n}\sum_{\substack{d|n \\ k|d}} f(k)\mu(n/d) = \sum_{k|n}\sum_{m|(n/k)} f(k)\mu(n/km).$$

By Lemma 1.2, the sum over m vanishes unless $k = n$, so the first claim follows. The second claim is a consequence of the first, Lemma 1.1, and the multiplicativity of μ. ■

1.2 Partial summation

We now discuss a simple trick that is put to a great use in analytic number theory.

Lemma 1.4 (Abel)**.** *Suppose that a_n are complex numbers and $f(x)$ is continuously differentiable on $[\alpha, \beta]$. Then*

$$\sum_{\alpha<n\le\beta} a_n f(n) = A(\beta)f(\beta) - \int_\alpha^\beta A(x)f'(x)\,dx,$$

where $A(x) = \sum_{\alpha<n\le x} a_n$.

Proof. Using Stieltjes integration by parts, we have

$$\sum_{\alpha<n\le\beta} a_n f(n) = \int_{\alpha^+}^{\beta^+} f(x)\,dA(x) = f(x)A(x)\Big|_\alpha^\beta - \int_\alpha^\beta A(x)\,df(x),$$

and the desired result follows. ■

Corollary 1.5. *There is a constant* c_1 *such that*

$$\sum_{n\le x}\frac{1}{n} = \log x + c_1 + O\big(x^{-1}\big).$$

Corollary 1.6. $\displaystyle\sum_{n\le x}\log n = x\log x - x + O(\log x).$

Remark. The constant c_1 is known as *Euler's constant* and usually is denoted by γ:

$$\gamma = \lim_{x\to\infty}\left(\sum_{n\le x}\frac{1}{n} - \log x\right) \approx 0.5772\dots. \tag{1.2}$$

Next, we define three arithmetic functions that play an important role in prime number theory. These are *von Mangoldt's function*

$$\Lambda(n) = \begin{cases}\log p & \text{if } n \text{ is a power of a prime } p,\\ 0 & \text{otherwise,}\end{cases} \tag{1.3}$$

and the functions

$$\theta(x) = \sum_{p\le x}\log p \qquad \text{and} \qquad \psi(x) = \sum_{n\le x}\Lambda(n), \tag{1.4}$$

which were introduced first by Chebyshev. Our first result about these functions is a Chebyshev-type bound for $\psi(x)$.

Theorem 1.7. *Suppose that* $0 < c_2 < \log 2$ *and* $c_3 > \log 4$*. Then for sufficiently large* x*,*

$$c_2 x \le \psi(x) \le c_3 x.$$

Proof. Define

$$T(x) = \sum_{n\le x}\log n.$$

Taking logarithms in the prime factorization of n, we see that

$$\log n = \sum_{d|n}\Lambda(d),$$

so we can rewrite $T(x)$ as

$$T(x) = \sum_{n\le x}\sum_{d|n}\Lambda(d) = \sum_{d\le x}\Lambda(d)\sum_{\substack{n\le x\\ d|n}}1 = \sum_{d\le x}\Lambda(d)\left[\frac{x}{d}\right].$$

Thus,

$$T(x) - 2T(x/2) = \sum_{d\le x}\Lambda(d)\left(\left[\frac{x}{d}\right] - 2\left[\frac{x}{2d}\right]\right).$$

We now note that

$$0 \leq \left[\frac{x}{d}\right] - 2\left[\frac{x}{2d}\right] \leq 1$$

and

$$\left[\frac{x}{d}\right] - 2\left[\frac{x}{2d}\right] = 1 \qquad \text{for } x/2 < d \leq x.$$

Hence,

$$\psi(x) - \psi(x/2) \leq T(x) - 2T(x/2) \leq \psi(x).$$

On the other hand, by Corollary 1.6,

$$T(x) - 2T(x/2) = x\log 2 + O(\log x).$$

We deduce that

$$\psi(x) \geq x\log 2 + O(\log x)$$

and

$$\begin{aligned}
\psi(x) &\leq \psi(x/2) + x\log 2 + O(\log x)\\
&\leq \psi(x/4) + \left(1 + \tfrac{1}{2}\right) x\log 2 + O(\log x)\\
&\leq \psi(x/8) + \left(1 + \tfrac{1}{2} + \tfrac{1}{4}\right) x\log 2 + O(\log x)\\
&\ \vdots\\
&\leq \psi(x/2^r) + \left(1 + \tfrac{1}{2} + \tfrac{1}{4} + \cdots\right) x\log 2 + O(r\log x)\\
&\leq x\log 4 + O\big((\log x)^2\big),
\end{aligned}$$

on choosing r so that $2^r \leq x < 2^{r+1}$. ∎

Lemma 1.8. $\displaystyle\sum_{p\leq x} \frac{\log p}{p} = \log x + O(1).$

Theorem 1.9 (Mertens)**.** *There is an absolute constant B such that*

$$\sum_{p\leq x} \frac{1}{p} = \log\log x + B + O\big((\log x)^{-1}\big). \tag{1.5}$$

Proof. Define the function

$$R(x) = \sum_{2<p\leq x} \frac{\log p}{p} - \log x$$

and the sequence

$$a_n = \begin{cases} (\log n)/n & \text{if } n \text{ is a prime number,}\\ 0 & \text{otherwise.}\end{cases}$$

Then, by Lemma 1.4,

$$\begin{aligned}
\sum_{p\le x}\frac{1}{p} &= \sum_{2<n\le x}\frac{a_n}{\log n}+\frac{1}{2}\\
&= \frac{1}{\log x}\sum_{2<n\le x}\frac{\log p}{p}+\int_2^x\Bigg(\sum_{2<n\le y}\frac{\log p}{p}\Bigg)\frac{dy}{y(\log y)^2}+\frac{1}{2}\\
&= \frac{1}{\log x}\big(\log x+R(x)\big)+\int_2^x\frac{\log y+R(y)}{y(\log y)^2}\,dy+\frac{1}{2}\\
&= \log\log x+\frac{3}{2}-\log\log 2+\int_2^x\frac{R(y)}{y(\log y)^2}\,dy+\frac{R(x)}{\log x}.
\end{aligned}$$

Using Lemma 1.8 to bound $R(y)$, we obtain

$$\int_2^x\frac{R(y)}{y(\log y)^2}\,dy+\frac{R(x)}{\log x}=\int_2^\infty\frac{R(y)}{y(\log y)^2}\,dy+O\big((\log x)^{-1}\big),$$

and the desired conclusion follows with

$$B=\frac{3}{2}-\log\log 2+\int_2^\infty\frac{R(y)}{y(\log y)^2}\,dy.$$

■

The final result of this section quantifies the relation between the error term in the PNT and the difference $\psi(x)-x$.

Theorem 1.10. *Suppose that f is an integrable function such that $x^{1/2}\ll f(x)\ll x$ and*

$$\int_2^x\frac{f(t)}{t}\,dt\ll f(x)\log x.$$

Then

$$\psi(x)-x\ll f(x)\qquad\Leftrightarrow\qquad \pi(x)-\operatorname{Li}x\ll f(x)(\log x)^{-1}.$$

Proof. Theorem 1.7 implies that

$$\theta(x)=\psi(x)+O\big(x^{1/2}\big),$$

whence

$$\theta(x)=x+O(f(x)).$$

Let a_n be the sequence

$$a_n=\begin{cases}\log n & \text{if } n \text{ is a prime number,}\\ 0 & \text{otherwise.}\end{cases}$$

As in the proof of Theorem 1.9,

$$\begin{aligned}\pi(x) &= \sum_{2<n\le x} \frac{a_n}{\log n} + 1 \\ &= \frac{\theta(x)}{\log x} + \int_2^x \frac{\theta(y)\,dy}{y(\log y)^2} + O(1) \\ &= \frac{x}{\log x} + \int_2^x \frac{dy}{(\log y)^2} + O\Big(\frac{f(x)}{\log x}\Big) + O\Big(\int_2^x \frac{f(y)\,dy}{y(\log y)^2}\Big) \\ &= \mathrm{Li}\, x + O\big(f(x)(\log x)^{-1}\big),\end{aligned}$$

by (0.2) and the properties of f. This proves the direct implication, the converse is left as an exercise. ■

1.3 Dirichlet series

A *Dirichlet series* is an infinite series of the form

$$\sum_{n=1}^{\infty} a_n n^{-s}, \tag{1.6}$$

where a_n are complex numbers and $s = \sigma + \mathrm{i}t$ is a complex variable. The following lemma shows that if a Dirichlet series converges at any finite complex number $s_0 = \sigma_0 + \mathrm{i}t_0$, then it converges to a holomorphic function in the half-plane $\mathrm{Re}(s) > \sigma_0$.

Lemma 1.11. *Suppose that* $s_0 = \sigma_0 + \mathrm{i}t_0$ *and the series*

$$\sum_{n=1}^{\infty} a_n n^{-s_0}$$

converges. Then the Dirichlet series (1.6) *converges uniformly on the compact subsets of the half-plane* $\mathrm{Re}(s) > \sigma_0$ *and the sum-function*

$$f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$$

is holomorphic in that half-plane.

Proof. It suffices to show that (1.6) converges uniformly in the regions

$$\big\{s \in \mathbb{C} \;:\; \mathrm{Re}(s) \ge \sigma_0 + \delta,\ |\mathrm{Im}(s) - t_0| \le T\big\}$$

where $\delta, T > 0$. By Lemma 1.4 with $a_n = a_n n^{-s_0}$ and $f(n) = n^{-(s-s_0)}$,

$$\sum_{\alpha<n\le\beta} a_n n^{-s} \ll_{\delta,T} \max_{\alpha<x\le\beta} \Big|\sum_{\alpha<n\le x} a_n n^{-s_0}\Big|, \tag{1.7}$$

so the uniform convergence of (1.6) follows from the convergence of $\sum_n a_n n^{-s_0}$ and Cauchy's criterion. ■

The number

$$\inf\left\{\operatorname{Re}(s) : (1.6) \text{ converges}\right\}$$

is called the *abscissa of convergence* of the Dirichlet series (1.6). Here, of course, we allow the possibility that the infimum could be $\pm\infty$. The abscissa of convergence of the Dirichlet series $\sum_n |a_n|n^{-s}$ is called the *abscissa of absolute convergence* of (1.6). The two abscissas are related by the following inequality.

Lemma 1.12. *Suppose that σ_c and σ_a are the abscissa of convergence and the abscissa of absolute convergence of the Dirichlet series* (1.6). *Then $\sigma_c \le \sigma_a \le \sigma_c + 1$.*

Dirichlet series are an important class of generating functions in number theory. In the remainder of this section, we discuss their properties related to their use in number theory. We first consider the relation between the sum-function of a Dirichlet series,

$$f(s) = \sum_{n=1}^{\infty} a_n n^{-s},$$

and the running sums of its coefficient sequence,

$$A(x) = \sum_{n \le x} a_n.$$

The passage from $A(x)$ to $f(s)$ is easy (at least, when $\operatorname{Re}(s)$ is sufficiently large):

$$f(s) = \sum_{n=1}^{\infty} a_n \int_n^{\infty} sx^{-s-1}\,dx = \int_1^{\infty} \left(\sum_{n \le x} a_n\right) sx^{-s-1}\,dx = \int_1^{\infty} A(x)sx^{-s-1}\,dx.$$

The inverse relation requires a little bit more work.

Lemma 1.13 (Perron's formula)**.** *Suppose that $\alpha > 0$. Then*

$$\frac{1}{2\pi \mathrm{i}} \int_{\alpha-\mathrm{i}T}^{\alpha+\mathrm{i}T} \frac{u^s}{s}\,ds = \begin{cases} 1 + O\left(u^{\alpha}(T|\log u|)^{-1}\right) & \text{if } u > 1, \\ \frac{1}{2} + O\left(\alpha T^{-1}\right) & \text{if } u = 1, \\ O\left(u^{\alpha}(T|\log u|)^{-1}\right) & \text{if } 0 < u < 1. \end{cases}$$

Proof. This is an exercise in contour integration. ∎

Corollary 1.14. *Let $f(s)$ be the sum-function of the Dirichlet series* (1.6). *Suppose that $x \notin \mathbb{Z}$ and $\alpha > \sigma_a$, where σ_a is the abscissa of absolute convergence of* (1.6). *Then*

$$\sum_{n \le x} a_n = \frac{1}{2\pi \mathrm{i}} \int_{\alpha-\mathrm{i}T}^{\alpha+\mathrm{i}T} f(s)x^s s^{-1}\,ds + O\left(\frac{x^{\alpha}}{T} \sum_{n=1}^{\infty} \frac{|a_n|n^{-\alpha}}{|\log(x/n)|}\right).$$

Corollary 1.15. *Suppose that* $a_1, a_2, a_3, \ldots$ *and* $b_1, b_2, b_3, \ldots$ *are sequences of complex numbers and the holomorphic functions* $f(s)$ *and* $g(s)$ *are defined in the half-plane* $\mathrm{Re}(s) > \sigma_0$ *by*

$$f(s) = \sum_{n=1}^{\infty} a_n n^{-s} \qquad \textit{and} \qquad g(s) = \sum_{n=1}^{\infty} b_n n^{-s}.$$

If $f(s) = g(s)$ *whenever* $\mathrm{Re}(s) > \sigma_0$*, then* $a_n = b_n$ *for all* $n = 1, 2, 3, \ldots$.

Proof. We apply Corollary 1.14 with $x \notin \mathbb{Z}$, $\alpha = \sigma_0 + 2$ (this ensures the absolute convergence of the series on the line $\mathrm{Re}(s) = \alpha$), and $T = \infty$. We get

$$\sum_{n \le x} a_n = \frac{1}{2\pi \mathrm{i}} \int_{\alpha - \mathrm{i}\infty}^{\alpha + \mathrm{i}\infty} f(s) x^s s^{-1}\, ds = \frac{1}{2\pi \mathrm{i}} \int_{\alpha - \mathrm{i}\infty}^{\alpha + \mathrm{i}\infty} g(s) x^s s^{-1}\, ds = \sum_{n \le x} b_n.$$

Since this holds for all non-integer $x > 1$, it follows that $a_n = b_n$ for all $n = 1, 2, 3, \ldots$. ■

The next two lemmas and their corollaries illustrate why Dirichlet series are convenient generating functions in multiplicative number theory.

Lemma 1.16. *Suppose that* $f(n)$ *is a multiplicative function. Then the identity*

$$\sum_{n=1}^{\infty} f(n) n^{-s} = \prod_p \left(1 + f(p)p^{-s} + f(p^2)p^{-2s} + \cdots\right)$$

holds whenever the series on the left converges absolutely.

Proof. The absolute convergence of the series $\sum_n f(n)n^{-s}$ implies the absolute convergence of the series $\sum_m f(p^m)p^{-ms}$ for all primes p. Let $x \ge 2$ and $r = \pi(x)$. Then

$$\begin{aligned}\prod_{p \le x} \left(1 + f(p)p^{-s} + f(p^2)p^{-2s} + \cdots\right) &= \sum_{m_1=0}^{\infty} \cdots \sum_{m_r=0}^{\infty} f(p_1^{m_1}) \cdots f(p_r^{m_r})(p_1^{m_1} \cdots p_r^{m_r})^{-s} \\ &= \sum_{m_1=0}^{\infty} \cdots \sum_{m_r=0}^{\infty} f(p_1^{m_1} \cdots p_r^{m_r})(p_1^{m_1} \cdots p_r^{m_r})^{-s} \\ &= \sum_{\substack{n=1 \\ p \mid n \Rightarrow p \le x}}^{\infty} f(n) n^{-s},\end{aligned}$$

where we have used the multiplicativity of f. Noting that the last sum contains, in particular, all the terms with $n \le x$, we conclude that

$$\left| \sum_{n \le x} f(n)n^{-s} - \prod_{p \le x} \left(1 + f(p)p^{-s} + f(p^2)p^{-2s} + \cdots\right) \right| \le \sum_{n > x} |f(n)| n^{-\sigma},$$

which establishes the desired identity. ■

Corollary 1.17. *Suppose that* $f(n)$ *is a completely multiplicative function. Then the identity*

$$\sum_{n=1}^{\infty} f(n)n^{-s} = \prod_{p} \left(1 - f(p)p^{-s}\right)^{-1}$$

holds whenever the series on the left converges absolutely.

Lemma 1.18. *Suppose that* $a_1, a_2, a_3, \ldots$ *and* $b_1, b_2, b_3, \ldots$ *are sequences of complex numbers such that the Dirichlet series*

$$f(s) = \sum_{n=1}^{\infty} a_n n^{-s} \qquad \text{and} \qquad g(s) = \sum_{n=1}^{\infty} b_n n^{-s}$$

converge absolutely in the half-plane $\mathrm{Re}(s) > \sigma_0$*. Then the Dirichlet series*

$$h(s) = \sum_{n=1}^{\infty} c_n n^{-s}, \qquad c_n = \sum_{uv=n} a_u b_v,$$

is also absolutely convergent in $\mathrm{Re}(s) > \sigma_0$ *and* $h(s) = f(s)g(s)$*.*

Proof. Suppose first that $\sigma > \sigma_0$. Then

$$\begin{aligned}
\sum_{n \le x} |c_n| n^{-\sigma} &= \sum_{n \le x} \left| \sum_{uv=n} a_u b_v \right| n^{-\sigma} \\
&\le \sum_{n \le x} \left(\sum_{uv=n} |a_u b_v| \right) n^{-\sigma} = \sum_{uv \le x} |a_u b_v| (uv)^{-\sigma} \\
&\le \left(\sum_{u \le x} |a_u| u^{-\sigma} \right) \left(\sum_{v \le x} |b_v| v^{-\sigma} \right) \\
&\le \left(\sum_{u=1}^{\infty} |a_u| u^{-\sigma} \right) \left(\sum_{v=1}^{\infty} |b_v| v^{-\sigma} \right),
\end{aligned}$$

which proves the absolute convergence of $\sum_n c_n n^{-s}$ for $\mathrm{Re}(s) = \sigma$. In particular, we have that

$$\sum_{n > x} \left(\sum_{uv=n} |a_u b_v| \right) n^{-\sigma} \to 0 \qquad \text{as } x \to \infty,$$

so the second part of the lemma follows from the inequality

$$\left| \sum_{n \le x} c_n n^{-s} - \left(\sum_{u \le x} a_u u^{-s} \right) \left(\sum_{v \le x} b_v v^{-s} \right) \right| \le \sum_{n > x} \left(\sum_{uv=n} |a_u b_v| \right) n^{-\sigma}. \tag{1.8}$$

■

In the next series of corollaries $\zeta(s)$ is the Riemann zeta-function.

Corollary 1.19. *Suppose that* $\mathrm{Re}(s) > 1$*. Then*

$$\sum_{n=1}^{\infty} \mu(n) n^{-s} = \frac{1}{\zeta(s)}.$$

Proof. By Lemmas 1.2 and 1.18,

$$\left(\sum_{n=1}^{\infty} \mu(n) n^{-s} \right) \left(\sum_{n=1}^{\infty} n^{-s} \right) = 1.$$

∎

Corollary 1.20. *Suppose that* $\mathrm{Re}(s) > 1$*. Then*

$$\sum_{n=1}^{\infty} \Lambda(n) n^{-s} = -\frac{\zeta'(s)}{\zeta(s)}.$$

1.4 Divisor functions

In this section we collect several standard estimates for the number of divisors function $d(n)$ and its averages. First of all, we note that $d(n)$ is multiplicative (by Lemma 1.1) and satisfies $d(p^k) = k+1$. These two observations lead (after some work) to the following upper bound for $d(n)$.

Lemma 1.21. *For any* $\epsilon > 0$, $d(n) \ll_\epsilon n^\epsilon$.

The bound in Lemma 1.21 is not tight, but it is also not too far from the best possible general bound (see Exercise 21). On the other hand, the next lemma shows that for most values of n, $d(n)$ is significantly smaller: its average value is $\log n$.

Theorem 1.22 (Dirichlet)**.** *Suppose that* $x \geq 2$*. Then*

$$\sum_{n \leq x} d(n) = x \log x + (2\gamma - 1)x + O\left(x^{1/2}\right), \tag{1.9}$$

where γ *is Euler's constant.*

Proof. Let $D(x)$ denote the left side of (1.9). We have

$$D(x) = \sum_{n \leq x} \sum_{uv=n} 1 = \sum_{uv \leq x} 1 = \sum_{\substack{uv \leq x \\ u \leq \sqrt{x}}} 1 + \sum_{\substack{uv \leq x \\ v \leq \sqrt{x}}} 1 - \sum_{u,v \leq \sqrt{x}} 1 = D_1(x) + D_2(x) - D_3(x), \qquad \text{say.}$$

Thus, (1.9) follows from the estimates

$$\begin{aligned} D_1(x) = D_2(x) &= \sum_{u \leq \sqrt{x}} \left[\frac{x}{u} \right] = \sum_{u \leq \sqrt{x}} \frac{x}{u} + O\left(x^{1/2}\right) \\ &= x \log \sqrt{x} + \gamma x + O\left(x^{1/2}\right) \qquad \text{(by Corollary 1.5);} \\ D_3(x) &= \left[\sqrt{x} \right]^2 = x + O\left(x^{1/2}\right). \end{aligned}$$

∎

Remark. The estimation of the error term in (1.9) is a famous problem in analytic number theory. It is not too difficult to show that

$$\Delta(x) = \sum_{n \le x} d(n) - x \log x - (2\gamma - 1)x \ll x^{1/3} \log x.$$

Attempts to improve further on this and other similar bounds have stimulated the development of the theory of exponential sums (see Graham and Kolesnik [15] and Huxley [24]). The best result to date was obtained recently by Huxley [25]:

$$\Delta(x) \ll_\epsilon x^{131/416+\epsilon},$$

where $131/416 = 0.3149\ldots$. It is conjectured that

$$\Delta(x) \ll_\epsilon x^{1/4+\epsilon},$$

which if proven would be essentially best possible, as it is an old result of Hardy [16] that the bound $\Delta(x) \ll x^{1/4}$ does not hold for all x.

Often one needs upper bounds for higher moments of $d(n)$. The following theorem provides such an estimate.

Theorem 1.23. *Suppose that* $x \ge 1$ *and* $k \in \mathbb{N}$. *Then*

$$\sum_{n \le x} (d(n))^k \ll_k x(\log x)^{2^k - 1} + 1. \tag{1.10}$$

Proof. By induction on k. The case $k = 1$ follows from Theorem 1.22. Now suppose that (1.10) holds for some $k \ge 1$. Then

$$\sum_{n \le x} (d(n))^{k+1} = \sum_{uv \le x} (d(uv))^k \le \sum_{uv \le x} (d(u)d(v))^k,$$

where the last step uses that $d(mn) \le d(m)d(n)$. Hence, by the inductive hypothesis,

$$\begin{aligned}\sum_{uv \le x} (d(u)d(v))^k &\le \sum_{u \le x} (d(u))^k \sum_{v \le x/u} (d(v))^k \\ &\ll_k x(\log x)^{2^k - 1} \sum_{u \le x} \frac{(d(u))^k}{u} + \sum_{u \le x} (d(u))^k \ll_k x(\log x)^{2^{k+1} - 1},\end{aligned}$$

on using the bound

$$\sum_{u \le x} \frac{(d(u))^k}{u} \ll_k (\log x)^{2^k},$$

which follows from the inductive hypothesis by partial summation. ∎

Exercises

1. Prove (0.2).

2. Prove that Euler's function $\phi(n)$ is multiplicative.

3. Prove Lemma 1.1.

4. Prove Corollary 1.5. [HINT: The value of c_1 is $1 - \int_1^\infty \{x\} x^{-2}\, dx$.]

5. Prove Corollary 1.6.

6. Prove Lemma 1.8. [HINT: First show that the given sum equals $x^{-1}T(x) + O(1)$, where $T(x)$ is the sum appearing in the proof of Theorem 1.7.]

7. Prove that
$$\prod_{p \le x} \left(1 - \frac{1}{p}\right) = \frac{C}{\log x}\left(1 + O\left(\frac{1}{\log x}\right)\right),$$
where C is an absolute constant. (It can be shown that, in fact, $C = e^{-\gamma}$, where γ is Euler's constant.)

8. Let B be the constant appearing in Theorem 1.9 and C be the constant appearing in the last problem. Prove that
$$B + \log C = \sum_p \left(\frac{1}{p} + \log\left(1 - \frac{1}{p}\right)\right).$$

9. Prove that under the hypotheses of Theorem 1.10,
$$\int_2^x \frac{f(y)\, dy}{y(\log y)^2} \ll \frac{f(x)}{\log x}.$$

10. Prove the converse part of Theorem 1.10.

11. Modify the proof of Theorem 1.10 to show that the PNT is equivalent to the statement that $\psi(x) \sim x$ as $x \to \infty$.

12. Verify (1.7).

13. Suppose that in Lemma 1.11 the assumption that the series $\sum_n a_n n^{-s_0}$ converges is weakened to the assertion that the partial sums
$$\sum_{n \le N} a_n n^{-s_0} \qquad (N = 1, 2, 3, \ldots)$$
are bounded. Prove that the conclusion of the lemma stays true.

14. Prove Lemma 1.12.

15. Prove Lemma 1.13.

16. Prove (1.8).

17. Prove Corollary 1.20.

18. Prove the following identities:

(a) $\sum_{n=1}^{\infty} d(n)n^{-s} = \zeta^2(s)$ whenever $\mathrm{Re}(s) > 1$;

(b) $\sum_{n=1}^{\infty} |\mu(n)|n^{-s} = \zeta(s)/\zeta(2s)$ whenever $\mathrm{Re}(s) > 1$;

(c) $\sum_{n=1}^{\infty} \phi(n)n^{-s} = \zeta(s-1)/\zeta(s)$ whenever $\mathrm{Re}(s) > 2$.

19. Define a multiplicative function $f : \mathbb{N} \to \mathbb{C}$ by

$$f(p^k) = \binom{k-1/2}{k} = (-1)^k \binom{-1/2}{k},$$

where the generalized binomial coefficient $\binom{s}{k}$, $s \in \mathbb{C}$, is the coefficient of z^k in the Maclaurin expansion of $(1+z)^s$:

$$\binom{s}{k} = \frac{s(s-1)\cdots(s-k+1)}{k!}.$$

(a) Prove that the Dirichlet series $F(s) = \sum_n f(n)n^{-s}$ converges absolutely and uniformly on the compact subsets of the half-plane $\mathrm{Re}(s) > 1$.

(b) Prove that $F(s)^2 = \zeta(s)$ whenever $\mathrm{Re}(s) > 1$.

20. Prove that $d(n) \le \sqrt{3n}$ for all $n \in \mathbb{N}$.

21. (a) Prove that there exists an absolute constant $c_1 > 0$ such that

$$d(n) \ll \exp\left(\frac{c_1 \log n}{\log\log n}\right).$$

(b) Let $n = p_1 p_2 \cdots p_k$, where p_k denotes the kth prime. Prove that there exists an absolute constant $c_2 > 0$ such that

$$d(n) \gg \exp\left(\frac{c_2 \log n}{\log\log n}\right).$$