M365g

Spring 2017

Bowen

Name________________________________

Exam #2

Instructions. Please put your name at the top of the exam. Read over the entire exam before you begin; you should work on the problems you'll find easiest first. Continue your work on the backs of pages or on extra sheets. *If your solution runs over onto these pages, please indicate that clearly.* If you use extra sheets, be sure to staple them to the rest of your exam and put your name on them.

Questions

1. True/False

   (a) Define $S = \{(x, y, z) \in \mathbb{R}^3 : xyz = 0\}$. Then S is a smooth surface.

   (b) Let $\Delta \subset S^2$ be a triangle whose sides lie along great circles. Then the sum of the angles formed at the vertices exceeds π.

   (c) Let $\sigma : U \to \mathbb{R}^3$ be a surface patch on an oriented surface $S \subset \mathbb{R}^3$ and

   $$I = Edu^2 + 2Fdudv + Gdv^2$$
   $$II = Ldu^2 + 2Mdudv + Ndv^2$$

   the induced first and second fundamental forms. Suppose $F = M = 0$. Then the Gauss curvature vanishes.

   (d) The map σ is area-preserving if and only if $EG - F^2 = 1$. (Here, and in subsequent problems, we use the standard metric in the (u, v)-plane.)

   (e) The map σ is conformal if and only if $E = G$.

   (f) Let $S \subset \mathbb{R}^3$ be a surface and $p \in S$. Then there exists a Cartesian coordinate system (x, y, z) in $\mathbb{R}^3$ and a function $f : \mathbb{R}^2 \to \mathbb{R}$ such that S is the graph $\{(x, y, f(x, y)) \in \mathbb{R}^3 : x, y \in \mathbb{R}\}$.

2. Let $0 < r < R$ and consider the torus $S \subset \mathbb{R}^3$ described by the equations

$$x = (R + r\cos\phi)\cos\theta$$
$$y = (R + r\cos\phi)\sin\theta$$
$$z = r\sin\phi$$

where $0 \le \phi, \theta \le 2\pi$. Consider the parametrized curve $\gamma : [0, \pi] \to S$ described by the equations $\phi = \pi/2$, $\theta = t$ and the vector field ξ along γ given by $\xi_{(x,y,z)} = (x, y, 0)$. Compute the covariant derivative $\nabla_\gamma \xi = \nabla_{\gamma'} \xi$ of ξ along γ.

3. Consider the helicoid, parametrized by

$$\sigma(u, v) = (v\,\cos u, v\,\sin u, \lambda u),$$

where λ is a positive constant, $0 < v < 2$, and $-10\pi < u < 10\pi$. Complete the following for each point on the surface

   (a) Compute the first fundamental form.

   (b) Compute the second fundamental form.

   (c) Compute the Gauss and mean curvatures.

   (d) Compute the principal curvatures.

   (e) Write an expression for the total area of the image of σ. Do not evaluate the integrals.

   (f) Let C_1 be the image of the curve $t \mapsto \sigma(t, 0)$ and C_2 the image of the curve $t \mapsto \sigma(0, t)$. Compute the angle of intersection of C_1 and C_2.

4. Let S^2 be the unit sphere and γ be the latitudinal circle which is the intersection of the plane $z = z_0$ with S^2 (for some $z_0 \in (-1, 1)$. Compute $|\kappa_g|$ = the absolute value of the geodesic curvature of γ.