the identification of a natural scale-invariant problem and a way of localizing the evolution. For the former, we consider peeling a Poission cloud inside an infinite paraboloid. For the latter, we sharpen the original estimates of Dalal. The Martingale argument implies homogenization of the scale-invariant problem. One concludes the theorem by invoking the uniqueness of viscosity solutions via the perturbed test function method.

References

[1] K. Dalal, *Counting the onion*, Random Structure & Algorithms **24** (2004), no. 2, 155-165

[2] S. Armstrong and P. Cardaliaguet, *Stochastic homogenization of quasilinear Hamilton-Jacobi equations and geometric motions*, preprint, arXiv:1504.02045

Sharp stability for the Euclidean concentration inequality and droplets formation in statistical mechanics

Francesco Maggi

(joint work with Eric A. Carlen, Alessio Figalli, Connor Mooney)

The starting point of this study if the analysis of liquid-vapor phase transitions in a model from statistical mechanics, based on the minimization of the *Gates-Penrose-Lebowitz* (GPL) free energy

$$GPL(u) = \frac{1}{2}\int_{T_L} dx \int_{T_L} J(|x-y|)\,|u(x)-u(y)|^2\,dy + \int_{T_L} W(u)\,.$$

Here T_L denotes a n-dimensional flat torus of side length L, $J(r)$ is a bounded decreasing interaction kernel with compact support on $[0,1]$ ($L \gg 1$) such that $\int_{\mathbb{R}^n} J(|x|)\,dx = 1$, $u : T_L \to (-1,1)$ represents a particle-hole density, and $W : (-1,1) \to [0,\infty)$ is an even, smooth, double-well potential, with $W(\pm m_0) = 0$ and $W''(m_0) > 0$ for some $m_0 \in (0,1)$. We stress that the length scale L is large compared to the length scale of the interaction kernel, which was set to unit by requiring $\mathrm{spt}\, J = [0,1]$.

Given a volume fraction $m \in (-1,1)$, one minimizes $GPL(u)$ under the constraint that $L^{-n}\int_{T_L} u = m$. Very much like in the case of the Cahn-Hilliard free energy, the double-well favors two constant states (namely, $u \equiv m_0$ and $u \equiv -m_0$) and the interaction energy penalizes variations. In particular, when $m = \pm m_0$ there is no doubt that the constant states are the unique minimizers. For other volume fractions m we expect to see a competition between the two terms in the energy, leading to transition profiles u between a m_0-phase and a $(-m_0)$-phase.

Because of this competition, in both models, one expects the formation of almost spherical "droplets", whenever $m \in (-m_0, m_0)$ and L is large enough. An heuristic analysis shows that this should also happen when $m \to \pm m_0$ as $L \to \infty$, and precisely for $m = -m_0 + K\,L^{-n/(n+1)}$ with K larger than some critical K_*. This kind of study for the Cahn-Hilliard model has been addressed, independently, in [3, 4]. There are two significant differences between the Cahn-Hilliard and the GPL models: first, since the interaction kernel J is not singular, minimizers of

GPL possess no smoothness property, and second, because of the statistical origin of the model, one is actually interested in understanding all near-minimizers of GPL, as the most likely observed states of the system. On a deeper level, almost sphericity of droplets is related to the Euclidean isoperimetric inequality in the Cahn-Hilliard case, and to the Euclidean concentration inequality in the GPL case. As explained below, a quantitative analysis of near-minimizers is definitely subtler for Euclidean concentration than for Euclidean isoperimetry.

Why round droplets? Guessing that near-minimizing u are sharp transitions between the constant densities m_0 and $-m_0$, concentrated along the boundary of $\{u \geq m_0\}$, and with $\operatorname{diam}(\{u \geq m_0\})$ way smaller than L, one should be able to argue as if $T_L \approx \mathbb{R}^n$. On the whole space, it makes sense to compare u by its spherically symmetric decreasing rearrangement u^*, whose super-level sets are balls with same volume as the corresponding super-level sets of u. This equimensurability property guarantees that $\int_{\mathbb{R}^n} g(u) = \int_{\mathbb{R}^n} g(u^*)$ for every $g : \mathbb{R} \to \mathbb{R}$, and thus, by combining the identity

$$GPL(u) = \int_{\mathbb{R}^n} u^2 - \int_{\mathbb{R}^n} dx \int_{\mathbb{R}^n} J(|x-y|)\, u(x)\, u(y)\, dy$$

(recall that $\int_{\mathbb{R}^n} J(|x|)\, dx = 1$) with the *Riesz rearrangement inequality*

$$\int_{\mathbb{R}^n} dx \int_{\mathbb{R}^n} J(|x-y|)\, u(x)\, u(y)\, dy \leq \int_{\mathbb{R}^n} dx \int_{\mathbb{R}^n} J(|x-y|)\, u^*(x)\, u^*(y)\, dy$$

we deduce that $GPL(u) \geq GPL(u^*)$. In particular, if u is a minimizer or a near-minimizer, so it is u^*. The quantitative analysis of radially decreasing near-minimizers of the GPL model in the spherical droplet regime has been addressed in [1, 2, 5]. The next step is thus understanding how far a generic near minimizer u is from being almost spherical, i.e. how to control the distance of u from u^* in terms of $GPL(u) - GPL(u^*)$. Considering the discussion of equality cases in the Riesz rearrangement inequality can be addressed in terms of a discussion of equality cases for the Euclidean concentration inequality, there are two main problems to address:

(i) provide a stability estimate for the Euclidean concentration inequality;
(ii) exploit such an estimate to obtain a robust improvement of the Riesz rearrangement inequality.

Both problems are addressed in the joint paper [6] with Eric Carlen, by exploiting suitable geometric arguments. These results pave the a way to a quantitative description of every near-minimizer of the GPL free energy in the droplet regime.

The paper [6] also indicate some interesting problems in the theory of geometric inequalities. For example, the arguments presented in [6] are not sufficient to produce a *sharp* stability estimates for Euclidean concentration. From the mathematical viewpoint, this last problem is particularly interesting because it seems out of reach for all the three different approaches developed in proving the closely related sharp stability estimate for Euclidean isoperimetry [13, 10, 7]. A new approach is thus required, and this is the content of the joint paper [9] with Alessio Figalli and Connor Mooney.

Let us recall that the *Euclidean concentration inequality* states that if E is a subset of $\mathbb{R}^n$, E^* is a ball with same volume as E, and $N_r(E) = \{x \in \mathbb{R}^n : \mathrm{dist}(x, E) < r\}$ denotes the r-neighborhood of E, then

$$(1)\qquad |N_r(E)| \geq |N_r(E^*)|\,, \qquad \forall r > 0\,,$$

with equality if and only if, up to a zero volume set, E is a ball. The main result proved in [9] is the existence of $c(n) > 0$ such that whenever $|E| = |B|$, then there exists $x \in \mathbb{R}^n$ with

$$(2)\qquad \max\left\{r, \frac{1}{r}\right\}\left(\frac{|N_r(E)|}{|N_r(E^*)|} - 1\right) \geq c(n)\,|E\Delta(x+B)|^2\,, \qquad \forall r > 0\,.$$

The factor $\max\{r, r^{-1}\}$ is needed for the inequality to be true, as otherwise the left-hand side of the inequality tends to 0 as $r \to 0^+$ or $r \to +\infty$. Notice also that in the limit $r \to 0^+$, (2) implies the sharp quantitative isoperimetric inequality: there exists $c_*(n) > 0$ such that whenever $|E| = |B|$, then there exists $x \in \mathbb{R}^n$ with

$$(3)\qquad P(E) - P(B) \geq c_*(n)\,|E\Delta(x+B)|^2\,,$$

provided $P(E)$ denotes the perimeter of E (i.e., the $(n-1)$-dimensional measure of the boundary of E).

The approach to (3) developed in [13] is based on dimension induction through the localization of the isoperimetric deficit $P(E) - P(B)$ on hyperplane slices of E. This kind of argument, clearly, does not combine smoothly with the nonlocal nature of the operation of forming the Minkowski sum $N_r(E) = E + B_r$. Although one can use localization by slicing and dimension induction to obtain *non-sharp* quantitative versions of the Brunn-Minkowski inequality, see [8], it seems quite hard to optimize this approach to the extent of proving sharp inequalities. The mass transportation approach to (3) developed in [10] can be used to prove (2) in the special case that E is convex. This is already detailed in [10] and, with a more direct argument, in [11]. Extending this analysis to the case when E is non-convex seems hard because it would require, for example in the case $r = 1$ and with T denoting the Brenier map between E and B, to control the distance of E from its convex envelope in terms of the non-negative quantity $|S(E)| - |N_1(B)|$, where $S = \mathrm{Id} + T$. Finally, the quite versatile approach to (3) proposed in [7] is based on the regularity theory for local minimizers of the perimeter functional, an ingredient that is completely missing when the functional under consideration is the volume of the r-neighborhood of a set.

The proof of (2) given in [9] is based on two separate arguments, one degenerating as r becomes larger, the one valid only if r is large enough. Both arguments move from a "regularization by viscosity" procedure based on taking an envelope of E by balls of radius r contained in its complement. The estimate degenerating for r large is obtained by combining the strong form of (2) obtained in [12] with the reduction to this notion of r-convex envelope. In large r-regime, one shows by a geometric construction that any set with $|E| = |B|$ and $r\,(|N_r(E)|/|N_r(B)| - 1)$ small enough must have positive reach of order one in the sense of Federer. The proof is then completed by combining the Steiner-Federer formula for sets of positive reach with (3).

References

[1] G. Alberti, *Some remarks about a notion of rearrangement*, Ann. Scuola Norm. Sup. Pisa Cl. Sci XXIX, 457–472 (2000).

[2] G. Alberti and G. Bellettini, *A nonlocal anisotropic model for phase transitions Part i: the optimal profile problem*, Math. Annalen. **310**, (1998), 527–560.

[3] G. Bellettini, M. S. Gelli, S. Luckhaus and M. Novaga, *Deterministic equivalent for the Cahn-Hilliard energy of a scaling law in the Ising model*, Calc. of Variations and PDE, **26**, no 4, (2006), 429–445

[4] E. A. Carlen, M. C. Carvalho, R. Esposito, J. L. Lebowitz and R. Marra, *Droplet minimizers for the Cahn Hilliard free energy functional*, J. Geometric Analysis **16** (2006), 233–264.

[5] E. A. Carlen, M. C. Carvalho, R. Esposito, J. L. Lebowitz and R. Marra, *Droplet minimizers for the Gates- Lebowitz-Penrose free energy functional*, Nonlinearity **22**(2009), 2919–2952.

[6] E. A. Carlen and F. Maggi, *Stability for the Brunn-Minkowski and Riesz rearrangement inequalities, with applications to Gaussian concentration and finite range non-local isoperimetry*, to appear on Canadian J. Math.

[7] M. Cicalese and G. P. Leonardi, *A selection principle for the sharp quantitative isoperimetric inequality.* Arch. Ration. Mech. Anal. **206** (2012), no. 2, 617–643.

[8] A. Figalli and D. Jerison, *Quantitative stability for the Brunn-Minkowski inequality*, preprint arXiv:1502.06513.

[9] A. Figalli, F. Maggi and C. Mooney, *The sharp quantitative Euclidean concentration inequality*, preprint arXiv:1601.04100.

[10] A. Figalli, F. Maggi and A. Pratelli, *A mass transportation approach to quantitative isoperimetric inequalities.* Invent. Math. **182** (2010), no. 1, 167–211.

[11] A. Figalli, F. Maggi and A. Pratelli, *A refined Brunn-Minkowski inequality for convex sets.* Ann. Inst. H. Poincar Anal. Non Linéaire **26** (2009), no. 6, 2511–2519.

[12] N. Fusco and V. Julin, *A strong form of the quantitative isoperimetric inequality.* Calc. Var. Partial Differential Equations **50** (2014), no. 3–4, 925–937.

[13] N. Fusco, F. Maggi, A. Pratelli, *The sharp quantitative isoperimetric inequality*, Ann. of Math. **168** (2008), 941–980.

Asymptotic behavior of the inverse mean curvature flows in the hyperbolic spaces

Pei-Ken Hung

(joint work with Mu-Tao Wang)

The solution of the inverse mean curvature flow is a family of smooth maps $F_t : \Sigma^{n-1} \to M^n$ satisfying the evolution equation

$$\frac{\partial F_t}{\partial t} = \frac{\nu}{H},$$

where H is the mean curvature and ν is the unit outer normal of $\Sigma_t = F_t(\Sigma)$. Geroch [2] introduced this parabolic flow and discovered that the Hawking mass of a surface is monotone nondecreasing along the flow provided the scalar curvature of M is nonnegative. Jang-Wald [5] observed that if there is a smooth solution of the inverse mean curvature flow which starts from the horizon and exists for all time, then the Penrose inequality follows the Geroch monotonicity. However,