M365g

Spring 2017

Bowen

Name______________________________

Exam #1

Instructions. Please put your name at the top of the exam. Read over the entire exam before you begin; you should work on the problems you'll find easiest first. Continue your work on the backs of pages or on extra sheets. *If your solution runs over onto these pages, please indicate that clearly.* If you use extra sheets, be sure to staple them to the rest of your exam and put your name on them.

Questions

1. Suppose $f : S_1 \to S_2$ is a local diffeomorphism and $\gamma : (a, b) \to S_1$ is a regular curve. Show that $f \circ \gamma$ is a regular curve.

   Solution. It is a regular curve because $D(f \circ \gamma) = Df \circ \gamma'$, Df is injective (since it is a local diffeomorphism) and γ' never zero (since γ is regular).

   Common mistakes. Some students wrote f' instead of Df or even $\frac{df}{d\gamma}$, neither of which are are right since f is a map from one surface to another. Also it's not enough to say that f is a local diffeomorphism since that's given in the problem. One should state that Df is injective.

2. Consider the following four parametrized plane curves $\gamma_i : [0, 1] \to \mathbb{R}^2$, $i = A, B, C, D$:

$$\gamma_A(t) = \big(\cos(2\pi t), \sin(2\pi t)\big)$$
$$\gamma_B(t) = \big(2\cos(2\pi t), 2\sin(2\pi t)\big)$$
$$\gamma_C(t) = \big(t, t^2\big)$$
$$\gamma_D(t) = \left(t, \sin\left(\frac{1}{t + .001}\right)\right)$$

Order these curves by length and then by curvature. You need not compute the lengths and curvatures exactly to complete this task! A few sketches should give you the right idea.

Solution. By length: $C < A < B < D$. By curvature $B < A < C < D$.

Reasons: γ_D oscillates wildly from -1 to 1. In fact,

$$\frac{1}{t + 0.001} = k\pi$$

if and only if $t = \frac{1}{k\pi} - 0.001$. So γ_D will oscillate several hundred times between -1 and 1 for $0 < t < 1$. For this reason, it has the largest curvature and longest length.

γ_A is a circle with length 2π and curvature 1.

γ_B is a circle with length 4π and curvature $1/2$.

γ_C is parametrizing part of a parabola. You can compute its length and curvature exactly. Because it is convex and $\gamma_C(0) = (0,0), \gamma_C(1) = (1,1)$, it's length cannot be any more than 2. So it has the shortest length. Its curvature is

$$\kappa = \frac{2}{(1 + 4t^2)^{3/2}}$$

which is maximized at $t = 0$. So it has larger curvature that either of the two circles for t near 0. It has smaller curvature than first circle near $t = 1$.

My mistake. Sorry! I should have said maximum curvature instead of just "curvature".

3. Consider the helix $C \subset \mathbb{R}^3$ parametrized by

$$\gamma(t) = (\cos(\alpha t), \sin(\alpha t), \beta t)$$

where $\alpha, \beta > 0$ are such that $\alpha^2 + \beta^2 = 1$. So $C = \gamma(\mathbb{R})$ is the image of γ.

(a) Compute the Frenet frame $\vec{t}, \vec{n}, \vec{b}$ at an arbitrary point of C.

(b) Compute the curvature and torsion at an arbitrary point of C

Solution.

$$\gamma'(t) = \vec{t} = (-\alpha \sin \alpha t, \alpha \cos \alpha t, \beta), \quad \|\gamma'\| = 1$$

$$\vec{t}' = (-\alpha^2 \cos \alpha t, -\alpha^2 \sin \alpha t, 0), \quad \kappa = \|\vec{t}'\| = \alpha^2$$

$$\vec{n} = \kappa^{-1}\vec{t}' = (-\cos \alpha t, -\sin \alpha t, 0)$$

$$\vec{b} = \vec{t} \times \vec{n} = (\beta \sin \alpha t, -\beta \cos \alpha t, \alpha)$$

$$\vec{b}' = \alpha\beta(\cos \alpha t, \sin \alpha t, 0) = -\tau \vec{n}$$

$$\tau = \alpha\beta.$$

(c) Let $S \subset \mathbb{R}^3$ be the union of the normal lines to C. We can parametrize S by

$$\sigma(u, v) = \gamma(u) + v\vec{n}(u).$$

Is S a smooth surface? You do not have to check injectivity.

Solution. So σ is regular if and only if

$$\sigma_u = \gamma'(u) - v\kappa\vec{t}(u) + v\tau\vec{b}(u) = (1 - v\kappa)\vec{t} + v\tau\vec{b}$$

and $\sigma_v = \vec{n}(u)$ are linearly independent. We claim that σ_u is never zero. Because $\vec{b}, \vec{t}$ are orthonormal, σ_u is zero if and only if both $1 - v\kappa = 0$ and $v\tau = 0$ but this is impossible since $\kappa, \tau > 0$.

Since σ_u is never zero and $\vec{t}, \vec{n}, \vec{b}$ are orthonormal, σ_u and σ_v are independent.

4. Find a basis for the tangent space to the surface $x^2 + y^2 - z^2 = 1$ at the point $(2^{-1/2}, 2^{-1/2}, 0)$. Show that this tangent space contains the z-axis.

Solution. There are many different ways to parametrize this surface. Here's one:

$$\sigma(\theta, z) = (\sqrt{1 + z^2}\cos(\theta), \sqrt{1 + z^2}\sin(\theta), z).$$

Taking derivatives

$$\sigma_\theta = (-\sqrt{1 + z^2}\sin(\theta), \sqrt{1 + z^2}\cos(\theta), 0)$$

$$\sigma_z = (z(1 + z^2)^{-1/2}\cos(\theta), z(1 + z^2)^{-1/2}\sin(\theta), 1).$$

Note $\sigma(\pi/4, 0) = (2^{-1/2}, 2^{-1/2}, 0)$. So

$$\sigma_\theta(\pi/4, 0) = (-2^{1/2}, 2^{1/2}, 0)$$

$$\sigma_z(\pi/4, 0) = (0, 0, 1).$$

This is a basis for the tangent space at $(2^{-1/2}, 2^{-1/2}, 0)$.

Common mistakes. It doesn't work if you parametrize by $\sigma(u, v) = (u, v, \sqrt{u^2 + v^2 - 1})$. The problem is that this patch does not contain an open neighborhood around $(2^{-1/2}, 2^{-1/2}, 0)$ because it is not injective in a neighborhood of $(2^{-1/2}, 2^{-1/2}, 0)$. So if you use this parametrization, you will get infinite derivatives.

5. Suppose $F : \mathbb{R}^2 \to \mathbb{R}$ is a smooth function, and for every $(x_0, y_0) \in \mathbb{R}^2$ with $F(x_0, y_0) = 0$, we have $\dfrac{\partial F}{\partial y}(x_0, y_0) \neq 0$. Prove that the set of points

$$C = \{(x, y) \in \mathbb{R}^2 : F(x, y) = 0\}$$

is the graph $\{\big(x, f(x)\big) \in \mathbb{R}^2 : x \in \mathbb{R}\}$ of a smooth function $f : \mathbb{R} \to \mathbb{R}$.

Solution. Define $G : \mathbb{R}^2 \to \mathbb{R}^2$ by $G(x, y) = (x, F(x, y))$. Because

$$DG = \begin{pmatrix} 1 & 0 \\ F_x & F_y \end{pmatrix}$$

has determinant $F_y \neq 0$, it is invertible. So the inverse function theorem implies the existence of a local inverse which necessarily has the form $H(x, y) = (x, h(x, y))$ for some function h. We define $f(x) = h(x, 0)$ to finish the solution.

You might object that we have only shown that C is locally the graph of a smooth function. But actually, f is globally defined. This is because f is uniquely specified by the requirement that $F((x, f(x)) = 0$ since $F_y \neq 0$ and F is smooth implies that F_y does not change sign. So $F(x, y_2) - F(x, y_1) = \int_{y_1}^{y_2} F_y(x, u)\, du$ cannot be zero unless $y_1 = y_2$.

Common mistakes. There was a lot of confusion about how to apply the IFT. You cannot apply the IFT directly to F because its domain and range have different dimensions.