M365g

Spring 2017

Bowen

Name______________________________

Exam #2

Instructions. Please put your name at the top of the exam. Read over the entire exam before you begin; you should work on the problems you'll find easiest first. Continue your work on the backs of pages or on extra sheets. *If your solution runs over onto these pages, please indicate that clearly.* If you use extra sheets, be sure to staple them to the rest of your exam and put your name on them.

Questions

1. Let U be an open set in the u, v plane, and suppose we have a first fundamental form

$$E\,du^2 + 2F\,du\,dv + G\,dv^2$$

on U. Let $f : U \to \mathbb{R}$ be a smooth function. The **gradient** of f at p (with respect to this form) is a vector $\nabla f \in T_p(U)$ satisfying: if γ is any curve in U with $\gamma(0) = p$ then

$$\frac{d}{dt}\bigg|_{t=0} f(\gamma(t)) = \langle \nabla f, \gamma'(0) \rangle_p$$

where $\langle \cdot, \cdot \rangle_p$ means the first fundamental form. So

$$\langle (a, b), (c, d) \rangle_p = E(p)ac + F(p)(ad + bc) + G(p)bd.$$

Problem: find an expression for ∇f in terms of E, F, G, f_u, f_v.

Solution. Let $\gamma(t) = (u(t), v(t))$ be a curve in U with $\gamma(t_0) = (u_0, v_0)$. We must have

$$\sigma^* \langle \nabla f(u_0, v_0), \gamma'(t) \rangle = \frac{d}{dt}\bigg|_{t=t_0} f(\gamma(t)).$$

If $\nabla f(u_0, v_0) = (e_1, e_2)$ then the left hand side is

$$Ee_1 u'(t_0) + F(e_1 v'(t_0) + e_2 u'(t_0)) + Ge_2 v'(t_0).$$

In the special case in which $\gamma(t) = (u_0 + t, v_0)$, we have

$$\frac{d}{dt}\bigg|_{t=t_0} f(\gamma(t)) = f_u(u_0, v_0).$$

So

$$Ee_1 + Fe_2 = f_u(u_0, v_0).$$

Similarly, if $\gamma(t) = (u_0, v_0 + t)$, we have

$$\frac{d}{dt}\bigg|_{t=t_0} f(\gamma(t)) = f_v(u_0, v_0).$$

So

$$Fe_1 + Ge_2 = f_v(u_0, v_0).$$

In other words,

$$\begin{bmatrix} E & F \\ F & G \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} f_u \\ f_v \end{bmatrix}.$$

So we have

$$\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1} \begin{bmatrix} f_u \\ f_v \end{bmatrix} = \frac{1}{EG - F^2} \begin{bmatrix} G & -F \\ -F & E \end{bmatrix} \begin{bmatrix} f_u \\ f_v \end{bmatrix} = \frac{1}{EG - F^2} \begin{bmatrix} Gf_u - Ff_v \\ -Ff_u + Ef_v \end{bmatrix}.$$

2. Let $S \subset \mathbb{R}^3$ be a surface and $p \in S$. Fix a unit vector $\vec{v}_0$ in T_pS and let $\kappa_n(\theta)$ be the normal curvature at p along the unit vector $\vec{v}$ line at angle θ from $\vec{v}_0$. Prove that the mean curvature H at p is

$$H = \frac{1}{\pi} \int_0^\pi \kappa_n(\theta)\, d\theta.$$

In other words, the mean curvature is the average normal curvature.

Solution. The normal curvature along a unit vector $\vec{v}$ is

$$\langle\langle \vec{v}, \vec{v} \rangle\rangle = \langle \mathcal{W}(\vec{v}), \vec{v} \rangle$$

where $\mathcal{W}$ is the Weingarten map. Let $\vec{e}_0, \vec{e}_1$ be an orthonormal basis of eigenvectors of $\mathcal{W}_p$. If κ_0, κ_1 are the normal curvatures in the directions of $\vec{e}_0, \vec{e}_1$ respectively then

$$2H \;=\; \mathrm{trace}(\mathcal{W}) = \langle \mathcal{W}(\tilde{\mathrm{e}}_0), \tilde{\mathrm{e}}_0 \rangle + \langle \mathcal{W}(\tilde{\mathrm{e}}_1), \tilde{\mathrm{e}}_1 \rangle = \kappa_0 + \kappa_1.$$

On the other hand,

$$\kappa_n(\theta) = \langle \mathcal{W}(\cos(\theta)\vec{e}_0 + \sin(\theta)\vec{e}_1), \cos(\theta)\vec{e}_0 + \sin(\theta)\vec{e}_1\rangle = \cos^2(\theta)\kappa_0 + \sin^2(\theta)\kappa_1.$$

Therefore

$$\frac{1}{\pi}\int_0^\pi \kappa_n(\theta)\, d\theta \quad = \quad \frac{1}{\pi}\int_0^\pi \cos^2(\theta)\kappa_0 + \sin^2(\theta)\kappa_1\, d\theta$$

Now $\cos^2(\theta) - \sin^2(\theta) = \cos(2\theta)$ by the double-angle formula. So $2\cos^2(\theta) - 1 = \cos(2\theta)$ which means $\cos^2(\theta) = \frac{\cos(2\theta)+1}{2}$. Therefore,

$$\int_0^\pi \cos^2(\theta)\ d\theta = \frac{\sin(2\theta)}{4} + \theta/2\Big|_0^\pi = \pi/2.$$

Also

$$\int_0^\pi \sin^2(\theta)\ d\theta = \pi - \int_0^\pi \cos^2(\theta)\ d\theta = \pi/2.$$

So

$$\frac{1}{\pi}\int_0^\pi \cos^2(\theta)\kappa_0 + \sin^2(\theta)\kappa_1\, d\theta = (\kappa_0 + \kappa_1)/2 = H.$$

3. For each of the following geometric objects either give an example or show that no example exists. You should justify your answer in either case. Sketches may be helpful.

   (a) A surface of revolution with both positive and negative Gauss curvature.

   Solution. A torus is an example of this.

   (b) A surface S of positive Gauss curvature with a point $p \in S$ where all normal sections have vanishing (normal) curvature.

   Solution. This is not possible because the Gauss curvature is the product of the principal curvatures which are the max and min curvatures over all normal sections.

   (c) A regular surface patch $\sigma : U \to \mathbb{R}^3$ with vanishing first fundamental form.

   Solution. This is not possible because σ_u and σ_v cannot vanish since σ is regular.

   (d) A ruled surface which is not a surface of revolution.

   Solution. The helicoid is an example of this.

4. Let

$$\sigma(\theta, \phi) = ((2 + \cos(\phi)\cos(\theta), (2 + \cos(\phi)\sin(\theta), \sin(\phi))$$

parametrize a torus (this is the surface obtained by rotating the circle $(x-2)^2 + z^2 = 1$ around the z-axis). Consider the following curves on S:

$$\gamma_1(\theta) = \sigma(\theta, 0), \quad \gamma_2(\theta) = \sigma(\theta, \pi/2), \quad \gamma_3(\theta) = \sigma(\theta, \pi), \quad \gamma_4(\theta) = \sigma(\theta, 3\pi/2)$$

$$\lambda(\phi) = \sigma(\phi, 0)$$

for $\theta, \phi \in \mathbb{R}$. Which of these curves are (up to reparametrization) geodesics? asymptotic curves? lines of curvature?

Solution. All of these curves are lines of curvature (we checked a few lectures ago that all meridians and parallels of a surface of revolution are lines of curvature).

The curves γ_2, γ_4 are such that their normal vector fields are in the tangent space to the surface. So they are asymptotic curves. They are not geodesics because their normal vector fields are not normal to the surface.

The curves $\gamma_1, \gamma_3, \lambda$ are such that their normals are normal to the surface. So they are all geodesics. They are also lines of curvature They are not asymptotic curves because their normal curvature equals their curvature (up to sign) and their curvature (as space curves) does not vanish because they are not straight lines in $\mathbb{R}^3$!

5. In this exercise we are going to explore lengths and areas in slightly curved paraboloids. Let S be the surface $z = ax^2 + by^2$, where a and b are small constants, and we are using x and y as our coordinates. (That is, $\sigma(u, v) = (u, v, au^2 + bv^2))$

   (a) Compute the first fundamental form.

   Solution. $\sigma_u = (1, 0, 2au)$ and $\sigma_v = (0, 1, 2bv)$, so $E = 1 + 4a^2u^2$, $F = 4abuv$ and $G = 1 + 4b^2v^2$.

   (b) Set up (but don't evaluate) an explicit integral that computes exactly the length of the path $\gamma(t) = \sigma(\cos(t), \sin(t))$ as t goes from 0 to 2π.

   Solution.

$$\int_0^{2\pi} \sqrt{(1 + 4a^2\cos^2(t))\sin^2(t) - 8ab\cos^2(t)\sin^2(t) + (1 + 4b^2\sin^2(t))\cos^2(t)}\; dt$$

$$= \int_0^{2\pi} \sqrt{1 + (4a^2 - 8ab + 4b^2)\cos^2(t)\sin^2(t)}\ dt$$

$$= \int_0^{2\pi} \sqrt{1 + 4(a-b)^2\cos^2(t)\sin^2(t)}\ dt.$$

(c) Set up another explicit integral that computes exactly the area of the surface enclosed by this curve.

Solution.

$$EG - F^2 = (1 + 4a^2u^2)(1 + 4b^2v^2) - 16a^2b^2u^2v^2 = 1 + 4a^2u^2 + 4b^2v^2.$$

So the area is

$$\iint_D \sqrt{1 + 4a^2u^2 + 4b^2v^2}\ dudv$$

where D is the interior of the unit circle in the uv-plane. We can change this to polar coordinates via:

$$= \int_0^{2\pi} \int_0^1 \sqrt{1 + 4a^2r^2\cos^2(\theta) + 4b^2r^2\sin^2(\theta)}r\ drd\theta.$$

(d) Evaluate the integrals from part (b) and (c) to second order in a and b. That is, keep terms like a^2, ab and b^2, but ignore anything of higher order. You might find these approximations and identities useful:

$$\sqrt{1+x} \approx 1 + (x/2), \quad \int_0^{2\pi} \cos^2(\theta)\sin^2(\theta)\ d\theta = \pi/4, \quad \iint_D x^2\ dxdy = \pi/4$$

where D is the unit disk. Then compute the isoperimetric ratio $(area)/(length)^2$ to second order in a and b.

Solution.

$$L \approx \int_0^{2\pi} 1 + 2(a-b)^2\cos^2(t)\sin^2(t)\ dt = 2\pi(1 + (a-b)^2/4).$$

$$A \approx \iint_D 1 + 2a^2u^2 + 2b^2v^2\ dudv = \pi(1 + (a^2+b^2)/2).$$

So

$$\frac{A}{L^2} \approx \frac{\pi(1 + (a^2+b^2)/2)}{4\pi^2(1 + (a-b)^2/4)^2} \approx \frac{1+ab}{4\pi}.$$

The moral of the story: if a and b have the same sign, we have more area per unit perimeter than in the flat Euclidean plane case, while if a and b have different signs we have more perimeter per unit area.

6. Let $\Pi_\gamma^{pq} : T_p(S) \to T_q(S)$ denote parallel transport from p to q along γ. Show that for any vectors $\vec{v}, \vec{w} \in T_p(S)$,

$$\langle \vec{v}, \vec{w} \rangle_p = \langle \Pi_\gamma^{pq} \vec{v}, \Pi_\gamma^{pq} \vec{w} \rangle_q.$$

Solution. By abuse of notation, we will consider $\vec{v}, \vec{w}$ to be parallel vector fields along γ. Then it suffices to show that

$$\langle \vec{v}(t), \vec{w}(t) \rangle_{\gamma(t)}$$

is constant in t. To do this, we covariantly differentiate:

$$\frac{d}{dt} \langle \vec{v}(t), \vec{w}(t) \rangle_{\gamma(t)} = \langle \nabla_\gamma \vec{v}, \vec{w} \rangle + \langle \vec{v}, \nabla_\gamma \vec{w} \rangle = 0.$$

This finishes it.

7. Let $\sigma(\theta, \phi) = (\cos\theta\cos\phi, \cos\theta\sin\phi, \sin\theta)$ parametrize the unit sphere. Fix θ_0 and let $\gamma(\phi) = \sigma(\theta_0, \phi)$ parametrize a latitude. Let

$$\vec{v}(\phi) = -\cos\theta_0 \sin(\phi \sin\theta_0) \vec{\sigma_\theta} + \cos(\phi \sin\theta_0) \vec{\sigma_\phi}$$

be a vector field along γ. Compute the covariant derivative $\nabla_\gamma \vec{v}$.