M365g

Spring 2017

Bowen

Name________________________________

Practice Final

Instructions. Please put your name at the top of the exam. Read over the entire exam before you begin; you should work on the problems you'll find easiest first. Continue your work on the backs of pages or on extra sheets. *If your solution runs over onto these pages, please indicate that clearly.* If you use extra sheets, be sure to staple them to the rest of your exam and put your name on them.

Questions

1. For each of the following surfaces describe all parametrized geodesics. The surface is cut out by the indicated equation in (x, y, z)-space. Justify your answer carefully.

   (a) The plane $z = 0$.

   (b) The cylinder $x^2 + y^2 = 1$.

   (c) The sphere $x^2 + y^2 + z^2 = 1$.

2. Let γ be a smooth curve on a surface. What is the relationship between the covariant derivative $\nabla_\gamma \gamma'$ and the geodesic curvature κ_g?

3. Suppose $S = \sigma(U)$ is a surface such that every point is umbilic. Let $\kappa : S \to \mathbb{R}$ be the principal curvature at p.

   (a) Show $\vec{N}_u = -\kappa \sigma_u$ and $\vec{N}_v = -\kappa \sigma_v$.

   (b) By differentiating the above, show κ is constant.

   (c) If $\kappa = 0$, show $\vec{N}$ is constant (and hence S is contained in a plane).

   (d) If $\kappa \neq 0$ show $\vec{N} = -\kappa \sigma + \vec{b}$ for some constant $\vec{b}$ (and hence S is contained in a sphere).

4. Holonomy.

   (a) Prove the following simple theorem. Let $S_1, S_2 \subset \mathbb{E}^3$ be surfaces which are tangent along a curve $C \subset S_1 \cap S_2$. Let $p, q \in C$ and suppose $P_i : T_pS_i \to T_qS_i$ is parallel transport along C from p to q in the surface S_i, $i = 1, 2$. Then $P_1 = P_2$.

   (b) Compute the holonomy around the parallel $z = z_0$ on the unit sphere S^2 with $0 < z_0 < 1$.

   (c) Consider the cone $z = x^2 + y^2$. Compute the holonomy around the circle at level $z = z_1$ on this curve (for some fixed $z_1 > 0$). Hint: by using part (a) and part (b) you can reduce this to a simple computation.

5. A ruled surface has a parametrization of the form

$$\sigma(u, v) = \gamma(u) + v\delta(u),$$

   where $\gamma : (\alpha, \beta) \to \mathbb{E}^3$ is a curve of points and $\delta : (\alpha, \beta) \to \mathbb{R}^3$ is a curve of vectors. Assume that γ is unit speed and δ is unit length.

   (a) Compute the Gauss curvature K. Show that the ruled surface is flat, i.e., $K = 0$, if and only if the three vectors $\dot{\gamma}, \delta, \dot{\delta}$ are linearly dependent.

   (b) One way this can happen is if $\dot{\delta} = 0$. What is the surface in this case?

   (c) Another possibility is that $\dot{\gamma}$ is a multiple of δ. What is the surface in this case?

   (d) Give an example of a flat ruled surface not of either of the above types.

   (e) Give an example of a nonflat ruled surface.

6. Cylindrical curves. Consider a curve of the form $\gamma(t) = (\cos(t), \sin(t), f(t))$ where f is a smooth function.

   (a) Compute the speed and curvature of γ at time t in terms of the (unknown) function f and its derivatives. Hint:

$$\kappa = \frac{\|\gamma' \times \gamma''\|}{\|\gamma'\|^3}.$$

(b) Think of the curve as lying on the cylinder $x^2 + y^2 = 1$. Compute its geodesic and normal curvatures.

7. Let γ be a smooth closed curve with curvature bounded below by a positive constant $\delta > 0$. Let $\epsilon > 0$ and

$$\sigma(u, v) = \gamma(u) + \epsilon \cos(v)\vec{n}(u) + \epsilon \sin(v)\vec{b}(u)$$

where $\vec{n}, \vec{b}$ are the unit normal and binormal vectors to γ. Prove that: if $\epsilon > 0$ is sufficiently small then σ is a regular surface.