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Rigidity of equality cases in symmetrization inequalities

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(joint work with F. Cagnetti, M. Colombo, and G. De Philippis)

Symmetrization inequalities, and, in particular, necessary conditions for their equality cases, are commonly used to prove symmetry properties of minimizers in geometric variational problems. The archetypical example of this method is Steiner's proof of the isoperimetric inequality, as made rigorous in the context of sets of finite perimeter by De Giorgi [6]. We consider here the problem of understanding *rigidity* of equality cases in a given symmetrization inequality. Sufficient conditions for rigidity of equality cases have been obtained in the case of the Polya-Szego inequality for Dirichlet-type integrals (Brothers and Ziemer [2]), and in the case of the Steiner's perimeter inequality for perimeter of sets (Chlebík, Cianchi and Fusco [5]). We have obtained geometric conditions that actually *characterize* rigidity in two model examples: Ehrhard's symmetrization for Gaussian perimeter [3], and Steiner's symmetrization for Euclidean perimeter [4]. We focus here on the latter problem.

Consider a Borel function $v : \mathbb{R}^{n-1} \to [0, \infty)$, and let $F[v]$ be the sets of points $x = (x', x_n) \in \mathbb{R}^n$ such that $|x_n| < v(x')/2$. Given a set $E \subset \mathbb{R}^n$, one denotes by $E_z = \{t \in \mathbb{R} : (z, t) \in E\}$ the vertical section of E above $z \in \mathbb{R}^{n-1}$, says that E is v-distributed if $v(z) = \mathcal{H}^1(E_z)$ for $\mathcal{H}^{n-1}$-a.e. $z \in \mathbb{R}^{n-1}$, and sets $E^s = F[v]$ for the *Steiner's symmetral of* E. Steiner's inequality gives

$$P(E) \geq P(F[v]) \,, \qquad \text{for } E \subset \mathbb{R}^n \ v\text{-distributed} \,, \tag{0.1}$$

where $P(E)$ denotes the distributional perimeter of E. (We notice that $P(F[v]) < \infty$ if and only if

$$v \in BV(\mathbb{R}^{n-1}) \,, \qquad \mathcal{H}^{n-1}(\{v > 0\}) < \infty \,, \tag{0.2}$$

where $BV(\mathbb{R}^{n-1})$ is the space of functions of bounded variation on $\mathbb{R}^{n-1}$.) Let us denote by $\mathcal{M}(v)$ the set of equality cases in (0.1). The rigidity problem amounts in characterizing those function v as in (0.2) such that

$$\mathcal{M}(v) = \Big\{ t\, e_n + F[v] : t \in \mathbb{R} \Big\} \,. \tag{0.3}$$

The inclusion $\supset$ is, of course, trivial, while the inclusion $\subset$ may fail for various reasons: (i) the projection $\{v > 0\}$ of $F[v]$ could be "disconnected"; (ii) the set $\{v = 0\}$ may "disconnect" the projection; (iii) it could be that the jump set of

v "disconnects" the projection; (iv) the projection could contain an "integrable" Cantorian part of Dv. Chlebík, Cianchi and Fusco in [5] provide a sufficient condition for rigidity by ruling out these four possibilities.

Theorem 0.1 (Chlebík, Cianchi, Fusco [5])**.** *If (a) Ω is an open connected set, (b) $v \in W^{1,1}(\Omega)$, and (c) the Lebesgue representative of v is positive $\mathcal{H}^{n-2}$-a.e. on Ω, then $P(E;\Omega\times\mathbb{R}) = P(F[v];\Omega\times\mathbb{R})$ implies the existence of $t\in\mathbb{R}$ such that $E\cap(\Omega\times\mathbb{R}) = (t\,e_n + F[v])\cap(\Omega\times\mathbb{R})$.*

Notice that assumptions (a) and (c) – together with the choice of working with the localized Steiner's inequality over $\Omega\times\mathbb{R}$ – exclude problems (i) and (ii), while assumption (b) excludes the existence of jump or Cantorian parts of Dv inside Ω, and thus rules out problems (iii) and (iv). Simple examples shows that Theorem 0.1 does not characterize rigidity even in the case of polyhedra. In order to improve on this result one thus needs to understand rigidity in the presence of jumps or Cantorian parts of Dv, or of substantially large regions where v vanishes. Since the various sets involved in this heuristic statements are just Borel sets, we first need to specify in which sense a Borel set K disconnects another Borel set G. Precisely, in [3] we introduce the following definition: if K and G are Borel sets in $\mathbb{R}^m$, then K *essentially disconnects* G if there exists a non-trivial Borel partition $\{G_+, G_-\}$ of G such that

$$G^{(1)}\cap\partial^{\rm e}G_+\cap\partial^{\rm e}G_- \subset_{\mathcal{H}^{m-1}} K\,.$$

(Here, $G^{(t)}$ is the set of point of density $t\in[0,1]$ of G, and $\partial^{\rm e}G = \mathbb{R}^m\setminus(G^{(0)}\cup G^{(1)})$.) Notice that if $\mathcal{H}^{m-1}(K\Delta K') = \mathcal{H}^m(G\Delta G') = 0$, then K essentially disconnects G if and only if K' essentially disconnects G'. Moreover, we say that G is essentially connected if the empty set does not essentially disconnect G. When G is of finite perimeter, this is equivalent to asking that G is indecomposable in the sense of [7], [1]; see also [8, 4.2.25].

Let us now denote by $v^\wedge$ and $v^\vee$ the lower and upper approximate limits of v (so that $v^\wedge$ and $v^\vee$ are pointwise unambiguously defined on $\mathbb{R}^{n-1}$ in $\mathcal{H}^{n-1}$-equivalence class of v), let $[v] = v^\vee - v^\wedge$ denote the jump of v, and let $S_v = \{[v]>0\}$. Starting from a sharp regularity result for barycenter functions of sets with segments as sections, see Theorem 0.2, we can use these notions to formulate several characterizations of rigidity.

Theorem 0.2. *If E is a v-distributed set with segments as vertical sections and $b_E(z)$ denotes the barycenter of E_z, then $b_{M,\delta} = \tau_M(1_{\{v>\delta\}}\,b_E)\in BV(\mathbb{R}^{n-1})$ for a.e. $\delta, M>0$, where $\tau_M(s)=\max\{-s,\min\{M,s\}\}$, $s\in\mathbb{R}$. Moreover, $E\in\mathcal{M}(v)$ if and only if the approximate gradient ∇b_E of b_E vanishes $\mathcal{H}^{n-1}$-a.e. on $\mathbb{R}^{n-1}$, $2\,[b_E]\le[v]$ $\mathcal{H}^{n-2}$-a.e. on $\{v^\wedge>0\}$, and $D^cb_{M,\delta} = f_{M,\delta}\,D^cv$ for a Borel function $f_{M,\delta}:\mathbb{R}^{n-1}\to[-1/2,1/2]$.*

Theorem 0.3. *If v satisfies* (0.2) *and $D^sv\llcorner\{v^\wedge>0\}=0$, then rigidity holds if and only if $\{v^\wedge=0\}$ does not essentially disconnect $\{v>0\}$. This last condition is in turn equivalent in asking that $F[v]$ is indecomposable.*

Theorem 0.4. *If $F[v]$ is a generalized polyhedron (roughly speaking, v consists of finitely many Sobolev functions over finitely many indecomposable sets of finite perimeter), then rigidity holds if and only if $\{v^\wedge = 0\} \cup \{[v] > \epsilon\}$ does not essentially disconnect $\{v > 0\}$.*

Theorem 0.5. *If $v \in SBV(\mathbb{R}^{n-1})$ (i.e. $D^c v = 0$) and S_v is locally $\mathcal{H}^{n-2}$-finite, then every $E \in \mathcal{M}(v)$ is obtained by countably many vertical translations of $F[v]$ (above disjoint Borel sets in $\mathbb{R}^{n-1}$). In particular, rigidity holds if and only if v has the following mismatched stairway property: If $\{G_h\}_{h\in I}$ is a Borel partition of $\{v > 0\}$ with $\sum_{h\in I} P(G_h \cap B_R \cap \{v > \delta\}) < \infty$ for a.e. $\delta, R > 0$, and if $\{c_h\}_{h\in I} \subset \mathbb{R}$ is a sequence with $c_h \neq c_k$ whenever $h \neq k$, then there exist $h_0, k_0 \in I$ with $h_0 \neq k_0$, and a Borel set Σ with*

$$\Sigma \subset \partial^{\mathrm{e}} G_{h_0} \cap \partial^{\mathrm{e}} G_{k_0} \cap \{v^\wedge > 0\}\,, \qquad \mathcal{H}^{n-2}(\Sigma) > 0\,,$$

such that $[v](z) < 2|c_{h_0} - c_{k_0}|$ for every $z \in \Sigma$.

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Uniformization of surfaces with conical singularities

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(joint work with D.Bartolucci, A.Carlotto, F.De Marchis, D.Ruiz)

We study some singular equations, motivated by the problem of the Gaussian curvature prescription, and from some models in physics such as self-dual Chern-Simons theory or Electroweak theory: we prove some existence results exploiting the variational structure of the problem.

Consider a compact surface Σ endowed with a metric g: with the conformal change of metric $\tilde{g} = e^{2w} g$ one has

$$-\Delta_g w + K_g = K_{\tilde{g}} e^{2w},$$