Gibbs measures and random graphs, HW2

Chuwei Zhang

Exercise 2.54

Let G be the graph whose vertices are $\{0,1,2,3,\dots\}$ and whose edges are the pairs of vertices that differ by 1. Let c be some conductance function on the edges. Define θ on the edge set as follows. Define $\theta(n,n+1)=1$ and $\theta(n+1,n)=-1$ for all $n=0,1,2,\dots$. Clearly, θ is a unit flow from 0 to ∞. I claim that this is the only unit flow from 0 to ∞ on G. Indeed, any unit flow θ from 0 to ∞ must satisfy that $\theta(0,1)=1$ and thus $\theta(1,0)=-1$. Since $d^*\theta(1)=0$, $\theta(1,2)$ has to be 1. Thus, we see inductively that $\theta(n,n+1)=1$ and $\theta(n+1,n)=-1$ for all n.

The energy functional of this unique unit flow from 0 to ∞ is

$$\mathscr{E}(\theta)=\sum_{n=0}^{\infty}\frac{\theta(n,n+1)^2}{c(n,n+1)}=\sum_{n=0}^{\infty}\frac{1}{c(n,n+1)} \tag{1}$$

Define

$$c(n,n+1)=c(n,n-1)=\begin{cases}1 & \text{if } n\equiv 0 \mod 3\\ 3 & \text{if } n\equiv 1 \mod 3\\ 9 & \text{if } n\equiv 2 \mod 3\end{cases}$$

We can check that if $p(0,1)=1$,

$$p(n,n+1)=\begin{cases}\frac{1}{10} & \text{if } n\equiv 0 \mod 3\\ \frac{3}{4} & \text{otherwise}\end{cases}$$

and

$$p(n,n-1)=\begin{cases}\frac{9}{10} & \text{if } n\equiv 0 \mod 3\\ \frac{1}{4} & \text{otherwise}\end{cases}$$

Therefore, the network (G,c) defined as above corresponds to the first random walk. We know that the random walk is recurrent if for any unit flow from 0 to ∞, its energy is infinite. Since θ defined above is the only unit flow from 0 to ∞, it suffices to show that $\mathscr{E}(\theta)=\infty$. Indeed, by (1),

$$\mathscr{E}(\theta)=\sum_{n=0}^{\infty}\frac{1}{c(n,n+1)}=1+\frac{1}{3}+\frac{1}{9}+1+\frac{1}{3}+\frac{1}{9}+1+\frac{1}{3}+\frac{1}{9}+\cdots=\infty.$$

So the first random walk is recurrent.

Now let us consider the second random walk described in the exercise. If $n\geq 1$, and n is not a multiple of 3, then

$$p(n,n+1)=\mathbb{P}(\text{head})\Big(\frac{3}{4}\Big)+\mathbb{P}(\text{tail})\Big(\frac{1}{2}\Big)=\frac{1}{2}\cdot\frac{3}{4}+\frac{1}{2}\cdot\frac{1}{2}=\frac{5}{8}$$

and

$$p(n,n-1)=\mathbb{P}(\text{head})\Big(\frac{1}{4}\Big)+\mathbb{P}(\text{tail})\Big(\frac{1}{2}\Big)=\frac{1}{2}\cdot\frac{1}{4}+\frac{1}{2}\cdot\frac{1}{2}=\frac{3}{8}.$$

Likewise, if $n\geq 1$, and n is a multiple of 3, then $p(n,n+1)=3/10$ and $p(n,n-1)=7/10$. So now, for $n=0,1,2,\dots$, we define

$$c'(3n,3n+1)=\Big(\frac{25}{21}\Big)^n$$

and

$$c'(3n+1,3n+2)=\frac{5}{3}\Big(\frac{25}{21}\Big)^n$$

and

$$c'(3n+2,3n+3)=\frac{25}{9}\Big(\frac{25}{21}\Big)^n.$$

Now the network (G, c') corresponds to the second random walk described in the exercise. In this network, the energy of θ is, according to (1),

$$\sum_{n=0}^{\infty} \frac{1}{c'(3n, 3n+1)} + \sum_{n=0}^{\infty} \frac{1}{c'(3n+1, 3n+2)} + \sum_{n=0}^{\infty} \frac{1}{c'(3n+2, 3n+3)} = \sum_{n=0}^{\infty} \left(\frac{21}{25}\right)^n + \sum_{n=0}^{\infty} \frac{5}{3}\left(\frac{21}{25}\right)^n + \sum_{n=0}^{\infty} \frac{25}{9}\left(\frac{21}{25}\right)^n.$$

Note that the right-hand side is the sum of three convergent geometric series. Thus the energy of θ is finite and the random walk is transient.

I think finally I need to show that for the second random walk, X_n/n converges to some positive constant almost surely. I am unable to do this.

Exercise 2.71

(a)

Consider a random walk Y_n in G. $V(G)$ is a single communicating class, so all the vertices in $V(G)$ are recurrent. Let $x \in V(G)$.

$$\begin{aligned} 0 &= \mathbb{P}_a(Y_n \text{ never returns to } a) \\ &\geq \mathbb{P}_a(Y_n = x \text{ for some } n \text{ and } Y_n \text{ never returns to } a \text{ after its first visit to } x) \\ &= \mathbb{P}_a(Y_n = x \text{ for some } n)\mathbb{P}_x(Y_n \text{ never visits } a) \end{aligned}$$

by the strong Markov property. So

$$0 = \mathbb{P}_a(Y_n = x \text{ for some } n)\mathbb{P}_x(Y_n \text{ never visits } a).$$

Since $V(G)$ is a single communicating class, therefore

$$\mathbb{P}_a(Y_n = x \text{ for some } n) > 0.$$

So $\mathbb{P}_x(Y_n \text{ never visits } a)$ has to be 0. (Likewise, $\mathbb{P}_x(Y_n \text{ never visits } z) = 0$.)

Because $G_m \subset G_{m+1}$ for all m, therefore, for random walk starting at at x, if Y_n visits $V(G) \setminus V(G_{m+1})$ before hitting a or z, then it has to visit $V(G) \setminus V(G_m)$ before hitting a or z. Therefore,

$$\mathbb{P}_x(Y_n \text{ visits } V(G) \setminus V(G_m) \text{ before hitting } a \text{ or } z) \downarrow \mathbb{P}_x\Big(\bigcap_{k\geq 1}\{Y_n \text{ visits } V(G) \setminus V(G_k) \text{ before hitting } a \text{ or } z\}\Big) \quad (2)$$

as $m \to \infty$. If Y_n starting at x ever visits a or z, then it has only visited finitely many vertices before hitting $\{a, z\}$. Since $(G_m)_{m\geq 1}$ exhaust G, there must be an m large enough such that G_m contains all the states visited by Y_n until it hits $\{a, z\}$, i.e. Y_n does not exit G_m before hitting $\{a, z\}$. Therefore,

$$\begin{aligned} &\bigcap_{k\geq 1}\{Y_n \text{ starting at } x \text{ visits } V(G) \setminus V(G_k) \text{ before hitting } a \text{ or } z\} \\ \subset &\big\{Y_n \text{ starting at } x \text{ never visits } \{a, z\}\big\} \\ \subset &\{Y_n \text{ starting at } x \text{ never visits } a\} \end{aligned}$$

Since we already found out that $\mathbb{P}_x(Y_n \text{ never visits } a) = 0$, therefore,

$$\mathbb{P}_x\Big(\bigcap_{k\geq 1}\{Y_n \text{ visits } V(G) \setminus V(G_k) \text{ before hitting } a \text{ or } z\}\Big) = 0.$$

Therefore, by (2), $\mathbb{P}_x(Y_n \text{ exists } G_m \text{ before hitting } a \text{ or } z) \downarrow 0$ as $m \to \infty$.

Now fix $x \in V(G)$. Let $\epsilon > 0$. Because we already established that

$$\mathbb{P}_x(Y_n \text{ visits } V(G) \setminus V(G_m) \text{ before hitting } a \text{ or } z) \downarrow 0$$

as $m \to \infty$, there exists $M \in \mathbb{N}$ such that for all $m \geq M$,

$$\mathbb{P}_x(Y_n \text{ visits } V(G) \setminus V(G_m) \text{ before hitting } a \text{ or } z) < \epsilon. \quad (3)$$

Replacing M by a larger natural number if necessary, we may assume that $x \in V(G_M)$. Let $N \geq M$ be such that G_N contains all the edges in G that are incident with the vertices in G_M. Let $m \geq N$ and consider the a random walk X_k in G_m starting at x. We consider the probability that X_k hits a before it hits z.

$$\begin{aligned}&\mathbb{P}(X_k \text{ hits a before it hits } z)\\ =&\underbrace{\mathbb{P}_x(\tau_a < \tau_z \text{ and } X_k \text{ remains in } G_M \text{ until it hits } \{a,z\})}_{\clubsuit}\\ &+\underbrace{\mathbb{P}_x(\tau_a < \tau_z \text{ and } X_k \text{ visits } V(G_m)\setminus V(G_M) \text{ before it hits } \{a,z\})}_{\spadesuit}\end{aligned}$$

Because $m \geq N$ and N is chosen such that G_N contains all the edges in G that are incident with the vertices in G_M, the transition probabilities at each vertex $y \in G_M$ are the same for the random walks X_k and Y_k. Thus,

$$\begin{aligned}\clubsuit &= \mathbb{P}_x(Y_k \text{ hits } a \text{ before } z \text{ and remains in } G_M \text{ before hitting } \{a,z\})\\ &= \mathbb{P}_x(Y_k \text{ hits } a \text{ before } z) - \underbrace{\mathbb{P}_x(Y_k \text{ hits } a \text{ before } z \text{ and visits } V(G)\setminus V(G_M) \text{ before hitting } \{a,z\})}_{\heartsuit}\end{aligned}$$

Therefore,

$$\mathbb{P}_x(X_k \text{ hits } a \text{ before it hits } z) = \mathbb{P}_x(Y_k \text{ hits } a \text{ before } z) - \heartsuit + \spadesuit.$$

So by the triangle inequality,

$$\left|\mathbb{P}_x(Y_k \text{ hits } a \text{ before } z) - \mathbb{P}_x(X_k \text{ hits } a \text{ before it hits } z)\right| \leq |\heartsuit| + |\spadesuit| = \heartsuit + \spadesuit \tag{4}$$

as $\heartsuit, \spadesuit \geq 0$.

$$\begin{aligned}\spadesuit &\leq \mathbb{P}_x\big(X_k \text{ visits } V(G_m)\setminus V(G_M) \text{ before it hits } \{a,z\}\big)\\ &= \mathbb{P}_x\big(Y_k \text{ visits } V(G)\setminus V(G_M) \text{ before it hits } \{a,z\}\big)\end{aligned}$$

since the transition probabilities at each vertex $y \in G_M$ are the same for the random walks X_k and Y_k. Therefore, by (3),

$$\spadesuit < \epsilon.$$

Likewise,

$$\heartsuit \leq \mathbb{P}_x(Y_k \text{ visits } V(G)\setminus V(G_M) \text{ before hitting } \{a,z\}) < \epsilon.$$

Because $\spadesuit, \heartsuit < \epsilon$, by (4),

$$\left|\mathbb{P}_x(Y_k \text{ hits } a \text{ before } z) - \mathbb{P}_x(X_k \text{ hits } a \text{ before it hits } z)\right| < 2\epsilon.$$

Note that by the uniqueness of harmonic functions,

$$\mathbb{P}_x(X_k \text{ hits } a \text{ before it hits } z) = v_m(x).$$

Therefore,

$$\left|\mathbb{P}_x(Y_k \text{ hits } a \text{ before } z) - v_m(x)\right| \leq 2\epsilon.$$

This is true for any $m \geq N$. Since ϵ and x are arbitrary , therefore we conclude that for any $x \in V(G)$,

$$\lim_{n\to\infty} v_n(x) = \mathbb{P}_x(Y_k \text{ hits } a \text{ before } z).$$

If we define $v(x)$ to be $\lim_{n\to\infty} v_n(x)$, the

$$v(x) = \mathbb{P}_x(Y_k \text{ hits } a \text{ before } z).$$

(b)

Let $m \le n$. Observe that i_m can be extended to a unit flow from a to z in G_n by defining $i_m(e) = 0$ for any edge e in $E(G_n) \setminus E(G_m)$. (And note that the extension is indeed a unit flow from a to z in G_n.) Therefore, by the Thompson's energy principle,

$$\mathscr{E}(i_m; G_n) \ge \mathscr{E}(i_n; G_n).$$

Note that $\mathscr{E}(i_m; G_n) = \mathscr{E}(i_m; G_m)$. Therefore

$$\mathscr{E}(i_m; G_m) \ge \mathscr{E}(i_n; G_n).$$

Therefore, $\big(\mathscr{E}(i_n; G_n)\big)_{n\in\mathbb{N}}$ is a decreasing sequence of non-negative numbers. It therefore converges.

Fix $\epsilon > 0$. There exists $N \ge 0$ such that for $n > m \ge N$,

$$\lim_{k\to\infty} \mathscr{E}(i_k; G_k) + \epsilon > \mathscr{E}(i_m; G_m) \ge \mathscr{E}(i_n; G_n) \ge \lim_{k\to\infty} \mathscr{E}(i_k; G_k).$$

So

$$\big|\mathscr{E}(i_m; G_m) - \mathscr{E}(i_n; G_n)\big| < \epsilon.$$

So

$$\big|\mathscr{E}(i_m; G_n) - \mathscr{E}(i_n; G_n)\big| < \epsilon. \tag{5}$$

i_n is in the ★ space on G_n since it is the gradient of the associated voltage function. In G_n, $i_m - i_n$ has zero divergence, since i_n and (the extension of) i_m are both unit flows from a to z. Therefore, $i_m - i_n$ is in the orthogonal complement in $\ell^2_-\big(E(G_n)\big)$ to the ★ space on G_n. Therefore, by the Pythagorean theorem (on inner product spaces),

$$\mathscr{E}(i_m - i_n; G_n) + \mathscr{E}(i_n; G_n) = \mathscr{E}(i_m; G_n)$$

So

$$\mathscr{E}(i_m - i_n; G_n) = \mathscr{E}(i_m; G_n) - \mathscr{E}(i_n; G_n) < \epsilon \tag{6}$$

because of (5). Also note that i_m and i_n both can be extended to unit flows from a to z in G by defining $i_m(e) = 0$ for $e \in E(G) \setminus E(G_m)$ and likewise for i_n. Note that

$$\mathscr{E}(i_m - i_n; G_n) = \mathscr{E}(i_m - i_n; G).$$

Therefore, by (6),

$$\mathscr{E}(i_m - i_n; G) < \epsilon.$$

This is true as long as $n > m \ge N$. Since ϵ is arbitrary, therefore, (the extension to G of) i_n form a Cauchy sequence in $\ell^2_-\big(E(G)\big)$, which is complete. So i_n converges in $\ell^2_-\big(E(G)\big)$. Write this ℓ^2 limit as i. For any edge e_0 in $E(G)$,

$$r(e_0)\big(i_n(e_0) - i(e_0)\big)^2 \le \mathscr{E}(i_n - i; G) \to 0$$

as $n \to \infty$. Since $r(e_0) > 0$, therefore,

$$\big|i_n(e_0) - i(e_0)\big| \to 0$$

as $n \to \infty$. So i is also the pointwise limit for i_n.

Let θ be a finitely supported unit flow in G from a to z. Then there exists $M \ge 0$ such that for all $m \ge M$, θ is supported on G_m and $\theta\restriction_{G_m}$ is a unit flow from a to z in G_m. Since i_m and $\theta\restriction_{G_m}$ are both unit flows from a to z in G_m, their difference $\theta\restriction_{G_m} - i_m$ has zero divergence, and therefore lies in the orthogonal complement to the ★ space in $\ell^2_-\big(V(G_m)\big)$. Since i_m is in the ★ space on G_m, therefore the inner product of $\theta\restriction_{G_m} - i_m$ and i_m is zero. Note that the inner product in $\ell^2_-\big(V(G_m)\big)$ of $\theta\restriction_{G_m} - i_m$ and i_m equals the inner product in $\ell^2_-\big(V(G)\big)$ of $\theta - i_m$ and i_m. So

$$(i_m, \theta - i_m)_r = 0 \tag{7}$$

for all $m \ge M$. Here $(\cdot,\cdot)_r$ denotes the inner product on $\ell^2_-\big(V(G)\big)$.

We have deduced that $\theta\restriction_{G_m} - i_m$ lies in the orthogonal complement to the ★ space in $\ell^2_-\big(V(G_m)\big)$. Since G_m is finite, $\theta\restriction_{G_m} - i_m$ lie in the ♢ space on G_m. Since the ♢ space on G_m is spanned by cycles in G_m, which can be viewed by extension as cycles in G, therefore the ♢ space on G_m is a subspace of the ♢ space on G. Therefore, $\theta - i_m$ lies in the ♢ space on G.

Now that $\theta - i_m$ lies in the $\diamondsuit$ space on G, and $i_m \to i$ in $\ell^2_-\left(E(G)\right)$, and $\diamondsuit$ is a closed subspace of $\ell^2_-\left(E(G)\right)$, therefore $\theta - i$ is in $\diamondsuit$. Since the inner product on an inner product space is continuous, therefore,

$$(i, \theta - i)_r = \lim_{m\to\infty} (i_m, \theta - i_m)_r = 0$$

because of (7) .

Because $\theta - i$ lies in the $\diamondsuit$ space on G, and because $i \in \bigstar \subset \diamondsuit^\perp$ and because $i + (\theta - i) = \theta$, therefore

$$i = P_{\diamondsuit^\perp}(\theta).$$

(c)

In G_n, apply volatage u by imposing $u(a) = \mathscr{R}(a \leftrightarrow z)$ and $u(z) = 0$. Then ∇u is the resulting current flow from a to z and

$$u(a) - u(z) = d^*(\nabla u)(a)\mathscr{R}(a \leftrightarrow z; G_n).$$

Since $u(a) = \mathscr{R}(a \leftrightarrow z) > 0$ and $u(0) = 0$, therefore

$$d^*(\nabla u)(a) = 1.$$

So ∇u is a unit current flow from a to z. By Thompson's energy principle, the unit current flow from a to z in G_n is unique. So

$$\nabla u = i_n$$

i.e.

$$du = i_n r. \tag{8}$$

Since u and $\mathscr{R}(a \leftrightarrow z)v_n$ are both harmonic on $V(G_n) \setminus \{a, z\}$ and since they agree on $\{a, z\}$, by the uniqueness of harmonic functions,

$$\mathscr{R}(a \leftrightarrow z; G_n)v_n = u. \tag{9}$$

since i_n is a unit current flow from a to z, therefore

$$\mathscr{E}(i_n; G_n) = \mathscr{R}(a \leftrightarrow z; G_n).$$

So by (9),

$$\mathscr{E}(i_n; G_n)v_n = u.$$

So

$$\mathscr{E}(i_n; G_n)dv_n = du = i_n r \tag{10}$$

by (8). We already established in (a) and (b) that

$$\lim_{n\to\infty} \mathscr{E}(i_n; G_n) = \mathscr{E}(i; G)$$

and that

$$\lim_{n\to\infty} v_n = v$$

and that

$$\lim_{n\to\infty} i_n = i$$

pointwise. Therefore by (10),

$$\mathscr{E}(i; G)dv = ir.$$

(d)

In (c) we already established

$$\mathscr{E}(i_n; G_n) = \mathscr{R}(a \leftrightarrow z; G_n).$$

In (b) we established that $\left(\mathscr{E}(i_n; G_n)\right)_{n\in\mathbb{N}}$ is a decreasing sequence of non-negative numbers that converge to $\mathscr{E}(i; G)$. Therefore,

$$\mathscr{R}(a \leftrightarrow z; G_n) \searrow \mathscr{E}(i; G)$$

as $n \to \infty$.

(e)

Fix $x \in V(G)$. Let K be a natural number sufficiently large such that for all $k \geq K$, x and all the edges in G that are incident with x are contained in G_k. Note that $\pi(x; G_k) = \pi(x; G)$ for all $k \geq K$. We showed in (c) that the voltage $\mathscr{E}(i_k; G_k)v_k$ gives rise to unit curret flow from a to z in G_k. Therefore, for $k \geq K$,

$$\mathscr{G}_z(a, x; G_k) = \pi(x; G_k)\mathscr{E}(i_k; G_k)v_k(x). \tag{11}$$

Because $\pi(x; G_k) = \pi(x; G)$ for all $k \geq K$ and we have established in (a) and (b) that $\lim_{k\to\infty} \mathscr{E}(i_k; G_k) = \mathscr{E}(i; G)$ and that $\lim_{k\to\infty} v_k(x) = v(x)$ (and that these limits are both finite), therefore, it follows from (11) that

$$\mathscr{G}_z(a, x; G_k) \to \pi(x; G)\mathscr{E}(i; G)v(x) = \mathscr{E}(i)\pi(x)v(x)$$

as $k \to \infty$.

But I am unable to show that $\lim_{k\to\infty} \mathscr{G}_z(a, x; G_k)$ equals $\mathscr{G}_z(a, x; G)$.

(f)

Exercise 2.104

Since rough isometry is an equivalence relation, it suffices to show that the regular tree of degree 3 is roughly isometric to the regular tree of degree d for any $d > 3$. Fix $d > 3$.

Denote the 3-regular tree by T^3. We color the edges red/blue.

1. Begin with a path in T^3 of length $d - 3$. Color the edges on this path red. For each vertex on this red path, color all its incident edges that are currently uncolored blue.
2. The edges that have been colored so far form finite tree. For each leaf in this colored tree, find a path of length $d - 3$ starting at that vertex consisting of edges that are currently uncolored. Color the edges on these paths red. For each vertex on the newly added red paths, color all its incident edges that are currently uncolored blue.
3. Repeat step 2 iteratively.

The above defines an edge coloring of all of T^3. Now define a new graph $K = (V(K), E(K))$ where

$$V(K) = \{\text{red paths of length } d - 3 \text{ in } T^3\}$$

and

$$E(K) = \{\text{blue edges in } T^3\}$$

and a vertex in K, i.e. a red path in T^3 is incident with an edge in K, i.e. a blue edge in T^3 if and only if the red path and the blue edge share a vertex in T^3.

We can caculate how many blue edges are incident with a red path: a red path of length $d - 3$ contains $d - 2$ vertices. Each vertex is incident with one blue edge, except the two vertices at the ends of the red path are each incident with 2 blue edges. Thus a red path is incident with $d - 2 + 2 = d$ blue edges. Therefore, K is d-regular.

For any two red paths P_1 and P_2 in T^3, since T^3 is a tree, there exists a unique path P whose two ends coincide with P_1 and P_2. Therefore, there exists a unique path between any two verties in K. So K is a tree.

Therefore, K is the d-regular tree.

Define

$$\phi : V(T^3) \to V(K)$$

such that $\phi(v)$ is the unique red path lenght $d - 3$ that contains v.

Now we prove that ϕ is a rough isometry from T^3 to K.

First, note that ϕ effectively contracts all the red paths, and therefore,

$$\text{distance}\big(\phi(u), \phi(v)\big) \leq \text{distance}(u, v)$$

for all vertices u and v in T^3.

On the other hand, given any two red paths P_1 and P_2 of length $d - 3$ in T^3, pick vertices v_1 and v_2 in P_1 and P_2 respectively such that v_1 and v_2 are as far apart as possible. Let P be the unique path in T^3 from v_1 and v_2. Since v_1 and v_2 are chosen to be as far apart as possible, P_1 and P_2 must be the ends of P. P is a concatenation of alternating red and blue paths. Each red path is at most of lenth $d-3$. If we add an artifical blue edge to one end of P and obtain a path P', then in P' is a concatenation of alternating red and blue paths that begin with a red path and ends with a blue path. Thus we can pair up the maximal blue and red paths in P'. Within each pair, the red path is of length no greater than $d-3$ and the blue path is of length no smaller than 1. So the ratio between the number of red edges and the number of blue edges in P' is at most $d-3:1$. So the proportion of blue edges in P' is at least $1/((d-3)+1) = 1/(d+2)$. The distance between $\phi(v_1) = P_1$ and $\phi(v_2) = P_2$ is the number of blue edges contained in P, which is one fewer than the number of blue edges in P'. So

$$\text{distance}\big(\phi(v_1), \phi(v_2)\big) + 1 \geq \frac{1}{d+2}\text{length}(P') \geq \frac{1}{d+2}\text{length}(P) = \frac{1}{d+2}\text{distance}(v_1, v_2)$$

So

$$\text{distance}\big(\phi(v_1), \phi(v_2)\big) \geq \frac{1}{d+2}\text{distance}(v_1, v_2) - 1. \tag{12}$$

Since v_1 and v_2 were chosen in P_1 and P_2 respectively such that v_1 and v_2 are as far apart as possible, therefore, (12) holds for any vertices v_1 and v_2 in T^3. Therefore we have that for all vertices v_1 and v_2 in T^3,

$$\frac{1}{d+2}\text{distance}(v_1, v_2) - 1 \leq \text{distance}\big(\phi(v_1), \phi(v_2)\big) \leq \text{distance}(v_1, v_2).$$

ϕ is clearly a surjection. Therefore, ϕ is a rough isometry from T^3 to K. Since K is the d-regular tree, we have proved that the 3-regular tree is roughly isometric to the d-regular tree for any $d > 3$.