M365g

Spring 2017

Bowen

Name________________________________

Exam #2

Instructions. Please put your name at the top of the exam. Read over the entire exam before you begin; you should work on the problems you'll find easiest first. Continue your work on the backs of pages or on extra sheets. *If your solution runs over onto these pages, please indicate that clearly.* If you use extra sheets, be sure to staple them to the rest of your exam and put your name on them.

Questions

1. Consider the surface of revolution

$$\sigma(u, \phi) = (f(u)\cos(\phi), f(u)\sin(\phi), g(u)),$$

where $f(u)$ and $g(u)$ are smooth functions, $f(u) > 0$ and $f'(u)^2 + g'(u)^2 = 1$.

(a) Compute the first fundamental form in terms of the functions f and g. (With ϕ playing the role of the variable that we usually call v.)

(b) Let $\gamma(t) = \sigma(u(t), \phi(t))$ be a path in the surface. Set up the integral for its length.

(c) Fix a value of ϕ_0. Show that the shortest path along the surface from $\sigma(a, \phi_0)$ to $\sigma(b, \phi_0)$ has the form $\gamma(t) = \sigma(u(t), \phi_0)$ for some function $u(t)$.

Solution. Since

$$\sigma_u = (f'\cos(\phi), f'\sin(\phi), g'), \quad \sigma_\phi = (-f\sin(\phi), f\cos(\phi), 0),$$

$$E = \|\sigma_u\|^2 = (f')^2 + (g')^2 = 1, \quad F = \sigma_u \cdot \sigma_v = 0, G = \|\sigma_v\|^2 = f^2.$$

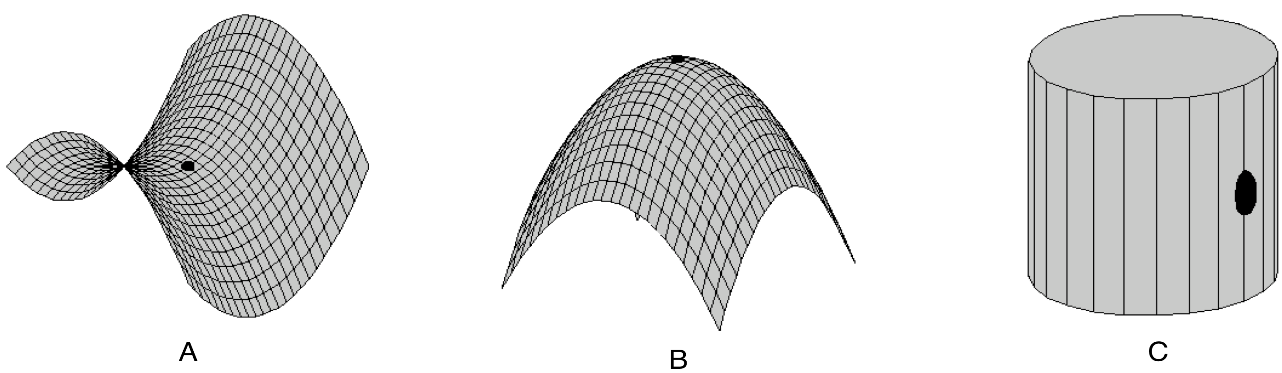
Since $F = 0$, the length of a path $\gamma(t) = \sigma(u(t), \phi(t))$ is

$$\int \sqrt{E(u(t))(du/dt)^2 + G(u(t))(d\phi/dt)^2}\ dt$$

which is at least as big as $\int \sqrt{E(u(t))}\ du$, which is the length of the path when ϕ constant. This argument doesn't depend at all on the form of E and G, just on the facts that $F = 0$ and that E does not depend on ϕ.

Common mistakes. It's a common mistake to think that because γ is a geodesic (if $u(t) = t$) that it must be length-minimizing. Geodesics are locally length-minimizing but there can be more than one. Also it is possible that there are no geodesics at all between some points (this happens with $\mathbb{R}^2 - \{(0,0)\}$).

2. Pictured below are three surfaces together with a distinguished point p. Order the pictures according to *increasing* Gauss curvature at p. In other words, write the letter of the surface of *least* Gauss curvature at p first.



Solution. $K_A < K_C < K_B$.

3. Let S be an oriented regular surface.

   (a) Prove: if $K > 0$ then $H \neq 0$.

   (b) Prove: if $K > 0$ then after re-choosing the unit normal $\vec{N}$ if necessary, we can assume that $H > 0$. In other words, the positivity or negativity of H depends only on the choice of unit normal.

   (c) Prove: if S is a regular surface and $K > 0$ everywhere on S then S is orientable. (You can assume without proof that H is a smooth function on S).

Solution. Since $K = \kappa_1\kappa_2 > 0$, κ_1, κ_2 have the same sign. So $H = \frac{\kappa_1+\kappa_2}{2} \neq 0$. If we change the choice of unit normal to the surface, then the Weingarten map gets multiplied by -1. This changes the sign of both κ_1 and κ_2. So H becomes $-H$. It follows that we can choose the unit normal so that $H > 0$ everywhere. This is a smooth choice because H is smooth (up to a choice of sign). So S is orientable.

Common mistakes. The point of this problem is that H is well-defined on surface patches (because every surface patch is orientable), it might be that S is non-orientable and then there is no smooth choice of unit normal. For example, this implies that the Möbius band must have nonpositive Gaussian curvature somewhere.

4. Suppose that S is a surface with $H = 0$ and $K \neq 0$ everywhere. Show that the Gauss map $\vec{N} : S \to S^2$ is conformal. Hint: to show that a map $F : S_1 \to S_2$ is conformal at a point p, it suffices to show that there is a basis $\{\vec{e}_1, \vec{e}_2\}$ for $T_p(S_1)$ and a constant $k \neq 0$ such that $\langle D_pF\vec{e}_i, D_pF\vec{e}_j\rangle = k\langle \vec{e}_i, \vec{e}_j\rangle$ for all i, j.

   Solution. Because $H = 0$, $\kappa_1 = -\kappa_2$ where κ_1, κ_2 are the principal curvatures. Therefore, if $\vec{e}_1, \vec{e}_2$ are unit eigenvectors of $\mathcal{W} = -D\vec{N}$ then

$$\langle D\vec{N}(\vec{e}_i), D\vec{N}(\vec{e}_j)\rangle = 0$$

   unless $i = j$ in which case $\langle D\vec{N}(\vec{e}_i), D\vec{N}(\vec{e}_j)\rangle = \kappa_i^2$. Since $\{\vec{e}_1, \vec{e}_2\}$ are a basis, this proves it. In detail: if $\vec{v}, \vec{w}$ are any unit vectors in $T_p(S)$ then

$$\vec{v} = a\vec{e}_1 + b\vec{e}_2, \quad \vec{w} = c\vec{e}_1 + d\vec{e}_2$$

   for some $a, b, c, d \in \mathbb{R}$. So $D\vec{N}(\vec{v}) = \kappa_1(a\vec{e}_1 - b\vec{e}_2)$, $D\vec{N}(\vec{w}) = \kappa_1(c\vec{e}_1 - d\vec{e}_2)$ and

$$\langle D\vec{N}(\vec{v}), D\vec{N}(\vec{w})\rangle = \kappa_1^2(ac + bd) = \kappa_1^2\langle \vec{v}, \vec{w}\rangle.$$

   Common mistakes. It is not enough to prove that

$$\langle D_pF\vec{e}_1, D_pF\vec{e}_2\rangle = k\langle \vec{e}_1, \vec{e}_2\rangle.$$

   In fact, $\langle \vec{e}_1, \vec{e}_2\rangle = 0$. However, if F is a map with $D_pF(\vec{e}_1) = 10\vec{e}_1$ and $D_pF(\vec{e}_2) = 100\vec{e}_2$ then it is true that

$$\langle D_pF\vec{e}_1, D_pF\vec{e}_2\rangle = k\langle \vec{e}_1, \vec{e}_2\rangle$$

   but this map is not conformal.

5. Let $\vec{v}, \vec{w}$ be smooth vector fields along a smooth curve γ in S so that $\vec{v}, \vec{w}$ are tangent to the surface everywhere. Show that

$$\frac{d}{dt}\langle \vec{v}, \vec{w} \rangle = \langle \nabla_\gamma \vec{v}, \vec{w} \rangle + \langle \vec{v}, \nabla_\gamma \vec{w} \rangle.$$

Solution. By the product rule

$$\frac{d}{dt}\langle \vec{v}, \vec{w} \rangle = \langle \frac{d}{dt}\vec{v}, \vec{w} \rangle + \langle \vec{v}, \frac{d}{dt}\vec{w} \rangle.$$

Now $\langle \frac{d}{dt}\vec{v}, \vec{w} \rangle = \langle \nabla_\gamma \vec{v}, \vec{w} \rangle$ because $\vec{w}$ is tangent to the surface and $\nabla_\gamma \vec{v}$ is the projection of $\frac{d}{dt}\vec{v}$ to the tangent plane of the surface. Similarly, $\langle \vec{v}, \frac{d}{dt}\vec{w} \rangle = \langle \vec{v}, \nabla_\gamma \vec{w} \rangle$. This implies the statement.

6. Suppose that γ is a unit-speed parametrization of the intersection of a plane Π with a surface. Suppose that Π contains $\vec{N}(p)$, the normal to the surface at $p \in S$, for every $p \in \Pi \cap S$.

   (a) Prove: the curvature of γ is the same as its normal curvature (up to sign).

   (b) Prove: γ is a geodesic.

   (c) Suppose $\mathcal{G} : S \to S^2$ is the Gauss map. What can you say about the image $\mathcal{G}(\gamma)$?

   (d) Show that γ is a line of curvature (this means γ' is an eigenvector of the Weingarten map).

Solution. Meusnier's Theorem implies that the curvature of γ is the same as its normal curvature (up to sign). So it is a geodesic. Also the Gauss map takes the curve to a circle contained in the parallel copy of Π that passes through the origin. So its derivative must take the tangent vector γ' into this same plane which means that γ' is an eigenvector of the Weingarten map. For example, the intersection of the plane xz-plane with the surface defined by $\sigma(u, v) = (u, v, \frac{au^2+bv^2}{2})$ satisfies these properties because

$$\sigma_u = (1, 0, au), \ \sigma_v = (0, 1, bv), \ \sigma_u \times \sigma_v = (-au, -bv, 1)$$

implies that the normal to the surface at $\sigma(u, 0) = (u, 0, au^2/2)$ is a scalar multiple of $(-au, 0, 1)$ which is contained in the xz-plane.