Homework #7

October 23, 2018

1 Background/Terminology

Let (X, μ) be a standard probability space. Let $\mathcal{R} \subset X \times X$ be a Borel subset that is also an equivalence relation. Then $\mathcal{R}$ is **discrete** if its classes are countable. Let μ_l, μ_r be the measures on $\mathcal{R}$ defined by

$$\mu_l(E) = \int \#\{y : (x,y) \in E\} \, d\mu(x)$$

$$\mu_r(E) = \int \#\{x : (x,y) \in E\} \, d\mu(y)$$

for $E \subset \mathcal{R}$. Then μ is **$\mathcal{R}$-invariant** if $\mu_l = \mu_r$. Also μ is **$\mathcal{R}$-ergodic** if for every measurable set $E \subset X$ that is a union of $\mathcal{R}$-classes, either $\mu(E) = 0$ or $\mu(X \setminus E) = 0$.

We will say that $(\mathcal{R}, X, \mu)$ is an **MER** if $\mathcal{R}$ is a discrete Borel equivalence relation on (X, μ) and μ is $\mathcal{R}$-invariant.

More terminology:

- $\mathcal{R}$ is **finite** if for a.e. x, the $\mathcal{R}$-class of x is finite.
- $\mathcal{R}$ is **hyperfinite** if there exists an increasing sequence $\mathcal{S}_1 \subset \mathcal{S}_2 \subset \cdots$ of finite equivalence relations such that $\mathcal{R} = \cup_i \mathcal{S}_i$ (everything taken mod measure 0 with respect to $\mu_l = \mu_r$).
- The $\mathcal{R}$-class of x is denoted $[x]_{\mathcal{R}}$.

- Two MERs $(\mathcal{R}_i, X_i, \mu_i)$ $(i = 1, 2)$ are **isomorphic** if there exists a measure-space isomorphism $\Phi : (X_1, \mu_1) \to (X_2, \mu_2)$ such that for a.e. $x \in X_1$, the restriction of Φ to $[x]_{\mathcal{R}_1}$ is a bijection onto $[\Phi(x)]_{\mathcal{R}_2}$.

- A **graphing** is a measurable subset $\mathcal{G} \subset \mathcal{R}$ such that if $(x, y) \in \mathcal{G}$ then $(y, x) \in \mathcal{G}$ and $\mathcal{R}$ is the smallest equivalence relation containing $\mathcal{G}$. We think of $\mathcal{G}$ as representing the edges of a graph with vertex set X. In this case, the classes $[x]_{\mathcal{R}}$ are the connected components of $\mathcal{G}$.

2 Homework problems

1. Show that any two hyperfinite ergodic MERs are isomorphic.

2. Now suppose $\mathcal{R}$ is an ergodic hyperfinite MER. Also suppose $\mathcal{G}$ is a graphing of $\mathcal{R}$ and there is a uniform bound on the degrees of all vertices (so there exists $D > 0$ such that for all x there are at most D y's such that $(x, y) \in \mathcal{G}$). Show that for a.e. x, the connected component of x in the graphing $\mathcal{G}$ is an amenable graph.

3. Give an example of a unimodular random graph (G, o) such that (G, o) is non-amenable as a unimodular random graph but almost surely (G, o) is amenable as a graph. Hint: Randomly subdivide the edges of the 3-regular tree to produce the unimodular random graph.