M365g

Spring 2017

Bowen

Name________________________________

Exam #2

Instructions. Please put your name at the top of the exam. Read over the entire exam before you begin; you should work on the problems you'll find easiest first. Continue your work on the backs of pages or on extra sheets. *If your solution runs over onto these pages, please indicate that clearly.* If you use extra sheets, be sure to staple them to the rest of your exam and put your name on them.

Questions

1. True/False

   (a) Define $S = \{(x, y, z) \in \mathbb{R}^3 : xyz = 0\}$. Then S is a smooth surface.

   Solution. This is false. S is the union of the three coordinate planes. It is not smooth on any of the coordinate axes.

   (b) Let $\Delta \subset S^2$ be a triangle whose sides lie along great circles. Then the sum of the angles formed at the vertices exceeds π.

   Solution. True. The area of Δ is the sum of its angles minus π.

   (c) Let $\sigma : U \to \mathbb{R}^3$ be a surface patch on an oriented surface $S \subset \mathbb{R}^3$ and

$$I = Edu^2 + 2Fdudv + Gdv^2$$
$$II = Ldu^2 + 2Mdudv + Ndv^2$$

   the induced first and second fundamental forms. Suppose $F = M = 0$. Then the Gauss curvature vanishes.

   Solution. False. The sphere parametrized by $\sigma(\theta, \phi) = (\sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), \sin(\phi))$ is a counterexample.

(d) The map σ is area-preserving if and only if $EG - F^2 = 1$. (Here, and in subsequent problems, we use the standard metric in the (u, v)-plane.)

Solution. True.

(e) The map σ is conformal if and only if $E = G$.

Solution. False, unless you assume that $F = 0$ too in which case it is true.

(f) Let $S \subset \mathbb{R}^3$ be a surface and $p \in S$. Then there exists a Cartesian coordinate system (x, y, z) in $\mathbb{R}^3$ and a function $f : \mathbb{R}^2 \to \mathbb{R}$ such that S is the graph $\{(x, y, f(x, y)) \in \mathbb{R}^3 : x, y \in \mathbb{R}\}$.

Solution. False. The unit sphere cannot be represented as the graph of a function. However, the statement is true locally.

2. Let $0 < r < R$ and consider the torus $S \subset \mathbb{R}^3$ described by the equations

$$x = (R + r\cos\phi)\cos\theta$$
$$y = (R + r\cos\phi)\sin\theta$$
$$z = r\sin\phi$$

where $0 \leq \phi, \theta \leq 2\pi$. Consider the parametrized curve $\gamma : [0, \pi] \to S$ described by the equations $\phi = \pi/2$, $\theta = t$ and the vector field ξ along γ given by $\xi_{(x,y,z)} = (x, y, 0)$. Compute the covariant derivative $\nabla_\gamma \xi = \nabla_{\gamma'} \xi$ of ξ along γ.

Solution. The curve γ parametrizes the "top circle" which is the intersection of the torus with the $z = 1$ plane. The vector field ξ parametrizes the outward unit normal to this plane curve (it is not normal to the surface). Its derivative (as a function of t) is $(-R\sin(t), R\cos(t), 0)$. In particular, its derivative already lies in the tangent space to the surface (the $z = 0$ plane). So

$$\nabla_{\gamma'}\xi = (-R\sin(t), R\cos(t), 0).$$

3. Consider the helicoid, parametrized by

$$\sigma(u, v) = (v\cos u, v\sin u, \lambda u),$$

where λ is a positive constant, $0 < v < 2$, and $-10\pi < u < 10\pi$. Complete the following for each point on the surface

(a) Compute the first fundamental form.

Solution.

$$\sigma_u = (-v\sin(u), v\cos(u), \lambda)$$

$$\sigma_v = (\cos(u), \sin(u), 0)$$

$$E = (v^2 + \lambda^2),\ \ F = 0,\ \ G = 1$$

$$\text{f.f.f.} = (\text{v}^2 + \lambda^2)\text{du}^2 + \text{dv}^2.$$

(b) Compute the second fundamental form.

$$\sigma_{uu} = (-v\cos(u), -v\sin(u), 0)$$

$$\sigma_{uv} = (-\sin(u), \cos(u), 0)$$

$$\sigma_{vv} = \vec{0}$$

$$\sigma_u \times \sigma_v = (-\lambda\sin(u), \lambda\cos(u), -v)$$

$$\vec{N} = \frac{(-\lambda\sin(u), \lambda\cos(u), -v)}{\lambda^2 + v^2}$$

$$L = \sigma_{uu} \cdot \vec{N} = 0$$

$$M = \sigma_{uv} \cdot \vec{N} = \frac{\lambda}{\lambda^2 + v^2}$$

$$N = \sigma_{vv} \cdot \vec{N} = 0.$$

So

$$\text{s.f.f.} = \frac{2\lambda}{\lambda^2 + \text{v}^2}\text{dudv}.$$

(c) Compute the Gauss and mean curvatures. It will be easier to compute the the matrix representing the Weingarten map in the basis $\{\sigma_u, \sigma_v\}$. This is

$$\frac{1}{v^2+\lambda^2}\begin{pmatrix} G & -F \\ -F & E \end{pmatrix}\begin{pmatrix} L & M \\ M & N \end{pmatrix} = \frac{\lambda}{(\lambda^2+v^2)^2}\begin{pmatrix} 1 & 0 \\ 0 & v^2+\lambda^2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \frac{\lambda}{(\lambda^2+v^2)^2}\begin{pmatrix} 0 & 1 \\ v^2+\lambda^2 & 0 \end{pmatrix}.$$

The eigenvalues are $\pm\lambda(\lambda^2+v^2)^{-3/2}$. The eigenvectors are

$$\begin{pmatrix} 1 \\ (\lambda^2+v^2)^{1/2} \end{pmatrix}, \begin{pmatrix} -1 \\ (\lambda^2+v^2)^{1/2} \end{pmatrix}.$$

So

$$K = \frac{LN-M^2}{EG-F^2} = -\frac{\lambda^2}{(\lambda^2+v^2)^3}.$$
$$H = 0.$$

(d) Compute the principal curvatures.

Solution. These are $\pm\lambda(\lambda^2+v^2)^{-3/2}$.

(e) Write an expression for the total area of the image of σ. Do not evaluate the integrals.

Solution. $\int_0^2 \int_{-10\pi}^{10\pi} \sqrt{\lambda^2+v^2}\, dudv$.

(f) Let C_1 be the image of the curve $t \mapsto \sigma(t,0)$ and C_2 the image of the curve $t \mapsto \sigma(0,t)$. Compute the angle of intersection of C_1 and C_2.

Solution. Because $\sigma_u \cdot \sigma_v = F = 0$, the angle is $\pi/2$.

4. Let S^2 be the unit sphere and γ be the latitudinal circle which is the intersection of the plane $z = z_0$ with S^2 (for some $z_0 \in (-1,1)$. Compute $|\kappa_g|$ = the absolute value of the geodesic curvature of γ.

Solution. The radius of γ is $\sqrt{1-z_0^2}$. So its curvature (as a space curve) is $\frac{1}{\sqrt{1-z_0^2}}$. On the other hand, the normal curvature of every curve on S^2 is $+1$ because the Weingarten map is the identity. Since

$$\kappa^2 = \kappa_g^2 + \kappa_n^2$$

this implies

$$\kappa_g^2 = \kappa^2 - \kappa_n^2 = \frac{1}{1-z_0^2} - 1.$$