In the problems below indicate your answers by drawing boxes around them. You must clearly show your work to get credit for a problem.

For Problems 1-4, let \( X_1, X_2, X_3 \) be a random sample from a geometric population with unknown parameter \( p \), i.e. the population distribution is given by

\[
f(x) = p(1 - p)^{x-1} \quad \text{for} \quad x \in \{1, 2, 3, \ldots\}
\]

1. Find the likelihood function for \( p \) corresponding to a sample \( (x_1, x_2, x_3) \).

\[
L(x_1, x_2, x_3 | p) = f(x_1 | p) \cdot f(x_2 | p) \cdot f(x_3 | p)
\]

\[
= p(1-p)^{x_1-1} \cdot p(1-p)^{x_2-1} \cdot p(1-p)^{x_3-1}
\]

\[
= p^3 (1-p)^{x_1 + x_2 + x_3 - 3}
\]
2. Let $X$ be the sample mean. That is, $X = (X_1 + X_2 + X_3)/3$. State the definition of sufficiency of $X$ for $\rho$ that uses the likelihood function for $\rho$. Use this to decide if $X$ is sufficient for $\rho$.

$X$ is sufficient for $\rho$ iff $L(\rho) = g(\bar{X}, \rho)$

i.e., the dependence of $L(\rho)$ on the data can be expressed through $\bar{X}$ alone.

$L(\rho) = \frac{\rho^3}{(1-\rho)^3} \rho^3 \bar{X} = g(\bar{X}, \rho)$
3. In the context above, let \( Y = X_1 + X_2 + X_3 \) be a statistic. Let \((1,1,2)\) be random sample. Find the probability of this sample given that \( Y = 4 \). Simplify as much as possible.

\[
P\left( X_1 = 1 \cap X_2 = 1 \cap X_3 = 2 \mid \Sigma = 4 \right) \\
= \frac{P\left( X_1 = 1 \cap X_2 = 1 \cap X_3 = 2 \mid \Sigma = 4 \right)}{P(\Sigma = 4)} \\
= \frac{P\left( X_1 = 1 \cap X_2 = 1 \cap X_3 = 2 \right)}{P(\Sigma = 4)} \\
= \frac{p(1-p)^7 p(1-p)^7 p(1-p)^2 \sum x_1 x_2 x_3^2}{p(1-p)^{x_1-1} p(1-p)^{x_2-1} p(1-p)^{x_3-1}} \\
= \frac{p^3 (1-p)^7 \sum x_1 x_2 x_3^2}{p^3 (1-p)^{x_1-1} p(1-p)^{x_2-1} p(1-p)^{x_3-1}} \\
= \frac{p (1-p)^7}{\sum x_1 x_2 x_3^2} = \frac{1}{N} \\
\text{Where } N = \# \text{ of } (x_1, x_2, x_3) \text{ s.t. } x_1 + x_2 + x_3 = 4 \\
N = 3 \text{ since } \\
\begin{array}{c|c|c|c}
  x_1 & x_2 & x_3 \\
  \hline
  1 & 1 & 2 \\
  1 & 2 & 1 \\
  2 & 1 & 1 \\
\end{array}
4. Again, let \( Y = X_1 + X_2 + X_3 \). Find the m.g.f. of \( Y \). Use the m.g.f. to compute \( E(Y) \).

The m.g.f. for a geometric population is \( \psi_X(t) = \frac{pe^t}{1-(1-p)e^t} \).

\[
\psi_Y(t) = \psi_{X_1, X_2, X_3}(t) = \psi_{X_1}(t) \cdot \psi_{X_2}(t) \cdot \psi_{X_3}(t) = \left( \frac{pe^t}{1-(1-p)e^t} \right)^3 = \frac{p^3e^{3t}}{(1-(1-p)e^t)^3}
\]

\[
E(Y) = \frac{d}{dt} \psi_Y(t) \bigg|_{t=0} = \frac{3p^3e^{2t}(1-(1-p)e^t)^2 + p^3e^{3t} \cdot 3(1-(1-p)e^t)^2(1-p)e^t}{(1-(1-p)e^t)^6} \bigg|_{t=0}
\]

\[
= \frac{3p^3(1-(1-p))^3 + p^3 \cdot 3(1-(1-p))^2(1-p)}{(1-(1-p))^6}
\]

\[
= \frac{3p^6 + 3p^5(1-p)}{p^6} = \frac{3}{p}
\]
5. Of the students in Luecke High, 40% come from Pebbles Middle School and the rest from Starlight Middle School. A student coming from Pebbles has a probability of .7 of graduating from Luecke High. A student from Starlight has a probability of .5 of graduating. You are talking to a Luecke High graduate. What is the probability that she went to Pebbles Middle School?

\[ P = \text{Pebbles M.S.} \]
\[ S = \text{Starlight M.S.} \]
\[ L = \text{graduating Luecke High} \]

\[
P(p|L) = \frac{P(p \cap L)}{P(L)} = \frac{P(L|p) \cdot P(p)}{P(L|p) \cdot P(p) + P(L|s) \cdot P(s)}
\]

\[
= \frac{.7 \cdot .4}{.7 \cdot .4 + .5 \cdot .6}
\]

\[
= \frac{.28}{.58}
\]

\[
= \frac{.483}{.58}
\]
6. I think it takes 25 minutes on average to jog from my house to Barton Creek and back. My dog feels like it takes much less time. I record 9 jogging times and find the average of these jogs to be 21 minutes with a variance of 16 minutes$^2$. To settle our disagreement, my dog poses to you the following question. Assume that my running times are normally distributed with mean 25 and variance 16. What is the probability that a random sample of 9 jogs would be as above; that is, that their average would be at most 21 minutes?

\[
\bar{X} = \frac{1}{9} (X_1 + \ldots + X_9)
\]

\[
\bar{X} \sim N(25, 16)
\]

\[
\bar{X} = \frac{1}{9} (X_1 + \ldots + X_9)
\]

\[
\sim N(25, \frac{16}{9})
\]

\[
P(\bar{X} \leq 21) = P\left( \frac{\bar{X} - 25}{\frac{4}{\sqrt{3}}} \leq \frac{21 - 25}{\frac{4}{\sqrt{3}}} \right)
\]

\[
= P( Z \leq -3)
\]

\[
= 0.0013
\]