M365g
Spring 2017
Bowen

Name________________________________

Practice Exam #1

Instructions. Please put your name at the top of the exam. Read over the entire exam before you begin; you should work on the problems you'll find easiest first. Continue your work on the backs of pages or on extra sheets. *If your solution runs over onto these pages, please indicate that clearly.* If you use extra sheets, be sure to staple them to the rest of your exam and put your name on them.

Questions

1. Is it possible to have a simple closed regular *space* curve with no vertices, i.e., with no points at which the derivative of the curvature vanishes? Either sketch a proof that it is not possible or sketch a construction of an example.

2. Show that if $\gamma : (a, b) \to \mathbb{R}^3$ is a curve whose image is contained in a regular surface patch $\sigma : U \to \mathbb{R}^3$ then $\gamma(t) = \sigma(u(t), v(t))$ for some smooth map $t \mapsto (u(t), v(t)) \in U$.

   In your proof, you can make free use of the inverse function theorem for maps $\mathbb{R}^n \to \mathbb{R}^n$, but you shouldnt assume any results about surfaces beyond the basic definitions (unless you prove them, of course).

3. (a) State carefully the definition of a *(regular smooth) surface patch*.

   (b) Find a domain U in the plane for which the function

   $$\sigma(u, v) = \big(u, v, u \sin(uv)\big)$$

   is a surface patch. Verify that σ is indeed a surface patch.

4. Answer each below as True or False. You need not provide further explanation.

(a) Let $\kappa : (a,b) \to \mathbb{R}$ be any (smooth) function. Then there exists a parametrized plane curve $\gamma : [a,b] \to \mathbb{R}^2$ whose signed curvature is κ.

(b) Let γ be a regular with nonzero curvature everywhere whose image lies on the unit sphere. Then γ has constant torsion.

(c) Suppose $F : \mathbb{R}^2 \to \mathbb{R}$ is a smooth function, and for every $(x_0, y_0) \in \mathbb{R}^2$ with $F(x_0, y_0) = 0$, we have $\dfrac{\partial F}{\partial y}(x_0, y_0) \neq 0$. Then the set of points

$$C = \{(x,y) \in \mathbb{R}^2 : F(x,y) = 0\}$$

is the graph $\{\big(x, f(x)\big) \in \mathbb{R}^2 : x \in \mathbb{R}\}$ of a smooth function $f : \mathbb{R} \to \mathbb{R}$.

(d) Let γ be a simple closed curve. Then its length Length(γ) and the area it encloses satisfy

$$\text{Area}(\gamma) \leq \frac{1}{4\pi}\text{Length}(\gamma)^2.$$

5. Let γ be a regular space curve. Define a vector-valued function $\vec{d}$ by

$$\vec{d} = \tau\vec{t} + \kappa\vec{b}.$$

Show that

$$\vec{t}' = \vec{d} \times \vec{t}$$

$$\vec{n}' = \vec{d} \times \vec{n}$$

$$\vec{b}' = \vec{d} \times \vec{b}.$$

$\vec{d}$ is called the **Darboux vector field**.