Name_________________________ **M381C Exam 2**

Instructions: Do as many problems as you can. Complete solutions (except for minor flaws) to 2-3 problems will be considered a good performance.

1. Suppose ϕ is a real-valued function on $\mathbb{R}$ such that

$$\phi\left(\int_0^1 f(x)\ dx\right) \leq \int_0^1 \phi(f(x))\ dx$$

for every real-valued bounded measurable f. Show that ϕ is convex.

Solution. Let $a < b \in \mathbb{R}$ and $t \in (0,1)$. Define $f(x) = a$ if $0 \leq x \leq t$ and $f(x) = b$ otherwise. Then $\int_0^1 f\ dx = ta + (1-t)b$. So the hypotheses imply

$$\phi(ta + (1-t)b) \leq t\phi(a) + (1-t)\phi(b)$$

which is equivalent to convexity.

2. Suppose (X, μ) is a measure space with $\mu(X) < \infty$. Show that for any bounded measurable function f on X,

$$\lim_{p\to\infty} \|f\|_p = \|f\|_\infty.$$

Solution. This is Theorem 8.1 in the book. For every number $0 \leq t < \|f\|_\infty$, there is a set $E \subset X$ such that $f \geq t\chi_E$. By direct computation, we see that $\|t\chi_E\|_p \to t$ as $p \to \infty$. By monotonicity, this implies $\liminf_{p\to\infty} \|f\|_p \geq t$. Therefore $\liminf_{p\to\infty} \|f\|_p \geq \|f\|_\infty$. On the other hand, since $f \leq \|f\|_\infty$, we have $\|f\|_p \leq \mu(E)^{1/p}\|f\|_\infty$ for every p which implies the oppositive inequality.

3. Let F be an increasing real-valued function on a bounded interval $I = (a, b)$. Suppose that for almost every $x \in I$, the limit

$$f(x) = \lim_{k\to\infty} \frac{F(x + 1/k) - F(x)}{1/k}$$

exists and is finite. Explain why f is measurable on I, and why

$$\int_a^b f\ dx \leq F(b^-) - F(a^+).$$

Solution. F is measurable because it is monotone (the inverse image of an interval of the form (r, ∞) is an interval of the form (s, ∞)). f is measurable because it is a limit of measurable functions. The last part was in a lecture and is also in the book.

4. Suppose that $\mathcal{H}$ is a Hilbert space,

- $\{v_n\}_{n=1}^\infty \subset \mathcal{H}$, $v_\infty \in \mathcal{H}$,
- $\lim_{n\to\infty}\langle v_n, w\rangle = \langle v_\infty, w\rangle$ for every $w \in \mathcal{H}$
- $\|v_n\| \to \|v_\infty\|$ as $n \to \infty$.

Show that $\lim_{n\to\infty} v_n = v_\infty$ (i.e., $\lim_{n\to\infty} \|v_n - v_\infty\| = 0$).

Solution.

$$\|v_n - v_\infty\|^2 = \|v_n\|^2 - 2\,\mathrm{Re}\langle v_n, v_\infty\rangle + \|v_\infty\|^2.$$

By hypothesis,

$$\langle v_n, v_\infty\rangle \to \langle v_\infty, v_\infty\rangle = \|v_\infty\|^2.$$

Therefore,

$$\lim_{n\to\infty} \|v_n - v_\infty\|^2 = \lim_{n\to\infty} \|v_n\|^2 - 2\|v_\infty\|^2 + \|v_\infty\|^2 = 0.$$

5. Suppose f is absolutely continuous on $[0, 1]$. Show that f has bounded variation.

   Solution. This is in the book.