m381c: Homework #10

October 29, 2014

1. Suppose $\mathcal{A}$ is an algebra and $\mu_0 : \mathcal{A} \to [0, \infty)$ is finitely additive and satisfies the following continuity property: if $A_1 \supset A_2 \supset \cdots$ are sets in $\mathcal{A}$ with $\cap_n A_n = \emptyset$ then $\lim_{n\to\infty} \mu_0(A_n) = 0$. Prove that μ_0 is countably additive.

2. Consider the Hilbert cube $[0,1]^{\mathbb{N}}$. A point $x \in [0,1]^{\mathbb{N}}$ is a sequence $x = (x_1, x_2, \ldots)$ with each $x_i \in [0,1]$. The Hilbert cube admits the metric
$$d(x,y) := \sum_{i=1}^{\infty} 2^{-i}|x_i - y_i|.$$
This is a compact space, but you don't have to show this. For any integer $n \geq 1$, let $\pi_n : [0,1]^{\mathbb{N}} \to [0,1]^n$ be the projection map $\pi_n(x) = (x_1, \ldots, x_n)$. Let $\mathcal{B}_n$ be the collection of all subsets of $[0,1]^{\mathbb{N}}$ of the form $\pi_n^{-1}(B)$ where $B \subset [0,1]^n$ is measurable. Let $\mathcal{A} = \cup_n \mathcal{B}_n$. Prove that $\mathcal{A}$ is an algebra. (This means that $\emptyset \in \mathcal{A}$ and $\mathcal{A}$ is closed under complementation, finite unions and finite intersections).

3. Prove that $\mathcal{A}$ is not a sigma-algebra.

4. Let m_n denote the Lebesgue measure on $[0,1]^n$. Define μ_0 on $\mathcal{A}$ by $\mu_0(\pi_n^{-1}(B)) = m_n(B)$ for any measurable subset $B \subset [0,1]^n$. Prove that μ_0 is countably additive. Thus Caratheodory's Extension Theorem implies that μ_0 extends to a measure, denoted by μ, on the sigma-algebra generated by $\mathcal{A}$. Hint: use exercise #1. Because $[0,1]^{\mathbb{N}}$ is compact, if $F_1 \supset F_2 \supset \cdots$ are decreasing closed sets with $\cap_n F_n = \emptyset$ then there must exists a k such that $F_k = \emptyset$. So in this case $\lim_{n\to\infty} \mu_0(F_n) = 0$.

5. Prove that every open subset of $[0,1]^{\mathbb{N}}$ is measurable. This implies that μ is a Borel measure (i.e. all Borel sets are measurable). Hint: it is enough to show that open balls are measurable because every open set is a countable union of open balls.

6. Let $B = \{x \in [0,1]^{\mathbb{N}} : \ \limsup_{i\to\infty} x_i = 1\}$. Show B is measurable and compute $\mu(B)$.