Name________________________ **M381C Exam 1**

Instructions: Do as many problems as you can. Complete solutions (except for minor flaws) to 4 problems will be considered a good performance.

1. Let $E \subset \mathbb{R}$ be a set with positive finite measure. Show that for any number $0 < c < 1$ there exists an open interval I such that

$$m(I \cap E) \geq cm(I).$$

Hint: recall that for every $\epsilon > 0$ there exists an open set $O \supset E$ with $m(O \setminus E) < \epsilon$.

Solution #1. Let $\epsilon > 0$. There exists an open set $O \supset E$ such that $m(O \setminus E) < \epsilon$.

Because O is open, it can be expressed as a countable union of pairwise disjoint intervals $O = \cup_{i=1}^{\infty} I_i$. Suppose that $\frac{m(I_i \cap E)}{m(I_i)} < c$ for all i. Then

$$m(E) = \sum_{i=1}^{\infty} m(I_i \cap E) = \sum_{i=1}^{\infty} \frac{m(I_i \cap E)}{m(I_i)} m(I_i) < c \sum_{i=1}^{\infty} m(I_i) = cm(O) \leq c(\epsilon + m(E)).$$

So we obtain $\frac{(1-c)}{c} m(E) \leq \epsilon$. However, ϵ is arbitrary. So if we choose it to be less than $\frac{(1-c)}{c} m(E)$ then this can't work. So there must be some interval satisfying the result.

Solution #2. Apply Lebesgue's Differentiation Theorem to χ_E: for a.e. x,

$$\chi_E(x) = \lim_{I \to x} \frac{1}{m(I)} \int_I \chi_E \, dm = \lim_{I \to x} \frac{m(I \cap E)}{m(I)}.$$

Consider the case $x \in E$ to obtain the result.

2. Let $Z \subset \mathbb{R}$ have measure zero. Prove that $Z^2 = \{x^2 : x \in Z\}$ also has measure zero.

Solution. Let $Z_n = Z \cap [-n, n]$. It suffices to show that $m(Z_n^2) = 0$ for each n. Let $\epsilon > 0$ and $O \supset Z_n$ be an open set such that $m(O) < \epsilon/n$ and $O \subset [-2n, 2n]$. Then $O^2 \supset Z^2$. If $I = [a, b]$ in an interval of O then $I^2 = [a^2, b^2]$ and

$$m(I^2) = b^2 - a^2 \leq n(b - a) = 2nm(I).$$

Since this is true for every interval, we have

$$m(Z_n^2) \leq m(O^2) \leq 2nm(O) \leq 2\epsilon.$$

Since this is true for every $\epsilon > 0$, $m(Z_n^2) = 0$.

3. Suppose $\{f_n\}$ is a sequence of measurable functions on $[0, 1]$ such that

$$\lim_{n,m \to +\infty} m(\{x \in [0, 1] : |f_n(x) - f_m(x)| > \epsilon\}) = 0$$

for every $\epsilon > 0$. Prove that there exists a measurable function f such that f_n converges to f in measure as $n \to \infty$.

Hint: Let $\{n_j\}$ be a subsequence such that for every $j < k$

$$m(\{x \in [0, 1] : |f_{n_j}(x) - f_{n_k}(x)| > 1/j\}) < 2^{-j}.$$

Show that $\{f_{n_j}\}$ converges pointwise a.e.

Solution. Let $\{n_j\}$ be a subsequence such that for every $j < k$

$$m(\{x \in [0,1] : \ |f_{n_j}(x) - f_{n_k}(x)| > 1/j\}) < 2^{-j}.$$

I claim that $\{f_{n_j}\}$ converges pointwise a.e. To see this, let

$$E_j = \{x \in [0,1] : \ |f_{n_j}(x) - f_{n_k}(x)| > 1/j\}$$

and

$$E_0 = \{x \in [0,1] : \ \limsup_{j,k\to\infty} |f_{n_j}(x) - f_{n_k}(x)| > 0\}.$$

Observe that

$$E_0 = \limsup_{j\to\infty} E_j = \cap_{n=1}^{\infty} \cup_{j\geq n} E_j.$$

It follows that

$$m(E_0) \leq m(\cup_{j\geq n} E_j) \leq 2^{-n+1}$$

for every n. Thus $m(E_0) = 0$ which implies $\{f_{n_j}\}$ converges pointwise a.e. Let $f_\infty = \lim_j f_{n_j}$ be the limit function. Because $\{f_{n_j}\}$ converges pointwise a.e. to f_∞, this sequence also converges in measure to f_∞. It follows from the triangle inequality that $\{f_n\}$ converges in measure to f_∞.

4. Let $f \geq 0$ be an integrable nonnegative function on $\mathbb{R}$. Define a measure μ on $\mathbb{R}$ by

$$\mu(E) = \int_E f \ dm$$

(for any measurable subset $E \subset \mathbb{R}$). Prove that if g is any bounded measurable function on $[0,1]$ then $\int g \ d\mu = \int fg \ dm$.

Solution. Observe that the equation is true if g is the characteristic function of a measurable set. By linearity, the equation must be true if g is simple. So

$$\int g \ d\mu = \sup_{\phi\leq g} \int \phi \ d\mu = \sup_{\phi\leq g} \int f\phi \ dm = \int fg \ dm$$

where the supremums are over all simple functions $\phi \leq g$. The last equality holds by Lebesgue's Dominated Convergence Theorem, or by the Bounded Convergence Theorem since we may assume, without loss of generality that the support of ϕ is contained in the support of g.

5. Let f be a continuous function on $[0,1]$. Find

$$\lim_{n\to\infty} n \int_0^1 x^n f(x) \ dx.$$

Justify your answer. Hint: the answer is not zero.

Solution. Because f is continuous $f(x)$ is close to $f(1)$ whenever x is close to 1. More precisely, for any $\epsilon > 0$ there is a $\delta > 0$ such that if $1 - \delta \leq x \leq 1$ then

$$|f(x) - f(1)| < \epsilon.$$

Observe that nx^n is monotone decreasing as $n \to \infty$ for every $x \in [0, 1-\delta]$ (if we start with n large enough so that $1 - \delta < n/(n+1)$). Moreover $nx^n \to 0$ for any $0 \le x < 1$ as $n \to \infty$. So Lebesgue's Dominated Convergence Thereom implies

$$\begin{aligned}
\lim_{n\to\infty} n \int_0^1 x^n f(x)\, dx &= \lim_{n\to\infty} n \int_0^{1-\delta} x^n f(x)\, dx + \lim_{n\to\infty} n \int_{1-\delta}^1 x^n f(x)\, dx \\
&= \lim_{n\to\infty} n \int_{1-\delta}^1 x^n f(x)\, dx.
\end{aligned}$$

By direct computation,

$$\lim_{n\to\infty} n \int_{1-\delta}^1 x^n\, dx = 1.$$

So,

$$\begin{aligned}
\limsup_{n\to\infty} |n \int_{1-\delta}^1 x^n f(x)\, dx - f(1)| &= \limsup_{n\to\infty} |n \int_{1-\delta}^1 x^n f(x)\, dx - n \int_{1-\delta}^1 x^n f(1)\, dx| \\
&\le \limsup_{n\to\infty} n \int_{1-\delta}^1 x^n |f(x) - f(1)|\, dx \\
&\le \limsup_{n\to\infty} n \int_{1-\delta}^1 x^n \epsilon\, dx = \epsilon.
\end{aligned}$$

Since this is true for every $\epsilon > 0$ we have

$$\lim_{n\to\infty} n \int_{1-\delta}^1 x^n f(x)\, dx = f(1).$$

6. Suppose $\{f_n\}$ are integrable functions on $[0,1]$ and that $\lim_{n\to\infty} f_n(x) = f_\infty(x)$ for a.e. x. Suppose also that $\|f_n\|_1 \to \|f_\infty\|_1 < \infty$ as $n \to \infty$. Show that $\int_0^1 |f_n(x) - f_\infty(x)|\, dx \to 0$ as $n \to \infty$.

   Hint: Because f_∞ is integrable, for every $\epsilon > 0$ there exists $\delta > 0$ such that if $E \subset [0,1]$ has $m(E) < \delta$ then $\int_E |f_\infty|\, dm < \epsilon$.

   Solution. Let $\epsilon > 0$. By absolute continuity of the integral, there exists a $\delta > 0$ such that if $E \subset [0,1]$ has $m(E) < \delta$ then $\int_E |f_\infty|\, dm < \epsilon$. By Egorov's Theorem, there exists a set $E \subset [0,1]$ with $m(E) < \delta$ such that f_n converges uniformly to f_∞ on $[0,1] \setminus E$. Therefore,

$$\limsup_n \int_0^1 |f_n - f_\infty|\, dm = \limsup_n \int_E |f_n - f_\infty|\, dm.$$

   Now $\int_E |f_\infty|\, dm < \epsilon$. Moreover,

$$\int_E |f_\infty| - |f_n|\, dm = \|f_\infty\|_1 - \|f_n\|_1 - \left(\int_{E^c} |f_\infty| - |f_n|\, dm \right) \to 0$$

   as $n \to \infty$. So $\limsup_n \int_E |f_n|\, dm < \epsilon$. Therefore,

$$\limsup_n \int_E |f_n - f_\infty|\, dm \le \limsup_n \int_E |f_n|\, dm + \int |f_\infty|\, dm < 2\epsilon.$$

So we have shown

$$\limsup_n \int_0^1 |f_n - f_\infty| \, dm \leq 2\epsilon.$$

Since ϵ is arbitrary, this proves it.