A brief introduction to sofic entropy theory

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Review of classical entropy theory

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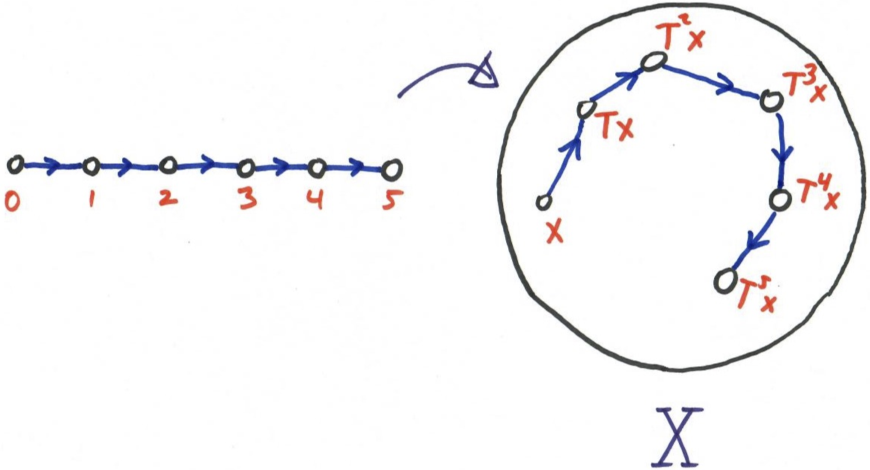
The **topological entropy** of T is the infimum over $\epsilon > 0$ of the exponential growth rate of the number of partial orbits that can be distinguished at scale $\epsilon > 0$.

Topological entropy ala Rufus Bowen

A **length-*n* partial orbit** is an n-tuple of the form
$\underline{x} = (x, Tx, T^2x, \ldots, T^{n-1}x)$.

A partial orbit



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$$h(X, T) := \sup_{\epsilon > 0} \limsup_{n \to \infty} n^{-1} \log \mathrm{cov}_\epsilon(\text{length-}n \text{ partial orbits}, \rho_\infty)$$

where $\mathrm{cov}_\epsilon(\cdot, \rho_\infty)$ is the minimum cardinality of a collection of length-n partial orbits that ϵ-cover the space of all length-n partial orbits.

Main Results

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3. A system (Y, S) is a **factor** of (X, T) if there is a surjective (T, S)-equivariant map $\Phi : X \to Y$:

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[d, "\Phi"'] & X \arrow[d, "\Phi"] \\ Y \arrow[r, "S"'] & Y \end{tikzcd}$$

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$$\begin{array}{ccc} X & \xrightarrow{T} & X \\ \Phi\downarrow & & \downarrow\Phi \\ Y & \xrightarrow[S]{} & Y \end{array}$$

   In this case, $h(X, T) \geq h(Y, S)$.
4. (Topological entropy was defined earlier in a different way by Adler, Konheim and McAndrew in 1965).

Overview

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Time is usually represented by either $\mathbb{Z}, \mathbb{N}, \mathbb{R}$ or $\mathbb{R}^{>0}$ but more general groups such as $\mathbb{Z}^d, \mathbb{R}^d$ can be and have been considered.

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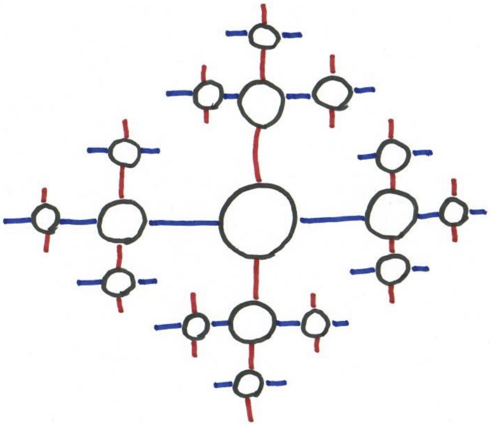
What happens if we replace the acting group with a free group $\mathbb{F}_2 = \langle a, b \rangle$?

The rank 2 free group



Important example: full shifts

If Γ is a countable group and K a Borel space then the **full K-shift** is

$$\Gamma \curvearrowright K^{\Gamma} = \{x : \Gamma \to K\}$$

$$(gx)_f = x_{g^{-1}f} \quad \forall g, f \in \Gamma, x \in K^{\Gamma}.$$

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So the full 2-shift over $\mathbb{Z}$ cannot factor onto the full 4-shift over $\mathbb{Z}$.

The Ornstein-Weiss Example

Theorem (Ornstein-Weiss, 1987)

If $\mathbb{F} = \langle a, b\rangle$ is the rank 2 free group then the full 2-shift over $\mathbb{F}$ factors onto the full 4-shift over $\mathbb{F}$.

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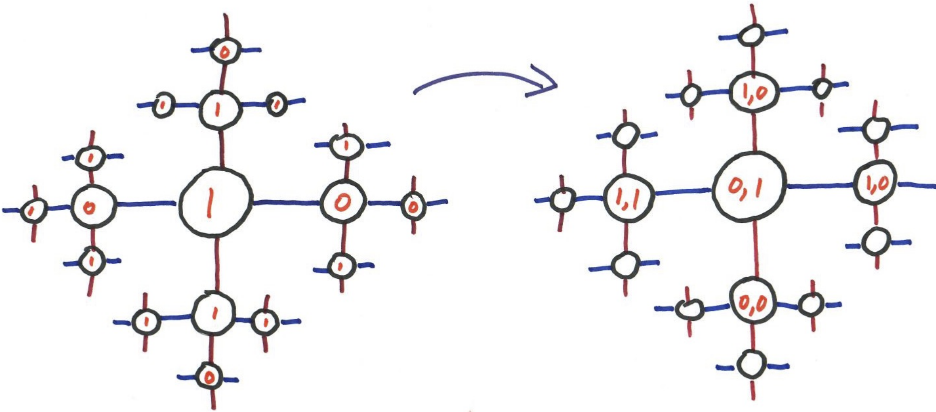
(Ornstein-Weiss, 1987): Is the full 2-shift over $\mathbb{F}$ measurably conjugate to the full 4-shift?

The Ornstein-Weiss map



Factors between topological Bernoulli shifts

Theorem (Bartholdi, 2016)

If Γ is any non-amenable group then there exist integers $2 \leq m < n$ and a continuous, shift-equivariant surjective map

$$(\mathbb{Z}/m\mathbb{Z})^\Gamma \to (\mathbb{Z}/n\mathbb{Z})^\Gamma.$$

A new approach to entropy theory for $\mathbb{Z}$-actions

Consider softening the notion of partial orbit.

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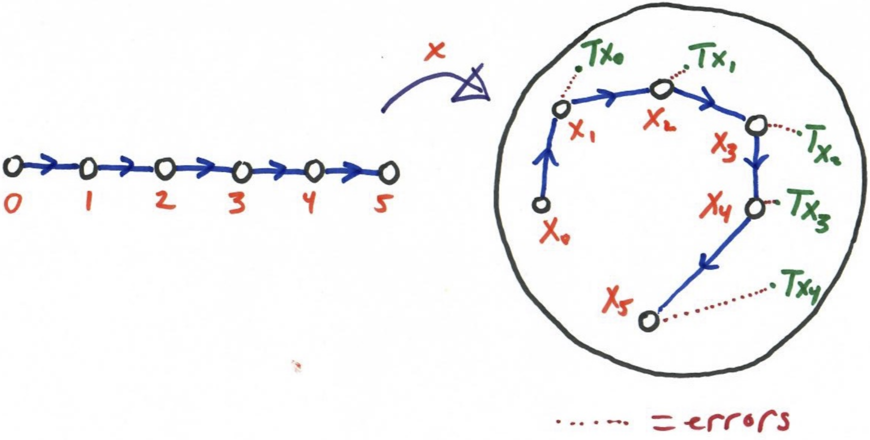
An (n,δ)-**pseudo orbit** is a tuple $\underline{x} = (x_1, \ldots, x_n) \in X^n$ such that

$$\frac{1}{n}\sum_{i=1}^{n-1} \rho(Tx_i, x_{i+1}) < \delta.$$

Note: we are using an ℓ^1 metric instead of an ℓ^∞ metric.

A pseudo-orbit



Entropy via pseudo-orbits

Theorem

$$h(X,T) = \sup_{\epsilon>0} \inf_{\delta>0} \limsup_{n\to\infty} n^{-1} \log \mathrm{cov}_\epsilon(\ (n,\delta)\textit{-pseudo orbits}, \rho_\infty).$$

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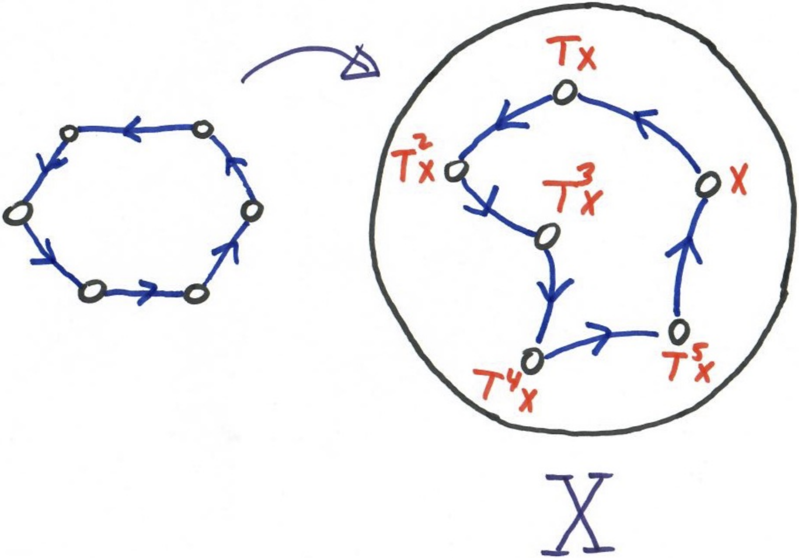
Proof sketch.

Use Markov's inequality to show that any (n,δ)-pseudo orbit is approximately shadowed by a union of a few long partial orbits. □

Periodic orbits

A **periodic orbit with period n** is a tuple $(x, Tx, \ldots, T^{n-1}x)$ such that $T^n x = x$ (up to cyclic reordering).

A periodic orbit



Entropy via periodic orbits?

The exponential rate of growth of the number of periodic points that can be distinguished at scale ϵ (and then send $\epsilon \searrow 0$) is a lower bound for entropy.

But in general, it is not equal to entropy.

How to compute entropy

$$\begin{tikzcd}
 & \textit{partial orbits} \arrow[dl] \arrow[dr] & \\
\textit{pseudo orbits} \arrow[dr] & & \textit{periodic orbits} \arrow[dl] \\
 & \textit{pseudo periodic orbits} &
\end{tikzcd}$$

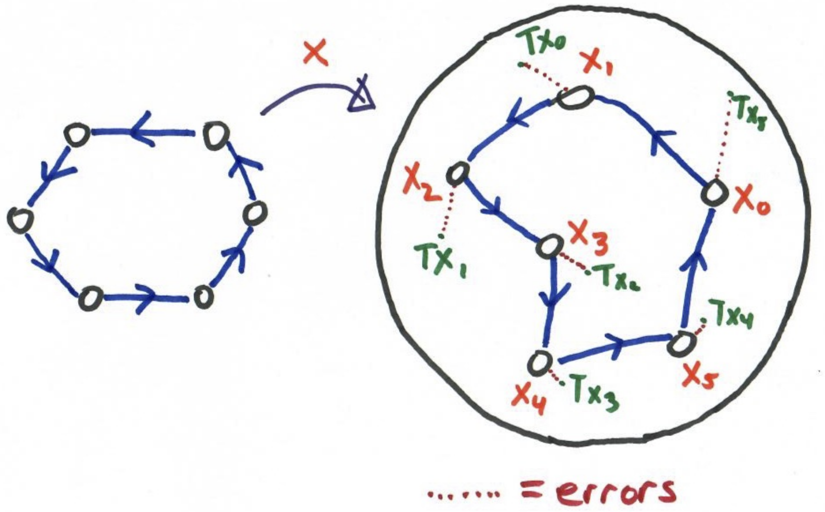
Pseudo-periodic orbits

An (n, δ)**-pseudo-periodic orbit** is a tuple $\underline{x} = (x_1, \ldots, x_n) \in X^n$ (up to cyclic order) such that

$$\frac{1}{n}\sum_{i=1}^{n} \rho(Tx_i, x_{i+1}) < \delta$$

(indices mod n).

Pseudo-periodic orbits



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Aside: pseudo-periodic orbits have also been called **microstates**, **good maps** or **sofic models for the action**.

A first step towards sofic entropy

Let Γ be a countable group, $\Gamma \curvearrowright X$ an action by homeomorphisms.

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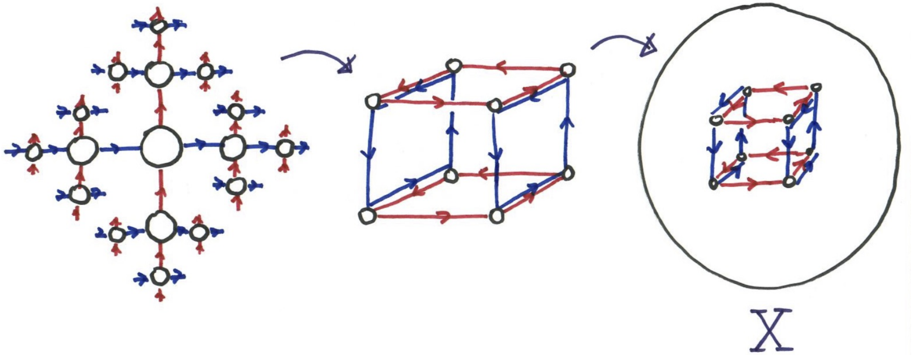
Preliminary definition : an **pseudo-periodic orbit** consists of an action $\Gamma \curvearrowright^{\sigma} V$ on a finite set and a map $\phi : V \to X$ that is approximately equivariant in an ℓ^1-sense:

$$|V|^{-1} \sum_{v \in V} \rho\Big(\phi\big(\sigma(g)v\big), g\phi(v)\Big) < \delta \quad \forall g \in F$$

where $F \subset \Gamma$ is finite.

More precisely, this is a (σ, δ, F)**-pseudo-periodic orbit** .

A pseudo-periodic orbit of a free group action



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Preliminary definition : the **sofic entropy of $\Gamma \curvearrowright X$ with respect to Σ** is

$$h_\Sigma(\Gamma \curvearrowright X) :=$$
$$\sup_{\epsilon>0} \inf_{\delta>0} \inf_{F \Subset \Gamma} \limsup_{n\to\infty} |V_n|^{-1} \log \mathrm{cov}_\epsilon((\sigma_n, \delta, F)\text{-pseudo-periodic orbits}, \rho_\infty).$$

Main Results (Kerr-Li, 2011)

1. If $\Gamma \curvearrowright X$ embeds into $\Gamma \curvearrowright Y$ then $h_\Sigma(\Gamma \curvearrowright X) \leq h_\Sigma(\Gamma \curvearrowright Y)$.

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2. In particular, Σ-entropy is a topological conjugacy invariant.

3. $h_\Sigma(\Gamma \curvearrowright K^\Gamma) = \log |K|$.

4. If Γ is amenable then sofic entropy agrees with classical entropy.

A boring example

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This isn't what is usually meant by entropy.

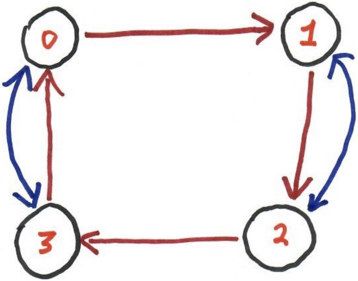
To fix this, require that the actions $\Gamma \curvearrowright^{\sigma_n} V_n$ witness Γ in the sense that: for all $g \in \Gamma \setminus \{1_\Gamma\}$,

$$|V_n|^{-1} \# \{v \in V_n : \ \sigma_n(g)v \neq v\} \to 1 \text{ as } n \to \infty.$$

With this assumption, Σ is said to be a **sofic approximation** to Γ.

A curious example

Given an action $\Gamma \curvearrowright V$,
form the graph on V:

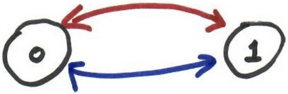


$\Gamma = \langle a, b \rangle$

$V = \{0, 1, 2, 3\}$

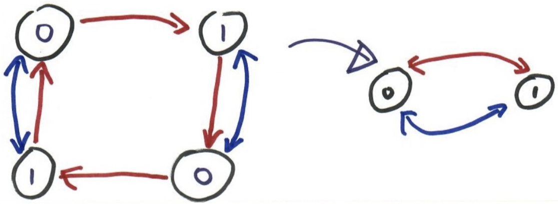
A curious example

Let's compute the sofic entropy of $\Gamma \curvearrowright \{0,1\}$



Suppose each $\Gamma \curvearrowright V_n$ gives a bi-partite graph.

A curious example



Then there exists a pseudo-periodic orbit (for any choice of δ, F) and $h_\Sigma(\Gamma \curvearrowright \{0,1\})$

$= 0.$

A curious example

Conversely, if there is a $(\sigma, \delta, \{a, b, a^{-1}, b^{-1}\})$-pseudo-periodic orbit,

$$\phi : V_n \to \{0, 1\}$$

then there is a δ-almost bi-partition $\{\phi^{-1}(0), \phi^{-1}(1)\}$ of the graph of $\Gamma \curvearrowright V_n$.

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So, if the graphs of $\Gamma \curvearrowright V_n$ do not have almost bi-partitions then

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Recap: Sofic entropy depends on the choice of sofic approximation and it can be $-\infty$.

The Ornstein-Weiss example revisited

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$$\begin{tikzcd}
 & (\mathbb{Z}/2\mathbb{Z})^{\mathbb{F}_2} \arrow[d, "OW"] \\
V_n \arrow[ur, dotted, "\exists\widetilde{\phi}?"] \arrow[r, "\phi"'] & (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})^{\mathbb{F}_2}
\end{tikzcd}$$

A cohomological interpretation of the Ornstein-Weiss example

Given

$$\phi : V_n \to (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})^{\mathbb{F}_2}$$

let

$$\phi^{edge} : E_n \to (\mathbb{Z}/2\mathbb{Z})$$

be the map defined on the *edges* of the action graph by

$$\phi(v)_e = (\phi^{edge}(v, va), \phi^{edge}(v, vb)).$$

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A pseudo-periodic orbit $\phi : V_n \to (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})^{\mathbb{F}_2}$ has an approximate lift if and only if ϕ^{edge} is (close to) a coboundary.

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Aside: the Ornstein-Weiss map has been generalized by Gaboriau-Seward from the free group $\mathbb{F}$ to an arbitrary group Γ and the increase in entropy is related to the first ℓ^2-betti number and cost of Γ.

What is this good for?

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If Γ is sofic then the conjecture is true.

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Proof by Kerr-Li, 2011.

- $h_\Sigma(\Gamma \curvearrowright k^\Gamma) = \log |k|$.
- $h_\Sigma(\Gamma \curvearrowright \Phi(k^\Gamma)) = \log |k|$.
- The sofic entropy of any proper subshift of k^Γ is $< \log |k|$.

$\square$

Partial actions

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Instead we require $\{\sigma_n : \Gamma \to \mathrm{sym}(V_n)\}$ to be a sequence of maps (not necessarily homomorphisms!) satisfying

- (asymptotic homorphism condition) $\forall g, h \in \Gamma$,

$$|V_n|^{-1}\#\{v \in V_n : \ \sigma_n(gh)v = \sigma_n(g)\sigma_n(h)v\} \to 1 \text{ as } n \to \infty$$

- (asymptotic freeness) $\forall g \in \Gamma \setminus \{1_\Gamma\}$,

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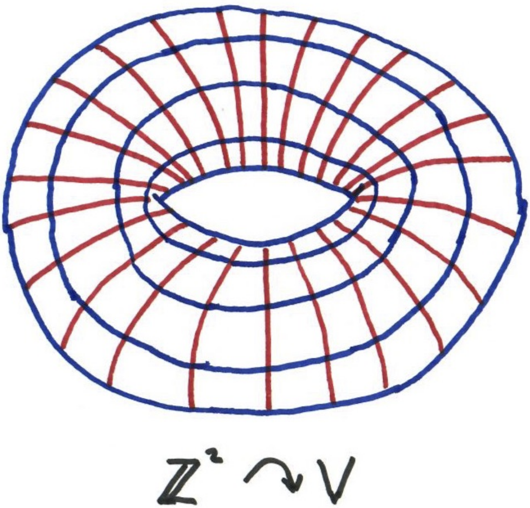
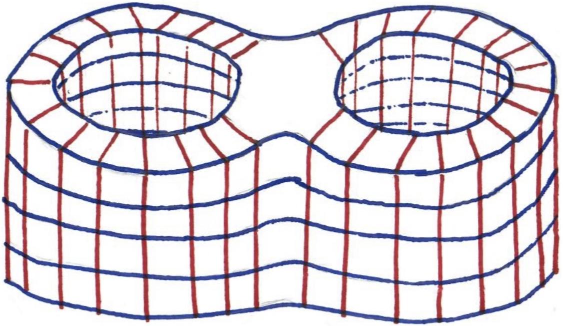
Such a sequence is a **sofic approximation** and Γ is **sofic** it has one.

Definition due to Gromov (1999), named and made accessible by Weiss (2000).

An action of $\mathbb{Z}^2$



A partial action of $\mathbb{Z}^2$



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- If G is sofic then G satisfies Gottshalk's surjunctivity conjecture, Connes embedding conjecture, the Determinant conjecture, Kaplansky's direct finiteness conjecture. (Gromov 1999, Weiss 2000, Elek-Szabo 2005)

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- Residually finite groups are sofic. Hence all linear groups are sofic.
- The class of sofic groups is closed under: subgroups, direct limits, inverse limits, direct products, extensions by amenable groups, free products with amenable amalgamation, wreathe products. (Elek-Szabo, Dykema-Kerr-Pichot, Paunescu, Hayes-Sale)
- If G is sofic then G satisfies Gottshalk's surjunctivity conjecture, Connes embedding conjecture, the Determinant conjecture, Kaplansky's direct finiteness conjecture. (Gromov 1999, Weiss 2000, Elek-Szabo 2005)
- Open: Is every countable group sofic?

The measure-conjugacy problem

Probability-measure-preserving actions

$$\Gamma \curvearrowright (X, \mu_X), \quad \Gamma \curvearrowright (Y, \mu_Y)$$

are **measurably conjugate** if there exists a measure-space isomorphism Φ

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "\Phi"'] & X \arrow[d, "\Phi"] \\ Y \arrow[r, "g"'] & Y \end{tikzcd}$$

intertwining the actions.

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Main Problem: Classify actions of Γ up to measure conjugacy.

Bernoulli shifts

- If κ is a probability measure on a space K then the full shift-action

$$\Gamma \curvearrowright (K^{\Gamma}, \kappa^{\Gamma})$$

with the product measure κ^{Γ} is the **Bernoulli shift** over G with base κ.

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- von Neumann's question: Is the full 2-shift over $\mathbb{Z}$ measurably conjugate to the full 3-shift?

$$\begin{tikzcd}
\{0,1\}^{\mathbb{Z}} \arrow[r] \arrow[d, "\exists\Phi?"'] & \{0,1\}^{\mathbb{Z}} \arrow[d, "\exists\Phi?"] \\
\{0,1,2\}^{\mathbb{Z}} \arrow[r] & \{0,1,2\}^{\mathbb{Z}}
\end{tikzcd}$$

Measure entropy ala Kerr-Li

Let $\Gamma \curvearrowright (X, \mu)$ be an action by homeomorphisms and μ an invariant probability measure.

Measure entropy ala Kerr-Li

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The **measure sofic entropy of** $\Gamma \curvearrowright (X, \mu)$ is the exponential growth rate of the number of approximately equidistributed periodic orbits that can be distinguished at scale ϵ (and then send $\epsilon \searrow 0$).

Measure entropy ala Kerr-Li

The **empirical distribution** of a map $\phi : V \to X$ is the probability measure

$$P_\phi := \frac{1}{|V|} \sum_{v \in V} \delta_{\phi(v)} \in \mathrm{Prob}(X).$$

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If $\mathcal{O} \subset \mathrm{Prob}(X)$ is an open neighborhood of μ then a $(\sigma, \delta, F, \mathcal{O})$**-pseudo-periodic orbit** is a map $\phi : V \to X$ such that ϕ is a (σ, δ, F)-pseudo-periodic orbit and $P_\phi \in \mathcal{O}$.

Measure entropy ala Kerr-Li

The **sofic entropy of $\Gamma \curvearrowright (X,\mu)$ with respect to Σ** is

$$h_\Sigma(\Gamma \curvearrowright (X,\mu)) :=$$
$$\sup_{\epsilon>0} \inf_{\delta,F,\mathcal{O}} \limsup_{n\to\infty} |V_n|^{-1} \log \mathrm{cov}_\epsilon((\sigma,\delta,F,\mathcal{O})\text{-pseudo-periodic orbits}, \rho_\infty).$$

Main Results

1. (Variational Principle, Kerr-Li) $h_\Sigma(\Gamma \curvearrowright X) = \sup_\mu h_\Sigma(\Gamma \curvearrowright (X,\mu))$.

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2. (B., Kerr-Li) Measure sofic entropy is a measure conjugacy invariant.
3. (B., Kerr-Li) If Γ is amenable then sofic entropy agrees with classical entropy.
4. The sofic entropy a Bernoulli shift action is the Shannon entropy of the base:

$$h_\Sigma(\Gamma \curvearrowright (K, \kappa)^\Gamma) = H(\kappa) := \sum_{k \in K} -\kappa(\{k\}) \log \kappa(\{k\}).$$

5. So the 2-shift over $\mathbb{F}_2$ is not isomorphic to the 4-shift over $\mathbb{F}_2$!

Classification of Bernoulli shifts

Conjecture: Assume $|\Gamma| = \infty$. Then

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- Γ amenable groups (folklore or Kieffer?, 1970s)
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$\Leftarrow$

- $\Gamma = \mathbb{Z}$ (Ornstein, 1970)
- Γ amenable (Ornstein-Weiss, 1980)
- $\mathbb{Z} \leq \Gamma$ (Stepin, 1975)
- $\forall \Gamma$, $|K| > 2$ and $|L| > 2$ (B. 2012)
- $\forall \Gamma$ (Seward, 2018)

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- Weak Pinsker Conjecture: Tim Austin recently posted a solution for actions of amenable groups. I have a counterexample in the case of free group actions based on sofic entropy theory and the hardcore model on random regular graphs.