STABILITY THEOREMS & APPLICATIONS TO PHASE TRANSITIONS

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TWO MODELS FOR DROPLETS FORMATION

1 GAUSS FREE ENERGY — BASED ON

SHARP INTERFACE MODEL — CIRAOLO-M. 2015

CLASSICAL CAPILLARITY THEORY — KRUMMEL-M. 2016

2 GATES-PENROSE-LEBOWITZ (GPL) FREE ENERGY

DIFFUSED INTERFACE MODEL

STATISTICAL MECHANICS

NONLOCAL RELATIVE OF ALLEN-CAHN FREE ENERGY

BASED ON CARLEN-M. 2015

FIGALLI-M.-MOONEY 2016

PART ONE

GAUSS FREE ENERGY - NO CONTAINER

$E \subseteq \mathbb{R}^{n+1}$ DROPLET REGION

VOLUME OF E = $|E| = m$ FIXED

$$\underbrace{\text{GAUSS FREE ENERGY - NO CONTAINER}} = \text{SURFACE TENSION} + \text{POTENTIAL ENERGY} = \underbrace{P(E)} + \int_E g(x)\,dx$$

PERIMETER OF $E = \mathcal{H}^n(\partial E)$

$\boxed{m \text{ IS SMALL!}}$ $\Rightarrow$ $\boxed{P(E) = O(m^{n/n+1}) \gg \int_E g = O(m)}$

GLOBAL MINIMIZERS $\Rightarrow$ ALMOST ISOPERIMETRIC

CRITICAL POINTS $\Rightarrow$ ALMOST CONSTANT MEAN CURV.

GLOBAL MINIMIZERS $\Rightarrow$ ALMOST ISOPERIMETRIC

$$1 + O(m^{1/(n+1)}) \geq \frac{P(E)}{c_{iso} |E|^{n/(n+1)}} \geq 1$$

IMPROVED ISOPERIMETRY

FUSCO M. PRATELLI (08)

$$\frac{P(E)}{c_{iso} |E|^{n/(n+1)}} \geq 1 + c(n) \min_{x \in \mathbb{R}^{n+1}} \left[\frac{|E \Delta B^{(m)}(x)|}{m} \right]^2$$

ADDITIONAL VARIATIONAL ARGUMENTS

FIGALLI M. (10)

∂E IS QUANTITAVELY C^2-CLOSE TO A SPHERE

IN A CONTAINER M.-MIHAILA (15)

CRITICAL POINTS $\Rightarrow$ ALMOST CONSTANT MEAN CURV.

$$\delta_{CMC}(E) = \left\| \frac{H_{\partial E}}{H^0_{\partial E}} - 1 \right\|_{C^0(\partial E)} = O\left(m^{1/n+1}\right)$$

C^0- CONST. MEAN CURV. DEFICIT

$H_{\partial E}$ MEAN CURV. OF ∂E

$$H^0_{\partial E} = \frac{n\, P(E)}{(n+1)\,|E|}$$

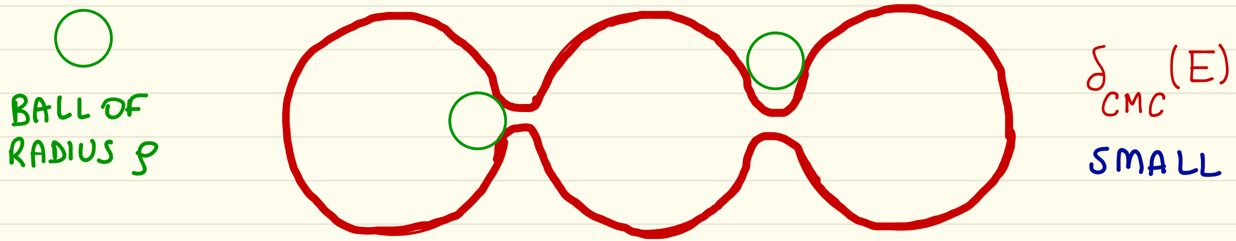
ALEXANDROU THM $\quad H_{\partial E} \equiv c \Longrightarrow \partial E$ SPHERE $\left(\& \ c = H^0_{\partial E}\right)$

CRITICAL POINTS $\Longrightarrow$ ALMOST CONSTANT MEAN CURV.

CIRAOLO-VEZZONI (15)
$$\begin{cases} \partial E \text{ EXT/INT BALL RADIUS } \rho > 0 \\ H^0_{\partial E} = n = H_{\partial B} \quad B = \text{UNIT BALL} \end{cases}$$

$$\Longrightarrow \quad \mathrm{hd}\left(\partial E, \partial B(x)\right) \le C(n, \rho, P(E))\, \delta_{CMC}(E)$$

EXT/INT BALL RADIUS $\rho > 0$ NOT TRUE ON ALMOST CMC

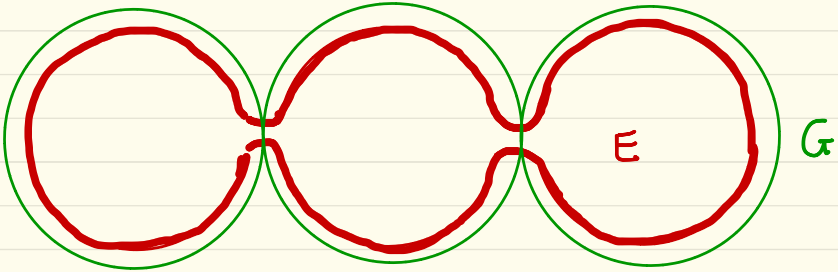


CRITICAL POINTS $\Rightarrow$ ALMOST CONSTANT MEAN CURV.

CIRAOLO M. (15)
$$\begin{cases} H^{\circ}_{\partial E} = n & P(E) \le (L+\tau) P(B) \quad L \in \mathbb{N} \quad 0<\tau<1 \\ \delta_{CMC}(E) \le \delta_0(n,L,\tau) \end{cases}$$

$\Rightarrow$ ∂E QUANTITAVELY $C^{1,\alpha}$ CLOSE TO UNION OF UNIT RADIUS TANGENT BALLS G

E.G. $$\frac{|E \Delta G|}{|E|} \le C(n) L^2 \delta_{CMC}(E)^{\frac{1}{2n+4}}$$ & MANY OTHER ESTIMATES



CRITICAL POINTS $\Rightarrow$ ALMOST CONSTANT MEAN CURV.

KRUMMEL-M. 16
$$\begin{cases} H^{\circ}_{\partial E} = n & P(E) \le (1+\tau)\,P(B) \quad 0<\tau<1 \\ \delta_{CMC}(E) \le \delta_0(n,\tau) \end{cases}$$

$$\Rightarrow \quad \partial E = \{ (1+u(x))\,x : x \in \partial B_1 \} \qquad \|u\|_{C^{1,\alpha}} \le C(n,\tau)\,\delta_{CMC}(E)$$

IN FACT we can use (AT LEAST IF $n \le 3$)

$$\delta^{*}_{CMC}(E) = \min \left\{ \left\| \left(\frac{H_{\partial E}}{H^{\circ}_{\partial E}} - 1 \right)^{+} \right\|_{C^{0}(\partial E)} , \left\| \left(\frac{H_{\partial E}}{H^{\circ}_{\partial E}} - 1 \right)^{-} \right\|_{L^2(\partial E)} \right\}$$

RELATED TO ALMGREN ISOPERIMETRIC PRINCIPLE

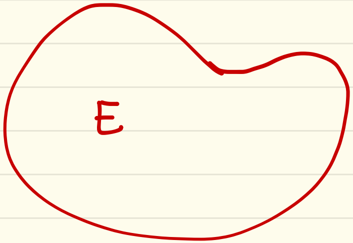
ALMGREN ISOPERIMETRIC PRINCIPLE

(CODIMENSION 1 VERSION)

IF $H_{\partial E} \leq n$ THEN $P(E) \geq P(B_1)$

ALMGREN ISOPERIMETRIC PRINCIPLE

IF $H_{\partial E} \le n$ THEN $P(E) \ge P(B_1)$



$$P(B_1) = \mathcal{H}^n(S^n) = \int_{\partial A} |D\nu_A|$$

$$= \int_{\partial A} K_{\partial A} = \int_{\partial A \cap \partial E} K_{\partial A}$$

$$\le \int_{\partial A \cap \partial E} \left(H_{\partial A}/n\right)^n$$

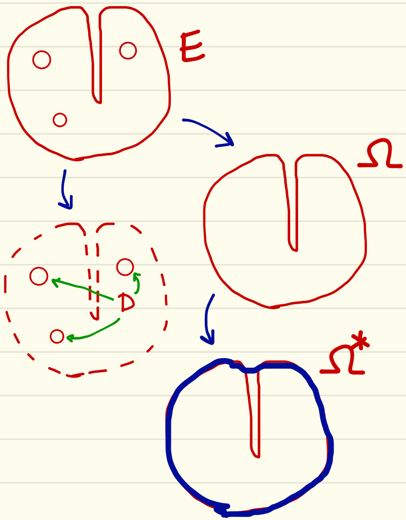
$$\le \mathcal{H}^n(\partial A \cap \partial E) \le \mathcal{H}^n(\partial E) = P(E)$$

RMK 1 EQUALITY HOLDS

$\iff E = B_1(x)$

RMK 2 YES! IT REMINDS A LOT ABP!

KRUMMEL-M. 16 IF $H_{\partial E} \le n$ & $\delta(E) = P(E) - P(B_1) \le \delta_0(n)$ SMALL



THEN $\partial E = \partial D \cup \partial \Omega$

D "DUST SET" $P(D) \le C(n)\,\delta(E)$

$\partial\Omega$ CONNECTED

$\Omega \subseteq \Omega^*$ $\quad |H_{\partial\Omega^*}| \le n$

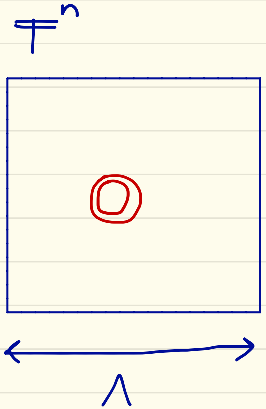
$$|\Omega^* \setminus \Omega| + \mathcal{H}^n(\partial\Omega^* \setminus \partial\Omega) \le C(n)\,\delta(E)$$

$$\partial\Omega^* = \{(1+u(x))\,x : x \in \partial B_1\}$$

$$\|u\|_{W^{1,1}} + \|u^+\|_{C^0} \le C(n)\,\delta(E)$$

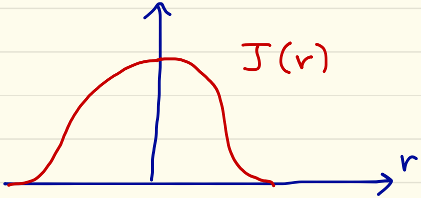
$$\|u\|_{C^0} \le C(n) \begin{cases} \delta(E) & n=1 \\ \delta(E)\log\left(C(2)/\delta(E)\right) & n=2 \\ \delta(E)^{1/n-1} & n \ge 3 \end{cases}$$

PART TWO GPL FREE ENERGY



$u : T^n \longrightarrow (-1,1)$

$\int_{T^n} u = m \in (-1,1)$



$$\mathcal{F}(u) = \iint_{T^n \times T^n} |u(x) - u(y)|^2 J(|x-y|)\,dx\,dy + \int_{T^n} W(u(x))\,dx$$

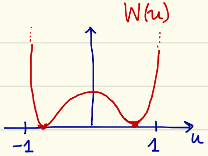
IF m/Λ SMALL* & u LOW ENERGY

THEN $u(x) \approx$ SHARP TRANSITION ALONG A SMALL SPHERE

LOW ENERGY = LIKELY TO BE OBSERVED

* BUT NOT TOO MUCH... OTHERWISE UNIFORM STATE ; SEE CARLEN SURVEY

PART TWO GPL FREE ENERGY



$$u : \mathbb{R}^n \to (-1,1) \qquad \int_{\mathbb{R}^n} u = m \in (-1,1) \qquad \int_{\mathbb{R}^n} J = 1$$

$$\mathcal{F}(u) = \iint_{\mathbb{R}^n \times \mathbb{R}^n} |u(x)-u(y)|^2 J(|x-y|)\,dx\,dy + \int_{\mathbb{R}^n} W(u(x))\,dx$$

$$\underbrace{\geq}_{\text{BY RIESZ REARRANGEMENT INQ.}} \mathcal{F}(u^*)$$

u^* SHWARTZ REARRANG. u

BY RIESZ REARRANGEMENT INQ.

CARLEN'S SURVEY FOR MORE GPL

① RADIALLY DECREASING LOW ENERGY STATES

CARLEN - CARVALHO - ESPOSITO

LEBOWITZ - MARRA (09)

② QUANTITATIVE RIESZ REARRANGEMENT INQ

CARLEN - M. (15)

BASED ON A QUANTITATIVE BRUNN-MINKOWSKI INEQUALITY

BM INEQUALITY $\qquad |E+F|^{1/n} \geq |E|^{1/n} + |F|^{1/n} \quad \forall E,F \subseteq \mathbb{R}^n$

$=$ E,F HOMOTHETIC CONVEX SETS

QUANTITATIVE BM

$$\delta(E,F) = \max\left\{\frac{|F|}{|E|}, \frac{|E|}{|F|}\right\}\left(\frac{|E+F|^{1/n}}{|E|^{1/n}+|F|^{1/n}} - 1\right)$$

$$\alpha(E,F) = \inf\left\{\frac{|E \triangle (x+\lambda F)|}{|E|} \text{ s.t. } |\lambda F| = |E|\right\}$$

① FIGALLI M. PRATELLI $\qquad E,F$ CONVEX $\qquad \delta(E,F) \geq c(n)\,\alpha(E,F)^2$

② FIGALLI JERISON CHRIST $\qquad E,F$ GENERIC $\quad$ NON-SHARP ESTIMATES

③ CARLEN-M. $\qquad E$ GENERIC, F CONVEX $\qquad \delta(E,F)\max\left\{1, \frac{|F|}{|E|}\right\}^{4+\frac{2}{n}} \geq c(n)\,\alpha(E,F)^4$

E GENERIC $F = B_r$ $r > 0$ $\Rightarrow$ BM BECOMES

EUCLIDEAN CONCENTRATION INEQUALITY

$$|I_r(E)| \geq |I_r(B_{r_E})| \quad \forall r > 0$$

$I_r(E) = \{$ DIST FROM $E < r \}$

SHARP QUANTITATIVE EUCLIDEAN CONCENTRATION INEQUALITY

FIGALLI-M.-MOONEY (16)

$$|I_r(E)| \geq |I_r(B_{r_E})| \left\{ 1 + c(n) \min\left\{ \frac{r}{r_E}, \frac{r_E}{r} \right\} \left[\frac{|E \Delta B_{r_E}(x)|}{|E|} \right]^2 \right\}$$

* NO SYMMETRIZATION (FUSCO M PRATELLI)

* NO MASS TRANSP. (FIGALLI M PRATELLI)

* NO REGUALIRITY (CICALESE LEONARDI)