# m381c: Homework #14

November 26, 2014

1. Show that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \pi^2/6$. Hint: consider the function $f(x) = \frac{\pi - x}{2}$ on $[0, 2\pi)$. Compute the Fourier coefficients of this function and use $\frac{1}{2\pi}\int_0^{2\pi} f(x)^2\, dx = \sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2$.

2. We'll show in class that the span of $\{e_n\}_{n\in\mathbb{Z}}$ is dense in $C(\mathbb{T})$ (where $e_n(x) = e^{inx}$ and we are identifying $\mathbb{T}$ with $[0, 2\pi)$). Use this (or any other method) to show that polynomials are dense in $C([0,1])$.

3. Suppose $f \in L^1(\mathbb{T})$. Prove that $f \in C^\infty(\mathbb{T})$ if and only if the Fourier coefficients $\hat{f}(n)$ "decrease rapidly" as $|n| \to \infty$. That is for any $\alpha > 0$ there exists a constant $C_\alpha$ such that
$$|\hat{f}(n)| \leq C_\alpha |n|^{-\alpha}$$
for all $n \in \mathbb{Z}$. Hint: if $f \in C^\infty$ use the relation $\widehat{f'}(n) = (in)\hat{f}(n)$ (we will prove this in class; you don't have to show it). If $\hat{f}$ decreases rapidly use the Inversion Theorem (see below):

   **Theorem 0.1** (Inversion Theorem)**.** *Let $f \in L^1(\mathbb{T})$ and assume that the Fourier series of $f$ converges absolutely: $\sum_{n\in\mathbb{Z}} |\hat{f}(n)| < \infty$. Then there exists a continuous function $g \in C(\mathbb{T})$ such that $f = g$ a.e. Moreover, the Fourier series $\sum_{n\in\mathbb{Z}} \hat{f}(n) e_n$ converges uniformly to $g$.*

   You don't have to prove this since we'll prove it in class.