M365g

Spring 2017

Bowen

Name______________________________

Quiz #4

Instructions. This a 30 minute take home quiz. Take it by yourself, without the aid of the book, the internet or other students. You will need extra paper to write your answers. It is due Wednesday April 26.

Questions

1. Prove a version of the surface-variation theorem for plane curves. That is, suppose $\gamma^\tau : [a,b] \to \mathbb{R}^2$ for $\tau \in (-\delta, \delta)$ is a smoothly varying family of plane curves. You may assume γ^0 has unit-speed. Let $\mathcal{L}(\tau)$ be the length of γ^θ. Prove that

$$\frac{d}{d\tau}\mathcal{L}(\tau) \restriction_{\tau=0} = \int_a^b f \; ds$$

where f is some explicit function of the curvature of $\gamma = \gamma^0$ and the variation $\phi = \frac{d}{d\tau}\gamma \restriction_{\tau=0}$. For simplicity, you may assume that $\phi = \alpha \vec{n}_s$ is a multiple of the signed unit normal $\vec{n}_s$ to γ.

Hints: (1) write down an integral expression for $\mathcal{L}(\tau)$. (2) differentiate with respect to τ inside the integral sign. At some point, an integration-by-parts argument will simplify the expression.

Solution. Let dot denote $\frac{d}{d\tau}$. Because

$$\mathcal{L}(\tau) = \int \|(\gamma^\tau)'\| \; ds$$

$$\frac{d}{d\tau}\mathcal{L}(\tau) = \int \frac{d}{d\tau}\|(\gamma^\tau)'\| \; ds.$$

We can write

$$\frac{d}{d\tau}(\|(\gamma^\tau)'\|) = \frac{d}{d\tau}(\langle(\gamma^\tau)', (\gamma^\tau)'\rangle^{1/2}) = \frac{\langle(\dot{\gamma})', \gamma'\rangle}{\langle(\gamma^\tau)', (\gamma^\tau)'\rangle^{1/2}}.$$

By definition, $\dot{\gamma} = \phi$. So

$$\frac{d}{d\tau}(\|(\gamma^\tau)'\|) \restriction_{\tau=0} = \frac{\langle\phi', \gamma'\rangle}{\|\gamma'\|} = \langle\phi', \gamma'\rangle.$$

The last equality uses our assumption that γ has unit-speed.

Now we use integration by parts. By assumption, $\langle\phi, \gamma'\rangle = 0$. Differentiating and using $\phi = \alpha\vec{n}_s$ we obtain

$$\langle\phi', \gamma'\rangle = -\langle\phi, \gamma''\rangle = -\langle\alpha\vec{n}_s, \gamma''\rangle = \alpha\kappa_s$$

where κ_s is the signed curvature. Therefore,

$$\frac{d}{d\tau}\mathcal{L}(\tau) \restriction_{\tau=0} = -\int_a^b \alpha\kappa_s \, ds.$$

For example, if γ is a positively oriented simple closed curve and α is constant then by Hopf's Umlaufsatz,

$$\frac{d}{d\tau}\mathcal{L}(\tau) \restriction_{\tau=0} = -2\pi\alpha.$$

2. (Extra Credit, 10 points). Give an intuitive explanation for the surface variation theorem based on your result in #1. For reference, the surface variation theorem states:

$$\frac{d\mathcal{A}}{d\tau} \restriction_{\tau=0} = -2\iint_{\mathrm{int}(\pi)} H(EG-F^2)^{1/2}(\phi\cdot\vec{N}) \, dudv.$$

Solution. For simplicity, suppose that no point on the surface is umbilic. This means that the principal curvatures are different. The lines of curvature meet at right angles. Suppose we consider a small "rectangle" whose sides are lines of curvature (this is correct only on the infinitesimal scale). The area of the rectangle is $R = l_1 l_2$ where l_1, l_2 are its side-lengths. For small $\epsilon > 0$, the previous result implies

$$l_1^\epsilon \approx l_1(1-\epsilon(\phi\cdot\vec{N})\kappa_1)$$

$$l_2^\epsilon \approx l_2(1-\epsilon(\phi\cdot\vec{N})\kappa_2).$$

So

$$\begin{aligned} R^\epsilon = l_1^\epsilon l_2^\epsilon &\approx l_1(1-\epsilon(\phi\cdot\vec{N})\kappa_1)l_2(1-\epsilon(\phi\cdot\vec{N})\kappa_2) \\ &\approx l_1 l_2(1-\epsilon(\phi\cdot\vec{N})(\kappa_1+\kappa_2)) \\ &\approx l_1 l_2(1-2\epsilon H(\phi\cdot\vec{N})) = R(1-2\epsilon H(\phi\cdot\vec{N})). \end{aligned}$$

This explains the surface variation theorem.