M365g

Spring 2017

Bowen

Name________________________________

Quiz #3

Instructions. This a 30 minute take home quiz. Take it by yourself, without the aid of the book, the internet or other students. You will need extra paper to write your answers. It is due Wednesday April 19.

Questions

1. Compute the Christoffel symbols Γ^1_{11} and Γ^2_{11}. Hint:

$$\sigma_{uu} = \Gamma^1_{11}\sigma_u + \Gamma^2_{11}\sigma_v + L\vec{N}.$$

Solution. Take the inner product with σ_u on both sides to obtain

$$\sigma_{uu} \cdot \sigma_u = \Gamma^1_{11}E + \Gamma^2_{11}F.$$

Take the inner product with σ_v on both sides to obtain

$$\sigma_{uu} \cdot \sigma_v = \Gamma^1_{11}F + \Gamma^2_{11}G.$$

Next, use an integration-by-parts argument to simplify the left-hand sides. Differentiate

$$\sigma_u \cdot \sigma_u = E$$

with respect to u to obtain

$$2\sigma_{uu} \cdot \sigma_u = E_u$$

and therefore

$$\sigma_{uu} \cdot \sigma_u = E_u/2.$$

Differentiate

$$\sigma_u \cdot \sigma_v = F$$

with respect to u to obtain

$$\sigma_{uu} \cdot \sigma_v + \sigma_u \cdot \sigma_{uv} = F_u.$$

So

$$\sigma_{uu} \cdot \sigma_v = F_u - \sigma_u \cdot \sigma_{uv}.$$

Also differentiate

$$\sigma_u \cdot \sigma_u = E$$

with respect to v to obtain

$$2\sigma_u \cdot \sigma_{uv} = E_v.$$

Therefore,

$$\sigma_{uu} \cdot \sigma_v = F_u - E_v/2.$$

Plug this back into the first two equations above to obtain

$$\begin{bmatrix} E_u/2 \\ F_u - E_v/2 \end{bmatrix} \begin{bmatrix} E & F \\ F & G \end{bmatrix} = \begin{bmatrix} \Gamma^1_{11} \\ \Gamma^2_{11} \end{bmatrix}.$$

Thus

$$\begin{bmatrix} \Gamma^1_{11} \\ \Gamma^2_{11} \end{bmatrix} = \begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1} = \begin{bmatrix} E_u/2 \\ F_u - E_v/2 \end{bmatrix} = (EG-F^2)^{1/2} \begin{bmatrix} E_uG/2 - FF_u + E_vF/2 \\ -E_uF/2 + EF_u - EE_v/2 \end{bmatrix}.$$

2. Suppose γ is both a geodesic and a line of curvature on a oriented surface S.

   (a) Show that $\vec{N} \times \gamma'$ is constant.

   (b) Show that γ lies on a plane.

Solution. For the first part, it suffices to show $(\vec{N} \times \gamma')' = 0$. By the product rule,

$$(\vec{N} \times \gamma')' = (\vec{N}' \times \gamma') + (\vec{N} \times \gamma'').$$

Because γ is a line of curvature, $\vec{N}' = -\mathcal{W}(\gamma')$ is parallel to γ'. Therefore $(\vec{N}' \times \gamma') = 0$. Because γ is a geodesic, γ'' is parallel to $\vec{N}$. Therefore $(\vec{N} \times \gamma'') = 0$. This shows $(\vec{N} \times \gamma')' = 0$.

For the second part, let $\vec{v} = (\vec{N} \times \gamma')$. This is a constant vector. Moreover, $\gamma' \cdot \vec{v} = 0$. Therefore $\gamma \cdot \vec{v} = C$ for some constant C. This shows that C lies in the plane $\{\vec{w} \in \mathbb{R}^3 : \vec{w} \cdot \vec{v} = C\}$.

Alternatively, one can show that the torsion of γ is zero. For example, if $\vec{n}$ is the normal to γ then because γ is a geodesic, $\vec{n} = \pm\vec{N}$. So $\vec{n}' = \pm\vec{N}'$. But because γ is a line of curvature, $\vec{N}'$ is a multiple of γ'. The Frenet-Serret equations state that $\vec{n}' = -\kappa\vec{t} + \theta\vec{b}$. Since $\vec{n}'$ is parallel to $\gamma' = \vec{t}$, this means that $\tau = 0$ and therefore γ is planar.