Sets of Finite Perimeter and Geometric Variational Problems

An Introduction to Geometric Measure Theory

FRANCESCO MAGGI
Università degli Studi di Firenze, Italy

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Contents

Preface

Everyone talks about rock these days;
the problem is they forget about the roll.
Keith Richards

The theory of sets of finite perimeter provides, in the broader framework of Geometric Measure Theory (hereafter referred to as GMT), a particularly well-suited framework for studying the existence, symmetry, regularity, and structure of singularities of minimizers in those geometric variational problems in which surface area is minimized under a volume constraint. Isoperimetric-type problems constitute one of the oldest and more attractive areas of the Calculus of Variations, with a long and beautiful history, and a large number of still open problems and current research. The first aim of this book is to provide a pedagogical introduction to this subject, ranging from the foundations of the theory, to some of the most deep and beautiful results in the field, thus providing a complete background for research activity. We shall cover topics like the Euclidean isoperimetric problem, the description of geometric properties of equilibrium shapes for liquid drops and crystals, the regularity up to a singular set of codimension at least 8 for area minimizing boundaries, and, probably for the first time in book form, the theory of minimizing clusters developed (in a more sophisticated framework) by Almgren in his AMS Memoir [Alm76].

Ideas and techniques from GMT are of crucial importance also in the study of other variational problems (both of parametric and non-parametric character), as well as of partial differential equations. The secondary aim of this book is to provide a multi-leveled introduction to these tools and methods, by adopting an expository style which consists of both heuristic explanations and fully detailed technical arguments. In my opinion, among the various parts of GMT, the theory of sets of finite perimeter is the best suited for this aim. Compared to the theories of currents and varifolds, it uses a lighter notation and, virtually, no preliminary notions from Algebraic or Differential Geometry. At the same time, concerning, for example, key topics like partial regularity properties of minimizers and the analysis of their singularities, the deeper structure of many fundamental arguments can be fully appreciated in this simplified framework. Of course this line of thought has not to be pushed too far. But it is my conviction that a careful reader of this book will be able to enter other parts of GMT with relative ease, or to apply the characteristic tools of GMT in the study of problems arising in other areas of Mathematics.

The book is divided into four parts, which in turn are opened by rather detailed synopses. Depending on their personal backgrounds, different readers may like to use the book in different ways. As we shall explain in a moment, a short "crash-course" is available for complete beginners.

Part I contains the basic theory of Radon measures, Hausdorff measures, and rectifiable sets, and provides the background material for the rest of the book. I am not a big fan of "preliminary chapters", as they often miss a storyline, and quickly become boring. I have thus tried to develop Part I as independent, self-contained, and easily accessible reading. In any case, following the above mentioned "crash-course" makes it possible to see some action taking place without having to work through the entire set of preliminaries.

Part II opens with the basic theory of sets of finite perimeter, which is presented, essentially, as it appears in the original papers by De Giorgi [DG54, DG55, DG58]. In particular, we avoid the use of functions of bounded variation, hoping to better stimulate the development of a geometric intuition of the theory. We also present the original proof of De Giorgi's structure theorem, relying on Whitney's extension theorem, and avoiding the notion of rectifiable set. Later on, in the central portion of Part II, we make the theory of rectifiable sets from Part I enter into the game. We thus provide another justification of De Giorgi's structure theorem, and develop some crucial cut-and-paste competitors' building techniques, first and second variation formulae, and slicing formulae for boundaries. The methods and ideas introduced in this part are finally applied to study variational problems concerning confined liquid drops and anisotropic surface energies.

Part III deals with the regularity theory for local perimeter minimizers, as well as with the analysis of their singularities. In fact, we shall deal with the more general notion of (Λ, r_0)-perimeter minimizer, thus providing regularity results for several Plateau-type problems and isoperimetric-type problems. Finally, Part IV provides an introduction to the theory of minimizing clusters. These last two parts are definitely more advanced, and contain the deeper ideas and finer arguments presented in this book. Although their natural audience will unavoidably be made of more expert readers, I have tried to keep in these parts the same pedagogical point of view adopted elsewhere.

As I said, a "crash-course" on the theory of sets of finite perimeter, of about 130 pages, is available for beginners. The course starts with a revision of the basic theory of Radon measures, temporarily excluding differentiation theory (Chapters 1–4), plus some simple facts concerning weak gradients from Section 7.2. The notion of distributional perimeter is then introduced and used to prove the existence of minimizers in several variational problems, culminating with the solution of the Euclidean isoperimetric problem (Chapters 12–14). Finally, the differentiation theory for Radon measures is developed (Chapter 5), and then applied to clarify the geometric structure of sets of finite perimeter through the study of reduced boundaries (Chapter 15).

Each part is closed by a set of notes and remarks, mainly, but not only, of bibliographical character. The bibliographical remarks, in particular, are not meant to provide a complete picture of the huge literature on the problems considered in this book, and are limited to some suggestions for further reading. In a similar way, we now mention some monographs related to our subject.

Concerning Radon measures and rectifiable sets, further readings of exceptional value are Falconer [Fal86], Mattila [Mat95], and De Lellis [DL08].

For the classical approach to sets of finite perimeter in the context of functions of bounded variation, we refer readers to Giusti [Giu84], Evans and Gariepy [EG92], and Ambrosio, Fusco, and Pallara [AFP00].

The partial regularity theory of Part III does not follow De Giorgi's original approach [DG60], but it is rather modeled after the work of authors like Almgren, Allard, Bombieri, Federer, Schoen, Simon, etc. in the study of area minimizing currents and stationary varifolds. The resulting proofs only rely on direct comparison arguments and on geometrically viewable constructions, and should provide several useful reference points for studying more advanced regularity theories. Accounts and extensions of De Giorgi's original approach can be found in the monographs by Giusti [Giu84] and Massari and Miranda [MM84], as well as in Tamanini's beautiful lecture notes [Tam84].

Readers willing to enter into other parts of GMT have several choices. The introductory books by Almgren [Alm66] and Morgan [Mor09] provide initial insight and motivation. Suggested readings are then Simon [Sim83], Krantz and Parks [KP08], and Giaquinta, Modica, and Souček [GMS98a, GMS98b], as well as, of course, the historical paper by Federer and Fleming [FF60]. Concerning the regularity theory for minimizing currents, the paper by Duzaar and Steffen [DS02] is a valuable source for both its clarity and its completeness. Finally (and although, since its appearance, various crucial parts of the theory have found alternative, simpler justifications, and several major achievements have been obtained), Federer's legendary book [Fed69] remains the ultimate reference for many topics in GMT.

I wish to acknowledge the support received from several friends and colleagues in the realization of this project. This book originates from the lecture notes of a course that I held at the University of Duisburg-Essen in the Spring of 2005, under the advice of Sergio Conti. The successful use of these unpublished notes in undergraduate seminar courses by Peter Hornung and Stefan Müller convinced me to start the revision and expansion of their content. The work with Nicola Fusco and Aldo Pratelli on the stability of the Euclidean isoperimetric inequality [FMP08] greatly influenced the point of view on sets of finite perimeter adopted in this book, which has also been crucially shaped (particularly in connection with the regularity theory of Part III) by several, endless, mathematical discussions with Alessio Figalli. Alessio has also lectured at the University of Texas at Austin on a draft of the first three parts, supporting me with hundreds of comments. Another important contribution came from Guido De Philippis, who read the entire book *twice*, giving me much careful criticism and many useful suggestions. I was lucky to have the opportunity of discussing with Gian Paolo Leonardi various aspects of the theory of minimizing clusters presented in Part IV. Comments and errata were provided to me by Luigi Ambrosio (his lecture notes [Amb97] have been a major source of inspiration), Marco Cicalese, Matteo Focardi, Nicola Fusco, Frank Morgan, Matteo Novaga, Giovanni Pisante and Berardo Ruffini. Finally, I wish to thank Giovanni Alberti, Almut Burchard, Eric Carlen, Camillo de Lellis, Michele Miranda, Massimiliano Morini, and Emanuele Nunzio Spadaro for having expressed to me their encouragement and interest in this project.

I have the feeling that while I was busy trying to talk about the rock without forgetting about the roll, some errors and misprints made their way to the printed page. I will keep an errata list on my webpage.

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Francesco Maggi

PART ONE

Radon measures on $\mathbb{R}^n$

Synopsis

In this part we discuss the basic theory of Radon measures on $\mathbb{R}^n$. Roughly speaking, if $\mathcal{P}(\mathbb{R}^n)$ denotes the set of the parts of $\mathbb{R}^n$, then a Radon measure μ on $\mathbb{R}^n$ is a function $\mu\colon \mathcal{P}(\mathbb{R}^n) \to [0,\infty]$, which is countably additive (at least) on the family of Borel sets of $\mathbb{R}^n$, takes finite values on bounded sets, and is completely identified by its values on open sets. The Lebesgue measure on $\mathbb{R}^n$ and the Dirac measure δ_x at $x \in \mathbb{R}^n$ are well-known examples of Radon measures on $\mathbb{R}^n$. Moreover, any locally summable function on $\mathbb{R}^n$, as well as any k-dimensional surface in $\mathbb{R}^n$, $1 \le k \le n-1$, can be naturally identified with a Radon measure on $\mathbb{R}^n$. There are good reasons to look at such familiar objects from this particular point of view. Indeed, the natural notion of convergence for sequences of Radon measures satisfies very flexible compactness properties. As a consequence, the theory of Radon measures provides a unified framework for dealing with the various convergence and compactness phenomena that one faces in the study of geometric variational problems. For example, a sequence of continuous functions on $\mathbb{R}^n$ that (as a sequence of Radon measures) is converging to a surface in $\mathbb{R}^n$ is something that cannot be handled with the notions of convergence usually considered on spaces of continuous functions or on Lebesgue spaces. Similarly, the existence of a tangent plane to a surface at one of its points can be understood as the convergence of the (Radon measures naturally associated with) re-scaled and translated copies of the surface to the (Radon measure naturally associated with the) tangent plane itself. This peculiar point of view opens the door for a geometrically meaningful (and analytically powerful) extension of the notion of differentiability to the wide class of objects, the family of rectifiable sets, that one must consider in solving geometric variational problems.

Part I is divided into two main portions. The first one (Chapters 1–6) is devoted to the more abstract aspects of the theory. In Chapters 1–4, we introduce the main definitions, present the most basic examples, and prove the fundamental representation and compactness theorems about Radon measures. (These results already suffice to give an understanding of the basic theory of sets of finite perimeter as presented in the first three chapters of Part II.) Differentiation theory, and its applications, are discussed in Chapters 5–6. In the second portion of Part I (Chapters 7–11), we consider Radon measures from a more geometric viewpoint, focusing on the interaction between Euclidean geometry and Measure Theory, and covering topics such as Lipschitz functions, Hausdorff measures, area formulae, rectifiable sets, and measure-theoretic differentiability. These are prerequisites to more advanced parts of the theory of sets of finite perimeter, and can be safely postponed until really needed. We now examine more closely each chapter.

In Chapters 1–2 we introduce the notions of Borel and Radon measure. This is done in the context of *outer measures*, rather than in the classical context of standard measures defined on σ-algebras. We simultaneously develop both the basic properties relating Borel and Radon measures to the Euclidean topology of $\mathbb{R}^n$ and the basic examples of the theory that are obtained by combining the definitions of Lebesgue and Hausdorff measures with the operations of restriction to a set and push-forward through a function.

In Chapter 3 we look more closely at Hausdorff measures. We establish their most basic properties and introduce the notion of Hausdorff dimension. Next, we show equivalence between the Lebesgue measure on $\mathbb{R}^n$ and the n-dimensional Hausdorff measure on $\mathbb{R}^n$, and we study the relation between the elementary notion of length of a curve, based on the existence of a parametrization, and the notion induced by one-dimensional Hausdorff measures.

In Chapter 4 we further develop the general theory of Radon measures. In particular, the deep link between Radon measures and continuous functions with compact support is presented, leading to the definition of *vector-valued* Radon measures, of weak-star convergence of Radon measures, and to the proof of the fundamental Riesz's representation theorem: every bounded linear functional on $C^0_c(\mathbb{R}^n;\mathbb{R}^m)$ is representable as integration with respect to an $\mathbb{R}^m$-valued Radon measure on $\mathbb{R}^n$. This last result, in turn, is the key to the weak-star compactness criterion for sequences of Radon measures.

Chapters 5–6 present differentiation theory and its applications. The goal is to compare two Radon measures ν and μ by looking, as $r \to 0^+$, at the ratios

$$\frac{\nu(B(x,r))}{\mu(B(x,r))},$$

which are defined at those x where μ is supported (i.e., $\mu(B(x,r)) > 0$ for every $r > 0$). The Besicovitch–Lebesgue differentiation theorem ensures that, for μ-a.e. x in the support of μ, these ratios converge to a finite limit $u(x)$, and that restriction of ν to the support of μ equals integration of u with respect to μ. Differentiation theory plays a crucial role in proving the validity of classical (or generalized) differentiability properties in many situations.

In Chapter 7 we study the basic properties of Lipschitz functions, proving Rademacher's theorem about the almost everywhere classical differentiability of Lipschitz functions, and Kirszbraun's theorem concerning the optimal extension problem for vector-valued Lipschitz maps.

Chapter 8 presents the area formula, which relates the Hausdorff measure of a set in $\mathbb{R}^n$ with that of its Lipschitz images into any $\mathbb{R}^m$ with $m \geq n$. As a consequence, the classical notion of area of a k-dimensional surface M in $\mathbb{R}^n$ is seen to coincide with the k-dimensional Hausdorff measure of M. Some applications of the area formula are presented in Chapter 9, where, in particular, the classical Gauss–Green theorem is proved.

In Chapter 10 we introduce one of the most important notions of Geometric Measure Theory, that of a k-dimensional rectifiable set in $\mathbb{R}^n$ $(1 \leq k \leq n-1)$. This is a very broad generalization of the concept of k-dimensional C^1-surface, allowing for complex singularities but, at the same time, retaining tangential differentiability properties, at least in a measure-theoretic sense. A crucial result is the following: if the k-dimensional blow-ups of a Radon measure μ converge to k-dimensional linear spaces (seen as Radon measures), then it turns out that μ itself is the restriction of the k-dimensional Hausdorff measure to a k-dimensional rectifiable set.

In Chapter 11, we introduce the notion of tangential differentiability of a Lipschitz function with respect to a rectifiable set, extend the area formula to this context, and prove the divergence theorem on C^2-surfaces with boundary.

PART TWO

Sets of finite perimeter

Synopsis

The starting point of the theory of sets of finite perimeter is a generalization of the Gauss–Green theorem based on the notion of vector-valued Radon measure. Precisely, we say that a Lebesgue measurable set $E \subset \mathbb{R}^n$ is a set of locally finite perimeter if there exists a $\mathbb{R}^n$-valued Radon measure μ_E on $\mathbb{R}^n$, called the Gauss–Green measure of E, such that the generalized Gauss–Green formula

$$\int_E \nabla\varphi = \int_{\mathbb{R}^n} \varphi \, d\mu_E \,, \qquad \forall \varphi \in C^1_c(\mathbb{R}^n) \,, \tag{1}$$

holds true. The total variation measure $|\mu_E|$ of μ_E induces the notions of relative perimeter $P(E;F)$ of E with respect to a set $F \subset \mathbb{R}^n$, and of (total) perimeter $P(E)$ of E, defined as

$$P(E;F) = |\mu_E|(F) \,, \qquad P(E) = |\mu_E|(\mathbb{R}^n) \,.$$

In particular, E is a set of finite perimeter if and only if $P(E) < \infty$. These definitions are motivated by the classical Gauss–Green theorem, Theorem 9.3. Indeed, if E is an open set with C^1-boundary with outer unit normal $\nu_E \in C^0(\partial E; S^{n-1})$, then Theorem 9.3 implies

$$\int_E \nabla\varphi = \int_{\partial E} \varphi \, \nu_E \, d\mathcal{H}^{n-1} \,, \qquad \forall \varphi \in C^1_c(\mathbb{R}^n) \,, \tag{2}$$

and thus E is a set of locally finite perimeter with

$$\mu_E = \nu_E \, \mathcal{H}^{n-1} \llcorner \partial E \,, \qquad |\mu_E| = \mathcal{H}^{n-1} \llcorner \partial E \,, \tag{3}$$

$$P(E;F) = \mathcal{H}^{n-1}(F \cap \partial E) \,, \qquad P(E) = \mathcal{H}^{n-1}(\partial E) \,, \tag{4}$$

for every $F \subset \mathbb{R}^n$; see Figure 1. One of the main themes of this part of the book is showing that these definitions lead to a geometrically meaningful generalization of the notion of open set with C^1-boundary, with natural and powerful applications to the study of geometric variational problems.

We start this programme in Chapter 12, where the link with the theory of Radon measures established by (1) is exploited to deduce some basic lower semicontinuity and compactness theorems for sequences of sets of locally finite perimeter; see Sections 12.1 and 12.4. In particular, these results make it

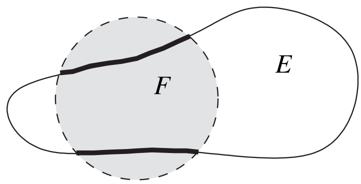

Figure 1 The perimeter $P(E; F)$ of E relative to F is the $(n-1)$-dimensional measure of the intersection of the (reduced) boundary of E with F.

possible to apply the Direct Method in order to prove the existence of minimizers in several geometric variational problems, see Section 12.5.

In Chapter 13 we discuss the possibility of approximating sets of finite perimeter by sequences of open sets with smooth boundary. The resulting approximation theorems appear often as useful technical devices, but also possess another merit. Indeed, generally speaking, they imply the coincidence of the minimum values of the different formulations of the same variational problems that are obtained by minimizing either among sets of finite perimeter or among open sets with C^1-boundary. Another relevant content of Chapter 13 is the *coarea formula*, which is a generalization of Fubini's theorem of ubiquitous importance in Geometric Measure Theory.

In Chapter 14 we study the Euclidean isoperimetric problem: given $m > 0$, minimize perimeter among sets of volume m, namely

$$\inf\left\{P(E) : |E| = m\right\}.$$

Exploiting the lower semicontinuity, compactness, and approximation theorems developed in the two previous chapters, together with the notion of Steiner symmetrization, we shall characterize Euclidean balls as the (unique) minimizers in the Euclidean isoperimetric problem. A remarkable feature of this result and, more generally, of the results from the first three chapters of this part, is that they are only based on the tools from basic Measure Theory and Functional Analysis set forth in Chapters 1–4, and that they are obtained without any knowledge on the geometric structure of arbitrary sets of finite perimeter.

We next turn to the following, fundamental question: does the validity of (1) imply a set of locally finite perimeter E to possess, in some suitable sense, a $(n-1)$-dimensional boundary and outer unit normal allowing us, for example, to generalize (2), (3), and (4)? The first important remark here is that the notion of topological boundary is of little use in answering this question. Indeed, if E is of locally finite perimeter and E' is equivalent to E (i.e., $|E \Delta E'| = 0$), then, as the left-hand side of (1) is left unchanged by replacing E with E', we have that E' is a set of locally finite perimeter too, with $\mu_E = \mu_{E'}$. Of course, the topological boundaries of E and E' may be completely different (for example, even if E is an open set with C^1-boundary, we may take $E' = E \cup \mathbb{Q}^n$, and have $\partial E' = \mathbb{R}^n$, $\mu_{E'} = \nu_E \mathcal{H}^{n-1} \llcorner \partial E$). For this reason, when dealing with sets of finite perimeter, it is always useful to keep in mind the possibility of making modifications on and/or by sets of measure zero to find a representative which "minimizes the size of the topological boundary". In other words, if E is of locally finite perimeter, then we always have $\operatorname{spt}\mu_E \subset \partial E$, and we can always find E' equivalent to E such that $\operatorname{spt}\mu_{E'} = \partial E'$; see Proposition 12.19. But even with these specifications in mind, we have to face the existence of sets of finite perimeter $E \subset \mathbb{R}^n$, $n \geq 2$, with $|E| < \infty$, $|\operatorname{spt}\mu_E| > 0$, and thus, in particular, $\mathcal{H}^{n-1}(\operatorname{spt}\mu_E) = \infty$; see Example 12.25. In conclusion, even after the suitable "minimization of size", the topological boundary of a set of finite perimeter may have Hausdorff dimension equal to that of its ambient space!

The key notion to consider in order to understand the geometric structure of sets of finite perimeter is that of *reduced boundary*, which may be explained as follows. If E is an open set with C^1-boundary, then the continuity of the outer unit normal ν_E allows us to characterize $\nu_E(x)$ in terms of the Gauss–Green measure $\mu_E = \nu_E \mathcal{H}^{n-1} \llcorner \partial E$ as

$$\nu_E(x) = \lim_{r\to 0^+} \fint_{B(x,r)\cap\partial E} \nu_E \, d\mathcal{H}^{n-1} = \lim_{r\to 0^+} \frac{\mu_E(B(x,r))}{|\mu_E|(B(x,r))}, \qquad \forall x \in \partial E .$$

If now E is a generic set of locally finite perimeter, then $|\mu_E|(B(x,r)) > 0$ for every $x \in \operatorname{spt}\mu_E$ and $r > 0$, and thus it makes sense to define the reduced boundary $\partial^* E$ of E as the set of those $x \in \operatorname{spt}\mu_E$ such that the limit

$$\lim_{r\to 0^+} \frac{\mu_E(B(x,r))}{|\mu_E|(B(x,r))} \qquad \text{exists and belongs to } S^{n-1}. \tag{5}$$

In analogy with the regular case, the Borel vector field $\nu_E \colon \partial^* E \to S^{n-1}$ defined in (5) is called the *measure-theoretic outer unit normal to E*. The reduced boundary and the measure-theoretic outer unit normal depend on E only through its Gauss–Green measure, and are therefore left unchanged by modifications of E on and/or by a set of measure zero. It also turns out that $\partial^* E$ has the structure of an $(n-1)$-dimensional surface, that ν_E has a precise geometric meaning as the outer unit normal to E, and that (3) and (4) hold true on generic sets of finite perimeter by replacing topological boundaries and classical outer unit normals with reduced boundaries and measure-theoretic outer unit normals. Precisely, the following statements from *De Giorgi's structure theory*, presented in Chapter 15, hold true:

(i) The Gauss–Green measure μ_E is obtained by integrating ν_E against the restriction of $\mathcal{H}^{n-1}$ to $\partial^* E$, that is,

$$\mu_E = \nu_E \, \mathcal{H}^{n-1} \llcorner \partial^* E \,, \qquad |\mu_E| = \mathcal{H}^{n-1} \llcorner \partial^* E \,,$$
$$P(E;F) = \mathcal{H}^{n-1}(F \cap \partial^* E) \,, \qquad P(E) = \mathcal{H}^{n-1}(\partial^* E) \,,$$

for every $F \subset \mathbb{R}^n$, and the Gauss–Green formula (1) takes the form

$$\int_E \nabla\varphi = \int_{\partial^* E} \varphi \, \nu_E \, d\mathcal{H}^{n-1} \,, \qquad \forall \varphi \in C^1_c(\mathbb{R}^n) \,.$$

(ii) If $x \in \partial^* E$, then $\nu_E(x)$ is orthogonal to $\partial^* E$ at x, in the sense that

$$\mathcal{H}^{n-1} \llcorner \left(\frac{\partial^* E - x}{r} \right) \stackrel{*}{\rightharpoonup} \mathcal{H}^{n-1} \llcorner \, \nu_E(x)^\perp \qquad \text{as } r \to 0^+ ,$$

and it is an *outer* unit normal to E at x, in the sense that

$$\frac{E - x}{r} \stackrel{\text{loc}}{\rightarrow} \left\{ y \in \mathbb{R}^n : y \cdot \nu_E(x) \le 0 \right\} \qquad \text{as } r \to 0^+ \,.$$

(iii) The reduced boundary $\partial^* E$ is the union of (at most countably many) compact subsets of C^1-hypersurfaces is $\mathbb{R}^n$; more precisely, there exist at most countably many C^1-hypersurfaces M_h and compact sets $K_h \subset M_h$ with $T_x M_h = \nu_E(x)^\perp$ for every $x \in K_h$, such that

$$\partial^* E = N \cup \bigcup_{h \in \mathbb{N}} K_h \,, \qquad \mathcal{H}^{n-1}(N) = 0 \,.$$

Statement (iii) implies of course that the reduced boundary of a set of locally finite perimeter is a locally $\mathcal{H}^{n-1}$-rectifiable set. In Chapter 16 we undertake the study of reduced boundaries and Gauss–Green measures in the light of the theory of rectifiable sets developed in Chapter 10. We prove *Federer's theorem*, stating the $\mathcal{H}^{n-1}$-equivalence between the reduced boundary of E, the set $E^{(1/2)}$ of its points of density one-half, and the essential boundary $\partial^e E$, which is defined as the complement in $\mathbb{R}^n$ of $E^{(0)} \cup E^{(1)}$. This result proves a powerful tool, as sets of density points are much more easily manipulated than reduced boundaries. For example, it is starting from Federer's theorem that in Section 16.1 we prove some representation formulae for Gauss–Green measures of unions, intersections, and set differences of two sets of locally finite perimeter. These formulae allow us to easily "cut and paste" sets of finite perimeter, an operation which proves useful in building comparison sets for testing minimality conditions. As an application of these techniques, in Section 16.2 we prove upper and lower density estimates for reduced boundaries of local perimeter minimizers, which, combined with Federer's theorem, imply a first, mild, regularity property of local perimeter minimizers: the $\mathcal{H}^{n-1}$-equivalence between the reduced boundary and the support of the Gauss–Green measure, that is, as said, the topological boundary of “minimal size”.

In Chapter 17 we apply the area formula of Chapter 8 to study the behavior of sets of finite perimeter under the action of one parameter families of diffeomorphisms. We compute the first and second variation formulae of perimeter, and introduce distributional formulations of classical first order necessary minimality conditions, like the vanishing mean curvature condition.

In Chapter 18 we present a refinement of the coarea formula from Chapter 13, which in turn allows us to discuss slicing of reduced boundaries. In particular, slicing by hyperplanes is discussed in some detail in Section 18.3.

We close Part II by briefly introducing two important examples of geometric variational problems which can be addressed in our framework. Precisely, in Chapter 19 we discuss the equilibrium problem for a liquid confined inside a given container, while in Chapter 20 we consider anisotropic surface energies and address the so-called Wulff problem, originating from the study of equlibrium shapes of crystals.

PART THREE

Regularity theory and analysis of singularities

Synopsis

In this part we shall discuss the regularity of boundaries of those sets of finite perimeter which arise as minimizers in some of the variational problems considered so far. The following theorem exemplifies the kind of result we shall obtain. We recall from Section 16.2 that E is a local perimeter minimizer (at scale r_0) in some open set A, if $\operatorname{spt}\mu_E = \partial E$ (recall Remark 16.11) and

$$P(E; A) \le P(F; A), \tag{1}$$

whenever $E \Delta F \subset\subset B(x, r_0) \cap A$ and $x \in A$.

Theorem *If $n \ge 2$, A is an open set in $\mathbb{R}^n$, and E is a local perimeter minimizer in A, then $A \cap \partial^* E$ is an analytic hypersurface with vanishing mean curvature which is relatively open in $A \cap \partial E$, while the* **singular set** *of E in A,*

$$\Sigma(E; A) = A \cap (\partial E \setminus \partial^* E),$$

satisfies the following properties:

(i) if $2 \le n \le 7$, then $\Sigma(E; A)$ is empty;
(ii) if $n = 8$, then $\Sigma(E; A)$ has no accumulation points in A;
(iii) if $n \ge 9$, then $\mathcal{H}^s(\Sigma(E; A)) = 0$ for every $s > n - 8$.

These assertions are sharp: there exists a perimeter minimizer E in $\mathbb{R}^8$ such that $\mathcal{H}^0(\Sigma(E; \mathbb{R}^8)) = 1$; moreover, if $n \ge 9$, then there exists a perimeter minimizer E in $\mathbb{R}^n$ such that $\mathcal{H}^{n-8}(\Sigma(E; \mathbb{R}^n)) = \infty$.

The proof of this deep theorem, which will take all of Part III, is essentially divided into two parts. The first one concerns the regularity of the reduced boundary in A and, precisely, it consists of proving that the locally $\mathcal{H}^{n-1}$-rectifiable set $A \cap \partial^* E$ is, in fact, a $C^{1,\gamma}$-hypersurface for every $\gamma \in (0, 1)$. (As we shall see, its analiticity will then follow rather straightforwardly from standard elliptic regularity theory.) The second part of the argument is devoted to the analysis of the structure of the singular set $\Sigma(E; A)$. By the density estimates of Theorem 16.14, we already know that $\mathcal{H}^{n-1}(\Sigma(E; A)) = 0$. In order to improve this estimate, we shall move from the fact that, roughly speaking, the blow-ups $E_{x,r}$ of E at points $x \in \Sigma(E; A)$ will have to converge to *cones* which are local perimeter minimizers in $\mathbb{R}^n$, and which have their vertex at a singular point. Starting from this result, and discussing the possible existence of such singular minimizing cones, we shall prove the claimed estimates.

In fact, we shall not confine our attention to local perimeter minimizers, but we shall work instead in the broader class of (Λ, r_0)-*perimeter minimizers*. This is a generalization of the notion of local perimeter minimizer, which allows for the presence on the right-hand side of the minimality inequality (1) of a higher order term of the form $\Lambda\,|E\Delta F|$. The interest of this kind of minimality condition, originally introduced in a more general context and form by Almgren [Alm76], lies in the fact that, contrary to local perimeter minimality, it is satisfied by minimizers in geometric variational problems with volume-constraints and potential-type energies. At the same time, the smaller the scale at which the competitor F differs from E, the closer (Λ, r_0)-perimeter minimality is to plain local perimeter minimality, and thus the regularity theory and the analysis of singularities may be tackled in both cases with essentially the same effort.

In Chapter 21 we thus introduce (Λ, r_0)-perimeter minimality, we discuss its applicability in studying minimizers which arise from the variational problems presented in Part II, and prove the compactness theorem for sequences of (Λ, r_0)-perimeter minimizers. In Chapter 22 we introduce the fundamental notion of excess $\mathbf{e}(E, x, r)$, which is used to measure the integral oscillation of the measure-theoretic outer unit normal to E over $B(x, r) \cap \partial^* E$. We discuss the basic properties of the excess and prove that its smallness at a given point x and scale r implies the uniform proximity of $B(x, r) \cap \partial E$ to a hyperplane. Starting from this result, in Chapter 23, we show that the $\mathcal{H}^{n-1}$-rectifiable set $B(x, r) \cap \partial E$ can always be covered by the graph of a Lipschitz function u over an $(n-1)$-dimensional ball $\mathbf{D}_r$ of radius r, up to an error which is controlled by the size of $\mathbf{e}(E, x, r)$. Moreover, again in terms of the size of $\mathbf{e}(E, x, r)$, the function u is in fact close to minimizing the area integrand $\int_{\mathbf{D}_r} \sqrt{1 + |\nabla' u|^2}$, and $\int_{\mathbf{D}_r} |\nabla' u|^2$ is close to zero, so that, by Taylor's formula

$$\int_{\mathbf{D}_r} \sqrt{1 + |\nabla' u|^2} = \mathcal{H}^{n-1}(\mathbf{D}_r) + \frac{1}{2}\int_{\mathbf{D}_r} |\nabla' u|^2 + \dots\,,$$

u is in fact close to minimizing the Dirichlet integral $\int_{\mathbf{D}_r} |\nabla' u|^2$; that is, u is almost a *harmonic function*. Through the use of the *reverse Poincaré inequality* (Chapter 24), and exploiting some basic properties of harmonic functions, in Chapter 25 we use this information to prove some explicit decay estimates for the integral averages of ∇u which, in turn, are equivalent in proving the uniform decay of the excess $\mathbf{e}(E, x, r)$ in r. In Chapter 26 we exploit the decay of the excess to prove the $C^{1,\gamma}$-regularity of $A \cap \partial^* E$. As a by-product we obtain a characterization of the singular set $\Sigma(E; A)$ in terms of the excess, as well as a powerful C^1-convergence theorem for sequences of (Λ, r_0)-perimeter minimizers. The exposition of the regularity theory is concluded in Chapter 27, where the connection with elliptic equations in divergence form is used to improve the $C^{1,\gamma}$-regularity result on minimizers of specific variational problems. Finally, Chapter 28 is devoted to the study of singular sets and singular minimizing cones. We refer to the beginning of that chapter for a detailed overview of its contents.

NOTATION WARNING: Throughout this part we shall continuously adopt Notation 4. Moreover, we shall denote by $\mathbf{C}(x, r, \nu)$ the cylinder

$$\mathbf{C}(x, r, \nu) = x + \Big\{ y \in \mathbb{R}^n : |y \cdot \nu| < r\,, |y - (y \cdot \nu)\nu| < r \Big\}\,,$$

where $x \in \mathbb{R}^n$, $r > 0$ and $\nu \in S^{n-1}$.

PART FOUR

Minimizing clusters

Synopsis

A cluster $\mathcal{E}$ in $\mathbb{R}^n$ is a finite disjoint family of sets of finite perimeter $\mathcal{E} = \{\mathcal{E}(h)\}_{h=1}^N$ ($N \in \mathbb{N}$, $N \geq 2$) with finite and positive Lebesgue measure (note: the chambers $\mathcal{E}(h)$ of $\mathcal{E}$ are not assumed to be connected/indecomposable). By convention, $\mathcal{E}(0) = \mathbb{R}^n \setminus \bigcup_{h=1}^N \mathcal{E}(h)$ denotes the exterior chamber of $\mathcal{E}$. The perimeter $P(\mathcal{E})$ of $\mathcal{E}$ is defined as the total $(n-1)$-dimensional Hausdorff measure of the interfaces of the cluster,

$$P(\mathcal{E}) = \sum_{0 \leq h < k \leq N} \mathcal{H}^{n-1}\Big(\partial^* \mathcal{E}(h) \cap \partial^* \mathcal{E}(k)\Big).$$

Denoting by $\mathbf{m}(\mathcal{E})$ the vector in $\mathbb{R}^N_+$ whose hth entry agrees with $|\mathcal{E}(h)|$, we shall say that $\mathcal{E}$ is a **minimizing cluster in** $\mathbb{R}^n$ if $\operatorname{spt} \mu_{\mathcal{E}(h)} = \partial \mathcal{E}(h)$ for every $h = 1, \ldots, N$, and, moreover, $P(\mathcal{E}) \leq P(\mathcal{E}')$ whenever $\mathbf{m}(\mathcal{E}') = \mathbf{m}(E)$. By a **partitioning problem in** $\mathbb{R}^n$, we mean any variational problem of the form

$$\inf \Big\{ P(\mathcal{E}) : \mathbf{m}(\mathcal{E}) = \mathbf{m} \Big\},$$

corresponding to the choice of some $\mathbf{m} \in \mathbb{R}^N_+$. Proving the following theorem will be the main aim of Part IV. The existence and regularity parts will be addressed, respectively, in Chapter 29 and Chapter 30.

Theorem (Almgren's theorem) *If $n, N \geq 2$ and $\mathbf{m} \in \mathbb{R}^N_+$, then there exist minimizers in the partitioning problem defined by $\mathbf{m}$. If $\mathcal{E}$ is an N-minimizing cluster in $\mathbb{R}^n$, then $\mathcal{E}$ is bounded. If $0 \leq h < k \leq N$, then $\partial^* \mathcal{E}(h) \cap \partial^* \mathcal{E}(k)$ is an analytic constant mean curvature hypersurface in $\mathbb{R}^n$, relatively open inside $\partial \mathcal{E}(h) \cap \partial \mathcal{E}(k)$. Finally,*

$$\sum_{h=0}^N \mathcal{H}^{n-1}\Big(\partial \mathcal{E}(h) \setminus \partial^* \mathcal{E}(h)\Big) = 0\,.$$

This existence and almost everywhere regularity theorem is one of the main results contained in the founding work for the theory of minimizing clusters and partitioning problems, that is Almgren's AMS Memoir [Alm76]. This theory, despite the various beautiful results which have been obtained since then, still presents many interesting open questions. We aim here to provide the reader with the necessary background to enter into these problems. Indeed, the techniques and ideas introduced in the proof of Almgren's theorem prove useful also in its subsequent developments, and are likely to play a role in possible further investigations in the subject.

Slightly rephrasing Almgren's words [Alm76, VI.1(6)], the aims of the theory are: (i) to show the existence of minimizing clusters; (ii) to prove the regularity of their interfaces outside singular closed sets; (iii) to describe the structure of these interfaces close to their singular sets, as well as the structure of the singular sets themselves; (iv) to construct examples of minimizing clusters; (v) to classify "in some reasonable way" the different minimizing clusters corresponding to different choices of $\mathbf{m} \in \mathbb{R}^N_+$; and (vi) to extend the analysis of these questions to multi-phase anisotropic partitioning problems (which are introduced below). We will deal with part (i) and (ii) of this programme in the case of mono-phase isotropic problems. We now provide a brief and partial review on the state of the art concerning Almgren's programme, and refer the reader to Morgan's book [Mor09, Chapters 13–16] for further references and information.

Planar minimizing clusters In the planar case, the constant mean curvature condition satisfied by the interfaces

$$\mathcal{E}(h,k) = \partial^*\mathcal{E}(h) \cap \partial^*\mathcal{E}(k)\,, \qquad 0 \le h < k \le N\,,$$

implies that each $\mathcal{E}(h,k)$ is a countable union of circular arcs, all with the same curvature $\kappa_{h\,k} \in \mathbb{R}$ (here, a straight segment is a circular arc with zero curvature); moreover, the blow-up clusters $\mathcal{E}_{x,r} = (\mathcal{E}-x)/r$ of $\mathcal{E}$ at a point x belonging to the singular set $\Sigma(\mathcal{E})$ of $\mathcal{E}$,

$$\Sigma(\mathcal{E}) = \bigcup_{0\le h<k\le N} \Big(\partial\mathcal{E}(h) \cap \partial\mathcal{E}(k)\Big) \setminus \mathcal{E}(h,k) = \bigcup_{h=0}^{N} \Big(\partial\mathcal{E}(h) \setminus \partial^*\mathcal{E}(h)\Big)\,,$$

have the planar Steiner partition (see Figure 30.2) as their unique (up to rotations) possible limit in local convergence. Exploiting these two facts, in Theorem 30.7 we shall prove that the singular set is discrete, that every point $x \in \Sigma(E)$ is the junction of exactly three different interfaces, that the three circular arcs meeting at x form three 120-degree angles, and that each interface is made up of finitely many circular arcs (all with the same curvature); see Figure 1 and Section 30.3.

These general rules which planar minimizing clusters have to obey provide the starting point for attempting their characterization, at least in some special cases. For example, planar double bubbles have been characterized as the only planar minimizing 2-clusters in [FAB+93], and as the only *stable* (vanishing first variation and non-negative second variation) 2-clusters in [MW02]. There also exists a characterization of planar 3-clusters,

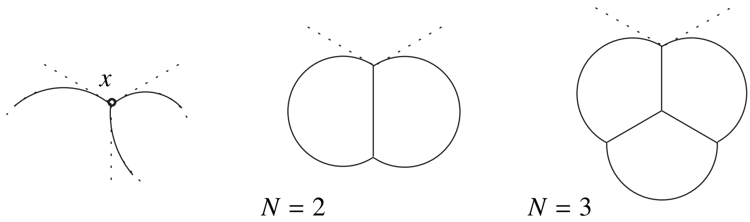

Figure 1 For planar clusters, the singular set is discrete, and the interfaces are circular arcs meeting in threes at singular points forming 120 degree angles. Starting from this information it is possible to characterize planar minimizing clusters with two and three chambers (the picture refers to the case in which the various chambers have equal areas).

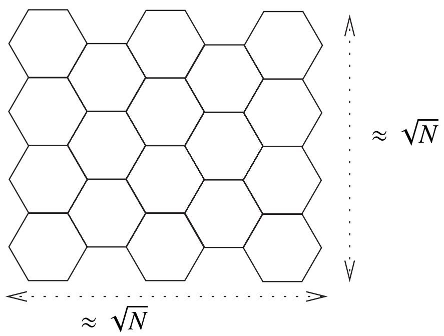

Figure 2 The honeycomb inequality: symmetric honeycombs (with unit area cells) provide the sharp asymptotic lower bound on the ratio perimeter over number of chambers for planar clusters with unit area chambers.

obtained in [Wic04]. Interestingly, no example of planar minimizing N-clusters is known if $N \geq 4$, although a list of possible candidates has been proposed in [CF10].

Describing the asymptotic properties of planar minimizing N-clusters as $N \to \infty$ provides another source of interesting questions. In this way, an interesting result is the so-called *honeycomb theorem* [Hal01]. A possible formulation of this result is as follows: if $\mathcal{E}$ is an N-cluster in $\mathbb{R}^2$ with $|\mathcal{E}(h)| = 1$ for every $h = 1, \dots, N$, then

$$\frac{P(\mathcal{E})}{N} > 2(12)^{1/4}\,. \tag{1}$$

This lower bound is sharp: if $\{\mathcal{E}_N\}_{N\in\mathbb{N}}$ denotes a sequence of planar N-clusters obtained by piling up approximately $\sqrt{N}$ rows consisting of approximately $\sqrt{N}$ many regular hexagons of unit area, then $P(\mathcal{E}_N)/N \to 2(12)^{1/4}$ as $N \to \infty$; see Figure 2. Moreover, in a sense that can be made precise, this is essentially the *unique* type of sequence $\{\mathcal{E}_N\}_{N\in\mathbb{N}}$ which asymptotically saturates (1).

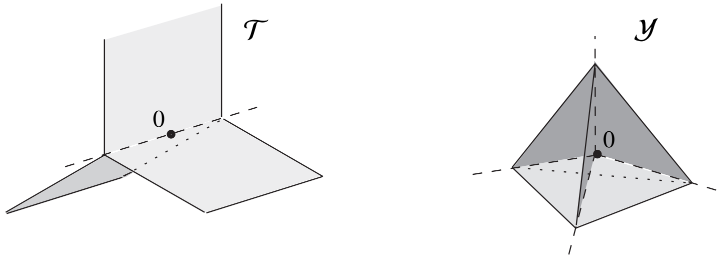

Figure 3 The only two possible tangent clusters at a singular point of a minimizing cluster in $\mathbb{R}^3$. The cone-like cluster $\mathcal{T}$ has as its interfaces three half-planes meeting along a line at 120 degree angles. The cone-like cluster $\mathcal{Y}$ has as its interfaces six planar angles of about 109 degrees of amplitude, which form the cone generated by the center of a regular tetrahedron and its edges. In a neighborhood of any of its singular points, a minimizing cluster in $\mathbb{R}^3$ is a $C^{1,\alpha}$-diffeomorphic image of either $B \cap \mathcal{T}$ or $B \cap \mathcal{Y}$.

Structure of singularities in higher dimension The analysis of singular sets of three-dimensional minimizing clusters was settled by Taylor in [Tay76]. This is considered a historical paper, since it provided the first complete mathematical justification of the equilibrium laws governing soap bubbles stated by the Belgian physicist Plateau in the nineteenth century. There it is proved that if x is a singular point for a minimizing cluster $\mathcal{E}$ in $\mathbb{R}^3$, then, up to rotations, the blow-up clusters $\mathcal{E}_{x,r} = (\mathcal{E} - x)/r$ of $\mathcal{E}$ at x locally converge as $r \to 0^+$ either to the 3-cluster $\mathcal{T}$ or the 4-cluster $\mathcal{Y}$ depicted in Figure 3. (Of course, according to our terminology, $\mathcal{T}$ and $\mathcal{Y}$ are not properly “clusters” as their chambers have infinite volumes.) Moreover, there exist an open neighborhood A of x in $\mathbb{R}^3$, $\alpha \in (0, 1)$, and a $C^{1,\alpha}$-diffeomorphism $f\colon \mathbb{R}^3 \to \mathbb{R}^3$, such that either

$$A \cap \mathcal{E} = f\Big(B \cap \mathcal{T}\Big) \qquad \text{or} \qquad A \cap \mathcal{E} = f\Big(B \cap \mathcal{Y}\Big),$$

where, as usual, B is the Euclidean unit ball in $\mathbb{R}^3$ centered at the origin. It should also be noted that Taylor’s theorem actually applies to describe the singularities of (roughly speaking) any $\mathcal{H}^2$-rectifiable set M in $\mathbb{R}^3$ satisfying a suitably perturbed area minimality condition. In this way, Taylor’s result has been extended to two-dimensional almost minimal rectifiable sets in $\mathbb{R}^n$ ($n \geq 3$) by David [Dav09, Dav10]. The extension of Taylor’s theorem to the case of minimizing clusters in higher dimensions has been announced by White [Whi]. We finally remark that not much is known about general qualitative properties of minimizing clusters in dimension $n \geq 3$. For example, Tamanini [Tam98] has proved the existence of a constant $k(n)$ bounding the number of chambers

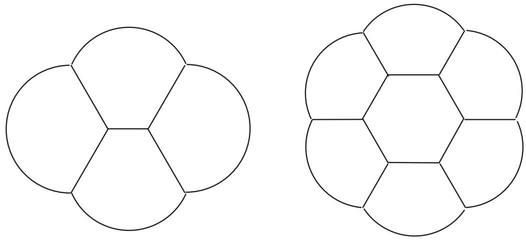

Figure 4 Conjectured minimizing clusters with 4 and 7 chambers which possess discrete groups of symmetries.

of a minimizing N-cluster which may meet at a given point of $\mathbb{R}^n$. However, no explicit bound on $k(n)$ is presently known.

Symmetry properties of minimizing clusters If $n \geq 2$ and $N \leq n - 1$, then, given N sets in $\mathbb{R}^n$ with finite Lebesgue measure, we may find $n - (N - 1)$ mutually orthogonal hyperplanes which cut each of the given sets into two halves of equal measure (this, by repeatedly applying a Borsuk–Ulam type argument). Notice that the intersection of $n - (N - 1)$ mutually orthogonal hyperplanes in $\mathbb{R}^n$ defines a $(N - 1)$-dimensional plane in $\mathbb{R}^n$. In this way, by standard reflection arguments (see, e.g., Section 19.5), we see that if $\mathbf{m} \in \mathbb{R}^N_+$ and $n - 1 \geq N$, then there always exists a minimizing N-cluster $\mathcal{E}$ in $\mathbb{R}^n$ with $\mathbf{m}(\mathcal{E}) = \mathbf{m}$, which is symmetric with respect to an suitable $(N - 1)$-dimensional plane of $\mathbb{R}^n$. Developing an idea due to White, Hutchings [Hut97] has actually proved that if $N \leq n - 1$, then *every* minimizing N-cluster in $\mathbb{R}^n$ is symmetric with respect to an $(N - 1)$-dimensional plane of $\mathbb{R}^n$ (a proof of this result in the language of sets of finite perimeter is presented in [Bon09]). The Hutchings–White theorem is the only general symmetry result for minimizing clusters known at present, although it is reasonable to expect that symmetries should appear also for special values of N and n outside the range $N \leq n - 1$; see Figure 4.

The double bubble theorem In dimension $n \geq 3$, the only characterization result for minimizing N-clusters concerns the case $N = 2$. The starting point is the Hutchings–White theorem, which guarantees minimizing 2-clusters in $\mathbb{R}^n$ $(n \geq 2)$ to be axially symmetric. In particular, the interfaces of a minimizing 2-cluster are constant mean curvature *surfaces of revolution*. This piece of information allows us to restrict the focus, so to say, on the mutual position of the various connected components of the chambers. Then, by careful first and second variations arguments, one comes to exclude all the alternative possibilities to the case of a double bubble, which is therefore the only minimizing

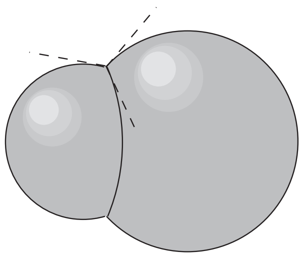

Figure 5 The topological boundary of a double bubble consists of three spherical caps which meet in a $(n-2)$-dimensional sphere forming three angles of 120 degrees.

2-cluster; see Figure 5. This beautiful result has been obtained by Hutchings, Morgan, Ritoré, and Ros in [HMRR02] in $\mathbb{R}^3$, and has been later extended to higher dimensions by Reichardt and collaborators [RHLS03, Rei08]. It should be noted that, at present, no characterization result for minimizing clusters is available in dimension $n \geq 3$ if $N \geq 3$. Another difficult problem is that of extending the honeycomb theorem (1) to higher dimensions. Following a conjecture by Lord Kelvin, it was believed for a long time that the asymptotic optimal tiling in $\mathbb{R}^3$ should be the one obtained by piling layers of relaxed truncated octahedra. Weaire and Phelan [WP94], however, disproved Kelvin's conjecture by showing a better competitor; see [Mor09, Chapter 15] for pictures and details.

Multi-phase anisotropic partition problems Finally, Almgren's existence and partial regularity theory applies to a wide class of partitioning problems, including the volume-constrained minimization of functionals of the type

$$\sum_{0\le h<k\le N} c_{hk} \int_{\mathcal{E}(h,k)} \Phi\Big(x, \nu_{\mathcal{E}(h)}(x)\Big)\, d\mathcal{H}^{n-1}(x)\,,$$

under suitable assumptions on the coefficients $c_{hk} > 0$ and on the anisotropy $\Phi\colon \mathbb{R}^n \times S^{n-1} \to [0,\infty)$. Given $\mathbf{m} \in \mathbb{R}^N_+$, the volume-constrained minimization of this energy in the isotropic case leads to the *immiscible fluids problem*,

$$\inf\Big\{ \sum_{0\le h<k\le N} c_{hk}\, \mathcal{H}^{n-1}\Big(\mathcal{E}(h,k)\Big) : \mathcal{E} \text{ is an } N\text{-cluster}, \mathbf{m}(\mathcal{E}) = \mathbf{m} \Big\}.$$

We thus see the different chambers of the clusters as the regions occupied by possibly different fluids. The relative strengths of the mutual interactions between these different fluids are then weighted by the positive constants c_{hk}.

The lower semicontinuity of the multi-phase interaction energy is equivalent to the validity of the triangular inequality $c_{hk} \le c_{hi} + c_{ik}$ (Ambrosio and Braides [AB90], White [Whi96]). Assuming the strict triangular inequality $c_{hk} < c_{hi} + c_{ik}$, the regularity of the interfaces for minimizers is then addressed by reduction to the regularity theory for volume-constrained perimeter minimizers. The key tool to obtain this reduction is an *infiltration lemma* [Whi96, Leo01], which will be discussed (in the simple case when the c_{hk} are all equal) in Section 30.1.

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