M365g

Spring 2017

Bowen

Name________________________________

Quiz #2

Instructions. This a 30 minute take home quiz. Take it by yourself, without the aid of the book, the internet or other students. You will need extra paper to write your answers. It is due Monday Feb 13.

Questions

1. Suppose that a plane curve C is specified in polar coordinates (r, θ) by giving a function $r = r(\theta)$ on a domain $\theta \in [a, b]$. Derive a formula for the curvature of C.

   Hints:

   $$x = r\cos(\theta), y = r\sin(\theta), \kappa = \frac{\|\gamma' \times \gamma''\|}{\|\gamma'\|^3}.$$
   $$\gamma(\theta) = (r(\theta)\cos(\theta), r(\theta)\sin(\theta)).$$

   Dropping the θ's for simplicity, we obtain:

   $$\gamma = r(\cos, \sin)$$

   $$\gamma' = r'(\cos, \sin) + r(-\sin, \cos)$$

   $$\gamma'' = (r'' - r)(\cos, \sin) + 2r'(-\sin, \cos).$$

   Solution.

   Notice that $(\cos, \sin) \times (\cos, \sin) = 0$ for example while $(\cos, \sin) \times (-\sin, \cos) = (0, 0, 1)$. This simplifies the computations because the cross product is bi-linear. So we get

   $$\gamma' \times \gamma'' = (0, 0, 2(r')^2 - r(r'' - r)).$$

   Since $(\cos, \sin)$ is orthogonal to $(-\sin, \cos)$,

   $$\|\gamma'\| = \sqrt{(r')^2 + r^2}.$$

   So

   $$\kappa = \frac{|2(r')^2 - r(r'' - r)|}{[(r')^2 + r^2]^{3/2}}.$$

2. Let $\gamma : (a, b) \to \mathbb{E}^3$ be a space curve and suppose that the curvature and torsion are nonzero at $\gamma(t_0)$ for some $t_0 \in (a, b)$. Demonstrate that the image of γ restricted to an arbitrary small interval around t_0 contains points on both sides of the osculating plane at $\gamma(t_0)$.

   Hint: use the Taylor series approximation of γ near $\gamma(t_0)$:

   $$\gamma(t_0 + \Delta) = \gamma(t_0) + \Delta\gamma'(t_0) + (\Delta^2/2)\gamma''(t_0) + (\Delta^3/6)\gamma'''(t_0) + \text{remainder}.$$

More Hints: $\vec{b} = \vec{t} \times \vec{n}, \vec{t}' = \kappa\vec{n}, \vec{n}' = -\kappa\vec{t} + \tau\vec{b}, \vec{b}' = -\tau\vec{n}$. The osculating plane is the plane containing the vectors $\vec{t}(t_0), \vec{n}(t_0)$ passing through the point $\gamma(t_0)$.

Solution. Without loss of generality, we may assume γ has unit-speed. Then $\gamma' = \vec{t}$, $\gamma'' = \kappa\vec{n}$ and

$$\gamma''' = \frac{d\kappa}{ds}\vec{n} + \kappa(\vec{n}') = \frac{d\kappa}{ds}\vec{n} + \kappa(-\kappa\vec{t} + \tau\vec{b}).$$

By the Taylor series approximation,

$$\gamma(t_0 + \Delta) = \gamma + \Delta\vec{t} + (\Delta^2/2)\kappa\vec{n} + (\Delta^3/6)(\frac{d\kappa}{ds}\vec{n} + \kappa(-\kappa\vec{t} + \tau\vec{b})) + \text{remainder}.$$

(I have removed the t_0's on the right hand side to make the equation easier to read). So

$$\gamma(t_0 + \Delta) = x + (\Delta^3/6)\kappa\tau\vec{b} + \text{remainder}$$

for some $x = x(t_0, \Delta)$ in the osculating plane.

The coefficient on $\vec{b}$ is $\Delta^3/6\kappa\tau$. So if $\Delta^3/6\kappa\tau$ is larger in absolute value than the remainder term then $\gamma(t_0 + \Delta)$ is not on the osculating plane. Moreover there is an $\epsilon > 0$ such that this condition holds if $|\Delta| < \epsilon$ because $(remainder)/\Delta^3 \to 0$ as $\Delta \to 0$. Also $\gamma(t_0 + \Delta)$ and $\gamma(t_0 - \Delta)$ will be on opposite sides of the osculating plane for $|\Delta| < \epsilon$ because the sign of the coefficient on $\vec{b}$ changes when the sign of Δ changes.