# m381c: Homework #13

November 20, 2014

1. Let $X$ be a compact metric space. Let $\{\mu_n\}_{n=1}^{\infty}$ and $\mu_\infty$ be regular Borel probability measures on $X$. Show that the following are equivalent:

   (a) for every $f \in C_c(X)$, $\lim_{n\to\infty} \mu_n(f) = \mu_\infty(f)$;

   (b) for every closed set $F \subset X$, $\limsup_{n\to\infty} \mu_n(F) \le \mu_\infty(F)$;

   (c) for every open set $O \subset X$, $\liminf_{n\to\infty} \mu_n(O) \ge \mu_\infty(O)$.

   This is called the **Portmanteau Theorem**.

2. Let $\mu$ be a regular Borel probability measure on a locally compact separable metric space $X$. Show that there exists a unique smallest closed set $F \subset X$ such that $\mu(X \setminus F) = 0$. 'Smallest' means that if $F' \subset F$ is any closed subset and $\mu(X \setminus F') = 0$ then $F = F'$. This set is called the **support** of $\mu$.

3. Construct a Borel probability measure on $\mathbb{R}$ whose support is the middle thirds Cantor set.

4. Construct a Borel probablity measure $\mu$ on $\mathbb{R}$ such that

   - $\mu$ is singular to Lebesgue measure and
   - the support of $\mu$ is $\mathbb{R}$.